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Boson bunching is not maximized by indistinguishable particles

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Boson bunching is among the most remarkable features of quantum physics. A celebrated example in optics is the Hong–Ou–Mandel effect, where the bunching of two photons arises from a destructive quantum interference between the trajectories where they both either cross a beamsplitter or are reflected. This effect takes its roots in the indistinguishability of identical photons. Hence, it is generally admitted—and experimentally verified—that bunching vanishes as soon as photons can be distinguished, for example, when they occupy distinct time bins or have different polarizations. Here we disprove this alleged straightforward link between indistinguishability and bunching by exploiting a recent finding in the theory of matrix permanents. We exhibit a family of optical circuits such that the bunching of photons into two modes can be substantially boosted by making them partially distinguishable via an appropriate polarization pattern. This boosting effect is already visible in a seven-photon interferometric process, making the observation of this phenomenon within reach of current photonic technology. This unexpected behaviour questions our understanding of multiparticle interference in the grey zone between indistinguishable bosons and classical particles.

In quantum physics, it is common knowledge that a gain of information results in the extinction of quantum interference. In the iconic double-slit experiment, the interference fringes originate from the absence of which-path information[1–3]. As emphasized, for instance, by Feynman[4], the fringes necessarily disappear as soon as the experiment allows us to learn that the particles have taken one or the other path. In quantum optics, photon bunching is another distinctive feature that follows from quantum interference. Specifically, the indistinguishability of photons makes it such that one cannot know which photon has followed a given trajectory in a linear interferometer (Fig. 1). The Hong–Ou–Mandel (HOM) effect[5], for example, arises from the fact that one cannot distinguish the trajectory where two photons have crossed a 50:50 beamsplitter from the trajectory where they have both been reflected. The net result is a tendency of indistinguishable photons to occupy the same mode—that is, to bunch—as a consequence of this lack of information. In accordance with Feynman's rule of thumb, quantum interference effects become less pronounced as soon as the photons become distinguishable, for example, if they occupy distinct temporal or polarization modes so that one gains information about their individual trajectories[6]. As a consequence, the HOM dip disappears for two photons with orthogonal polarization because their distinct trajectories can then be fully traced back. Hence, even in more general scenarios involving multiple independently prepared photons and larger interferometers, it is commonly admitted that bunching effects are maximum for fully indistinguishable photons and gradually decline when photons are made increasingly distinguishable[7–9].

In this Article, we find a quantum interferometric scenario that goes against this intuition and contradicts the very idea that distinguishability undermines photon bunching. We consider the probability of multimode bunching, that is, the probability that all photons entering a linear interferometer end up in a certain subset of the output modes. In accordance with a longstanding mathematical conjecture on matrix permanents due to Bapat and Sunder[10], this bunching probability must indeed be maximum if the input state consists of fully

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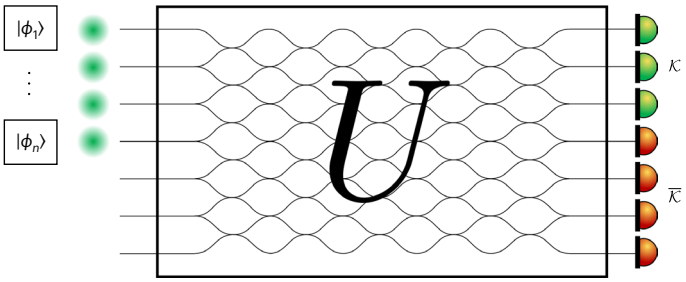

Fig. 1 | Interferometric set-up. Single photons are sent through the first n input modes of an m-mode linear interferometer U, which can always be decomposed into a network of two-mode couplers[44]. Each photon at input j carries internal degrees of freedom (polarization, arrival time, and so on) described by an (internal) wavefunction $|\phi_j\rangle$. Not perfectly overlapping wavefunctions (measured via the Gram matrix S) give rise to partial distinguishability among the photons, reducing the degree of quantum interference and bosonic effects such as bunching (Conjecture 1). We focus in particular on the probability that all n photons bunch into a subset $\mathcal{K}$ (corresponding to the green detectors) of the output modes, while the red detectors in $\overline{\mathcal{K}}$ do not click. This is a natural extension of the HOM experiment for more than two modes.

indistinguishable photons[8]. However, inspired by a counter-example to this conjecture recently discovered by Drury[11], we have found that multimode bunching may, against all odds, be enhanced if photons are partially distinguishable. For this, we not only convert Drury's counter-example into a linear interferometric experiment, but also find a natural generalization, leading to a family of physical set-ups where partial distinguishability enhances multimode bunching. Incidentally, on the mathematical side, this finding even implies new counter-examples to the Bapat–Sunder conjecture.

More specifically, we construct a family of interferometers such that a higher two-mode bunching probability is attained if the photons are prepared in a well-chosen polarization state (making them partially distinguishable) rather than in the same state (in which case they would all be indistinguishable). In other words, gaining partial information about the photons paradoxically results in a higher probability for them to coalesce on two output modes, contradicting common knowledge. We give an interpretation of the physical process behind this counterintuitive effect and prove that, in our set-up, the enhancement ratio of the two-mode bunching probability actually grows (at least linearly) with the system size. In the simplest case, an enhancement of 7% is already visible for seven photons in seven modes with a specific polarization pattern, which makes the observation of this remarkable phenomenon within reach of today's photonic technology.

Multimode boson bunching

Consider a general interferometric experiment in which n bosons are sent through a linear interferometer of m modes, described by the $m \times m$ unitary matrix U. Although our discussion is valid for any bosonic particle, we focus on photons here, because, in practice, controlled linear interferometric experiments are easier to carry out in photonics. In ref. 8, Shchesnovich provides compelling evidence for the following conjecture.

Conjecture 1 (generalized bunching). Consider any input state of n classically correlated photons. For any linear interferometer and any nontrivial subset $\mathcal{K}$ of output modes, the probability that all photons are found in $\mathcal{K}$ is maximal if the photons are perfectly indistinguishable.

Note that the considered class of input states must exclude entangled photons, so that it keeps a closer resemblance to the original HOM setting (indeed, a related conjecture with entangled input states was proven to be false in ref. 12). In fact, it is enough for our purposes to consider a further simplification of the statement of Conjecture 1 and assume that photons are prepared independently of each other, that is, they are uncorrelated, and that there is only one photon in each of the first n modes. Moreover, we may also assume that the state of each photon is pure. This setting, depicted in Fig. 1, is enough to demonstrate that Conjecture 1 is false. We refer to ref. 8 for a mathematical treatment of bunching probabilities in more general scenarios.

The state of each photon is not only described by the spatial mode it occupies but also by other degrees of freedom, such as its polarization and its spectral distribution, which we will refer to as internal degrees of freedom. Partial distinguishability can then be modelled by considering that the internal state of the photon entering mode j is described by an (internal) wavefunction $|\phi_j\rangle$. Let us denote the creation operator associated to this state as $\hat{a}^\dagger_{j,\phi_j}$. We make the common assumption that the interferometer acts only on spatial modes, leaving the internal states of the photons invariant[13–15]. More precisely, the interferometer is described by an operator $\hat{U}$ that acts as

$$\hat{U}\,\hat{a}^\dagger_{j,\phi_j}\,\hat{U}^\dagger = \sum_k U_{j,k}\,\hat{a}^\dagger_{k,\phi_j},\ \forall j. \tag{1}$$

Following refs. 13,14, it can be shown that the probabilities of the different outcomes of the linear interferometric process not only depend on the unitary U but also on the distinguishability matrix, defined as

$$S_{i,j} = \langle \phi_i | \phi_j \rangle. \tag{2}$$

This is an $n \times n$ Gram matrix constructed from all possible overlaps of the internal wavefunctions of the input photons. In particular, $S = \mathbb{1}$ if all photons are fully distinguishable, while $S = \mathbb{E}$ (with $\mathbb{E}_{i,j} = 1$ for all i,j) in the case where they are fully indistinguishable. Intermediate situations between these two extreme cases are called partial distinguishability. Note that the internal wavefunction of each photon may be multiplied by a phase $|\phi_j\rangle \to \mathrm{e}^{i\theta_j}|\phi_j\rangle$, which does not affect event probabilities. Hence, there is an equivalence class of distinguishability matrices S for each physical situation as $S_{i,j} \equiv \mathrm{e}^{i(\theta_j-\theta_i)}S_{i,j}$, so that we represent each equivalence class with a single matrix.

To compute the probability that all n photons are found in a subset $\mathcal{K}$ of the output modes, which we refer to as the multimode bunching probability $P_n(S)$, it is useful to define the matrix

$$H_{a,b} = \sum_{k\in\mathcal{K}} U^*_{k,a}\,U_{k,b}, \tag{3}$$

where $a, b \in \{1, \ldots, n\}$. For a fixed interferometer U and subset $\mathcal{K}$, the multimode bunching probability is a function of the distinguishability matrix S and can be expressed as

$$P_n(S) = \mathrm{perm}\,(H \odot S^{\mathrm{T}}) \tag{4}$$

that is, the permanent of the Hadamard (or elementwise) product $(H \odot S^{\mathrm{T}})_{i,j} \equiv H_{i,j}S_{j,i}$ (ref. 8 and Methods). It is important to remark that H, S, and $H \odot S^{\mathrm{T}}$ are all positive semidefinite matrices, which ensures their permanent is positive[16]. Moreover, note that for indistinguishable photons, $P_n^{(\mathrm{bos})} = \mathrm{perm}\,(H)$. Hence, Conjecture 1, when restricted to the setting that we consider (Fig. 1), takes the following mathematical form:

$$\mathrm{perm}\,(H \odot S^{\mathrm{T}}) \overset{?}{\leq} \mathrm{perm}\,(H), \tag{5}$$

with the equality holding if S corresponds to indistinguishable photons. The reasons to presume that this conjecture might be an actual physical law governing multiparticle interferences are manifold and, in what follows, we detail several evidences supporting this hypothesis.

Single-mode bunching

If the subset $\mathcal{K}$ is a single output mode, then the conjecture holds. In this case, if we choose $\mathcal{K} = \{1\}$, the single-mode bunching probability is given by

$$P_n(S) = \prod_{j=1}^{n} |U_{1,j}|^2 \,\mathrm{perm}\,(S) = P_n^{(\mathrm{dist})}\,\mathrm{perm}\,(S), \tag{6}$$

where $P_n^{(\mathrm{dist})} = P_n(\mathbb{1})$ corresponds to the single-mode bunching probability when the photons are fully distinguishable. Then, $P_n(S)$ is indeed maximum for fully indistinguishable photons, because the maximum value of perm(S) is attained when $S = \mathbb{E}$, with perm $(\mathbb{E}) = n!$. This is also a reason why perm(S) can be seen as a measure of indistinguishability of the input photons[13,17]. Note that the celebrated HOM effect can be simply recovered from equation (6): for a two-mode 50:50 beamsplitter, the maximum probability of bunching in a single output mode is attained for fully indistinguishable photons and given by $1/4 \times 2! = 1/2$.

In addition, even if we take $|\mathcal{K}| > 1$ but restrict to compare fully indistinguishable with fully distinguishable photons, it appears that the multimode bunching probability satisfies $P_n(\mathbb{E}) \geq P_n(\mathbb{1})$ (ref. 8), further suggesting that any partial distinguishability is bound to decrease bunching effects.

Fermion antibunching

Another physical motivation for Conjecture 1 is the fact that an analogous statement on fermion antibunching can be proved. Indeed, an input state of n fully indistinguishable fermions minimizes the probability that all of the n fermions are bunched in any subset $\mathcal{K}$ of the output modes. This can be shown using a famous result by Schur[8,18]. In the setting we consider, the fermionic multimode bunching probability obeys

$$P_n^{(\mathrm{ferm})}(S) = \det\,(H \odot S^{\mathrm{T}}) \geq \det H, \tag{7}$$

which follows from the Oppenheim inequality for determinants[19] stating that for any two positive semidefinite matrices A and B we have

$$\det\,(A \odot B) \geq \det A \,\det B. \tag{8}$$

Bapat–Sunder conjecture

In 1985, Bapat and Sunder questioned whether an analogue to the Oppenheim inequality holds for permanents[10]. They conjectured the following.

Conjecture 2 (Bapat–Sunder). For any two positive semidefinite $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we have

$$\mathrm{perm}\,(A \odot B) \leq \mathrm{perm}\,(A) \prod_{i=1}^{n} b_{ii}.$$

It is easy to see that if this statement was valid, it would imply equation (5), thus confirming the validity of Conjecture 1 in the specific setting of Fig. 1 (ref. 8).

Counter-example (Drury). The Bapat–Sunder conjecture was recently disproved by Drury, who found a seven-dimensional counter-example[11]. It consists in a positive semidefinite matrix A of dimension 7, whose diagonals are $a_{ii} = 1$, which is such that

$$\frac{\mathrm{perm}\,(A \odot A^{\mathrm{T}})}{\mathrm{perm}\,(A)} = \frac{1,237}{1,152} \approx 1.07, \tag{9}$$

thus implying a violation of Conjecture 2. Instead of presenting the matrix A, it will be useful to consider its Cholesky decomposition, which can always be found for any positive semidefinite matrix[20]. Thus, we write $A = M^{\dagger}M$, with

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & \sqrt{2} & 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 \end{pmatrix} \tag{10}$$

where $\omega = \exp(2i\pi/5)$ is the fifth root of unity. As we shall see, the existence of this counter-example implies that there are 7×7 matrices H and S for which equation (5) is false, hence contradicting Conjecture 1.

Consequently, in spite of the compelling evidences listed above suggesting that multimode bunching should be maximum for indistinguishable bosons, Drury's counter-example allows us to predict the existence of boosted boson bunching with partially distinguishable bosons. Before turning to the optical realization and physical mechanism behind this counterintuitive phenomenon, we stress that brute-force numerical trials really seem to support the (now proven wrong) Conjecture 1 and Conjecture 2. In particular, we generated 10^7 samples of dimension $n = 7$ and of rank $r = 2$ with the following physically inspired procedure: a unitary U is sampled randomly according to the Haar measure and used to compute the matrix H. The matrix S is constructed by taking random normalized vectors in a space of dimension r. We did not encounter a single counter-example, which suggests that Conjecture 1 holds in practically all cases, even when restricting to a dimension and rank where we know that a counter-example actually exists. Numerical trials of Conjecture 2 with two random Gram matrices also gave similar results. These observations are corroborated by the numerical searches reported in refs. 8,12 and thus make the finding of enhanced boson bunching via partial distinguishability even more surprising. Arguably, the violation of the extended Conjecture 1 for some entangled input photon states observed in ref. 12 could be viewed as an instance of enhanced boson bunching, but these counter-examples do not break any common assumption on multiphoton interference and it is unclear whether such states can be prepared experimentally using linear interferometry and currently available photon sources.

Optical realization of boosted bunching

As we now prove, Drury's counter-example entails the existence of a physical experiment that violates the generalized bunching conjecture. In Fig. 2a, we present a possible optical set-up realizing this violation and involving seven-photon interferometry. Details on how to construct the internal states of the photons $|\varphi_j\rangle$ and the unitary matrix U to obtain the desired H and S matrices are explained in Methods. In a nutshell, states $|\varphi_j\rangle$ can be read from the columns of matrix M in equation (10), while U is chosen to contain a rescaled version of M as a submatrix.

The resulting set-up is surprisingly simple: the seven-mode interferometer U is constructed with a five-mode discrete Fourier transform (DFT) supplemented with two additional beamsplitters (with the same transmittance $\eta = 2/7$). As S is of rank 2, the internal states live in a two-dimensional Hilbert space, so it is most natural to use photon polarization, which can easily be manipulated via waveplates (other encodings would also be possible, such as time-bin encoding).

A single photon is sent in each of the seven input modes with an appropriately chosen polarization state, making the seven photons partially distinguishable. The polarization pattern is shown in Fig. 2b and can be viewed as a five-star polarization state denoted by a star, ⋆ (for the five photons entering the DFT), supplemented with a horizontally polarized and a vertically polarized state. The probability of detecting all seven photons in the output bin (modes 0′ and 1′) is then given by $P_7^{(\star)} \approx 7.5 \times 10^{-3}$. By comparison, for fully indistinguishable photons (all with the same polarization), this probability is only $P_7^{(\mathrm{bos})} \approx 7 \times 10^{-3}$. This simple experimental set-up thus exhibits a bunching violation greater than 7%, in accordance with equation (9). The photon-number distribution in modes 0′ and 1′ is depicted in

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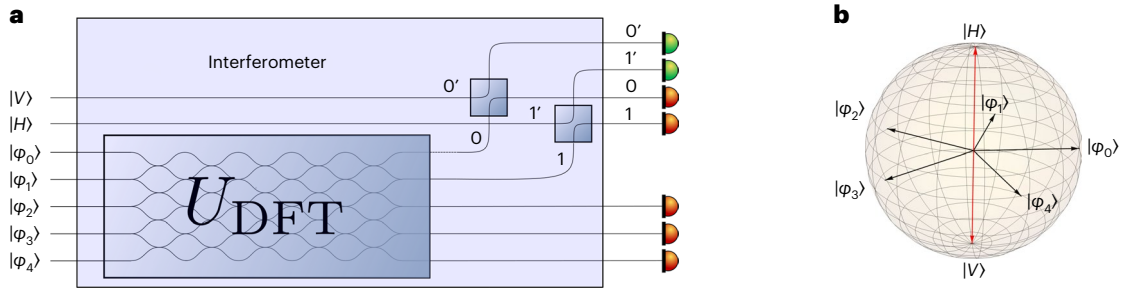

Fig. 2 | Boosted two-mode bunching. a, Seven-mode interferometer violating the generalized bunching conjecture for an appropriate input polarization pattern (depicted in **b**). Here seven photons are sent into the seven input modes, and the probability of detecting them all in the two output modes indicated with green detectors exceeds its value for indistinguishable photons (all with the same polarization). We assume the action of the interferometer is polarization-independent; see equation (1). This set-up can be generalized to n modes as follows. Defining $q = n - 2$, we first send q photons in a q-mode DFT interferometer $U_{jk} = \frac{1}{\sqrt{q}}\omega^{jk}$ with polarization states $|\varphi_j\rangle = \frac{1}{\sqrt{2}}(|H\rangle + \omega^j |V\rangle)$, where $j, k = 0, \ldots, q-1$ and $\omega = \exp(2i\pi/q)$. The upper two output modes (labelled 0 and 1) of the DFT are then sent to two beam splitters of equal transmittance $\eta = 2/n$, achieving interference respectively with a vertically polarized photon (in mode 0′) and horizontally polarized photon (in mode 1′). We measure the bunching of all n photons in the subset $\mathcal{K}$ corresponding to the output modes 0′ and 1′ indicated with green detectors; thus all red detectors do not click. For $n \geq 7$, we observe a boosted two-mode bunching probability by comparison with indistinguishable photons. **b**, Bloch-sphere representation of the input polarization pattern for $n = 7$. The polarization states of the five input photons of the DFT (indicated as black arrows) are equally spaced along the equator of the Bloch sphere. We call this special state a five-star polarization state (or q-star polarization state for general q) and denote it by a star. The two extra photons (indicated as red arrows) have antipodal—horizontal and vertical—polarization states.

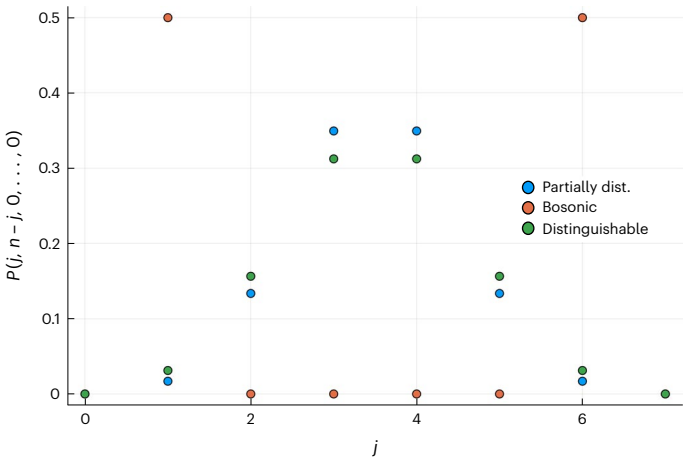

Fig. 3 | Photon-number distribution at the output of the circuit achieving boosted two-mode bunching. The probability distributions in output mode 0′ of the seven-mode circuit of Fig. 2 are plotted for three different scenarios. They are normalized; that is, the probabilities are conditioned on events where all seven photons end up in modes 0′ and 1′ (two-mode bunching events). Due to the symmetry of the circuit, the probability of event $(j, 7-j)$ is the same as event $(7-j, j)$. For fully indistinguishable bosons (red points), most events are interferometrically suppressed, originating from the fact that the output of the five-mode DFT is a NOON state; see equation (12). The only surviving events are (1, 6) and (6, 1). Using partially distinguishable bosons (blue points) erases these destructive interferences and, as proven in this work, enhances the overall two-mode bunching probability. This also leads to a qualitatively very different photon-number distribution, which has a Bell-like shape. Interestingly, fully distinguishable particles (green points) lead to a photon-number distribution of similar Bell-like shape but a considerably smaller two-mode bunching probability. The two-mode bunching probabilities in the three cases are $P_7^{(\mathrm{bos})} \approx 7 \times 10^{-3}$, $P_7^{(\star)} \approx 7.5 \times 10^{-3}$ and $P_7^{(\mathrm{dist.})} \approx 1.5 \times 10^{-4}$, respectively.

Fig. 3 for different scenarios (fully indistinguishable, partially distinguishable and fully distinguishable photons).

The outstanding recent progress in boson sampling experiments indicates that the experimental observation of such a boosted bunching due to partial distinguishability should be possible with present-day technology[21–26]. For example, five-photon coincidence rates in the hundreds of hertz range have already been demonstrated in optical circuits of a much bigger size than the one represented in Fig. 2 (ref. 24). Moreover, optical implementations of the DFT of dimensions 6 and 8 have already been reported[27,28]. The required photon-number resolution (up to seven photons) could be achieved with single-photon-resolving detectors by first multiplexing the output modes 0′ and 1′ into several spatial or temporal modes[29,30]. In fact, very recently, a photon-number resolution of up to a hundred photons was achieved[31]. As a further feasibility argument, we note that the experimental scheme realizing boosted bunching is stable under small perturbations to the matrix elements of the unitary U as well as to the distinguishability matrix S (Fig. 4). This is easy to understand intuitively as the permanent is a sum of products of matrix elements, and thus smooth and well behaved under Taylor expansion (for a formal proof, see 'Resilience to perturbations' in Supplementary Information). A fully realistic treatment of how perturbations affect the bunching violation would depend on the particular details of the physical implementation of the scheme and is out of the scope of the current work.

Physical mechanism and asymptotically large violations

The interferometer shown in Fig. 2 arguably provides only a small relative violation of the generalized bunching conjecture, as seen from equation (9). It is natural to ask whether larger relative violations can be obtained and what would be the corresponding physical set-up. Moreover, from a physics perspective, it is important to pinpoint the underlying mechanism that explains the violation, at least in some particular setting. We answer these questions and find a way to generalize the seven-mode circuit into an n-mode circuit, as described in the caption of Fig. 2. For this family of circuits, we show that the ratio of the bunching probabilities, which we refer to simply as the bunching violation ratio R_n, obeys the bound

$$R_n = \frac{P_n^{(\star)}}{P_n^{(\mathrm{bos})}} \geq \frac{n}{8} + \frac{1}{32}\frac{(n-2)^2}{n-1}, \tag{11}$$

for any $n \geq 4$. Hence, partial distinguishability can lead to an asymptotically larger multimode bunching probability with respect to fully indistinguishable bosons. Incidentally, this family of circuits also yields some previously unreported family of H and S matrices that violate the Bapat–Sunder conjecture. Although a detailed derivation of this bound

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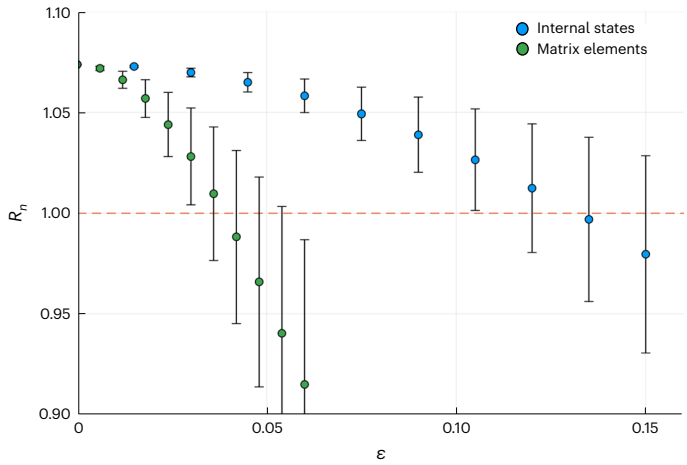

Fig. 4 | Perturbation effects on boosted two-mode bunching. The bunching violation ratio R_n (vertical axis) is plotted for the seven-mode circuit of Fig. 2. Blue symbols are associated with perturbations of the internal wavefunctions defined in the caption of Fig. 2 (for $n = 7$) by a random amount drawn from a Gaussian distribution with zero mean and standard deviation ϵ (horizontal axis). For each ϵ value, 10^4 samples are taken. We observe violations of Conjecture 1 (on average) up to $\epsilon_{\max} \approx 0.135$, which corresponds to the components of the internal wavefunctions being perturbed by $\epsilon_{\max}/(1+\epsilon_{\max}^2) \approx 13.3\%$. The vertical bars represent the standard deviation for each ϵ value. Green symbols are associated with perturbations of the matrix elements U_{ij} of the seven-mode interferometer. The same random Gaussian perturbations are added to the columns of matrix U, which are then Gram-orthonormalized. In this case, the matrix elements can be perturbed by $\epsilon_{\max} \approx 0.039$ to still exhibit a violation (on average). The optical scheme is thus resilient to perturbations both to the internal states of the photons and to the interferometer, making a good case for its experimental feasibility.

is given in Methods, we present here the main arguments by comparing the physical mechanism of bunching for fully indistinguishable and partially distinguishable photons.

Fully indistinguishable photons

The first step of the argument is to note that the only possibility for all of the n photons to be observed in the subset $\mathcal{K}$ (output modes 0′ and 1′) is if there are $q = n-2$ photons in the first two output modes of the DFT interferometer of dimension q and vacuum on the rest (we assume $q \geq 2$). The corresponding conditional (subnormalized) state of these two output modes is given by the NOON state[32]

$$\begin{aligned}\left|\psi_{\text{out}}^{(\text{bos})}\right\rangle &= \frac{1}{q^{q/2}}\prod_{j=0}^{q-1}\left(\hat{a}_0^\dagger + \omega^j \hat{a}_1^\dagger\right)|0\rangle \\ &= \frac{1}{q^{q/2}}\left((\hat{a}_0^\dagger)^q + (-1)^q(\hat{a}_1^\dagger)^q\right)|0\rangle,\end{aligned} \tag{12}$$

where $\omega = \exp(2i\pi/q)$ is the qth root of unity. The probability of having q photons in these two modes is simply given by the square norm of this state, namely $2q!/q^q$. The next part of the circuit realizes the interference between modes 0 and 0′ via beamsplitter $\hat{U}_{\text{BS}}^{0,0'}$, as well as between modes 1 and 1′ via beamsplitter $\hat{U}_{\text{BS}}^{1,1'}$. The action of these beamsplitters on state $\hat{a}_{0'}^\dagger \hat{a}_{1'}^\dagger \left|\psi_{\text{out}}^{(\text{bos})}\right\rangle$ followed by postselection on vacuum in both output modes 0 and 1 is analysed in Methods. The resulting probability is governed by the bunching mechanism sketched in the upper part of Fig. 5, where η denotes the transmittance of the two beamsplitters. The first term of state (equation (12)) describing q photons in mode 0 undergoes bunching with the extra photon in mode 0′ with probability $(q+1)\,\eta(1-\eta)^q$, while the extra photon in mode

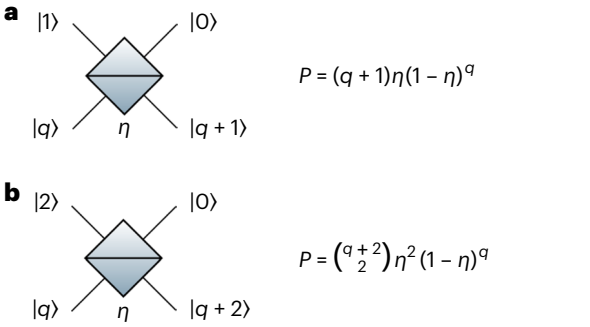

Fig. 5 | Mechanism at the origin of boosted bunching. a, For indistinguishable photons, the extra photon in mode $\hat{a}_{0'}^\dagger$ (or $\hat{a}_{1'}^\dagger$) bunches with q photons in mode $\hat{a}_0^\dagger$ (or $\hat{a}_1^\dagger$) coming from the NOON state (equation (12)). **b**, For partially distinguishable photons, the largest term contributing to the bunching probability (equation (21)) comes from the double-bunching of two photons in the delocalized mode $\hat{c}_+'^\dagger$ (equation (19)) with q photons in the delocalized mode $\hat{c}_+^\dagger$ (equation (15)). The asymptotics of the bunching violation ratio R_n as shown in equation (11) originates from the probabilities indicated on the right of the processes depicted here.

1′ is simply transmitted with probability η. The second term of state (equation (12)) behaves similarly. Consequently, the multimode bunching probability (in output modes 0′ and 1′) is given by

$$P_n^{(\text{bos})} = \frac{2(q+1)!}{q^q}\eta^2(1-\eta)^q \tag{13}$$

Partially distinguishable photons

We consider the q-star polarization pattern

$$\left|\psi_{\text{in}}^{(\star)}\right\rangle = \frac{1}{2^{q/2}}\prod_{j=0}^{q-1}\left(\hat{a}_{h,j}^\dagger + \omega^j \hat{a}_{v,j}^\dagger\right)|0\rangle \tag{14}$$

as the input state sent to the DFT interferometer. Here $\hat{a}_{h,j}^\dagger$ ($\hat{a}_{v,j}^\dagger$) are creation operators of a photon in spatial mode j and horizontal (vertical) polarization. This state is a generalization of the five-star polarization pattern shown in Fig. 2b. To understand why multimode bunching is boosted with this special input state, it is convenient to define the spatio-polarization modes

$$\hat{c}_\pm^\dagger = \frac{\hat{a}_{h,1}^\dagger \pm \hat{a}_{v,0}^\dagger}{\sqrt{2}}. \tag{15}$$

Similarly as for indistinguishable photons, we compute the conditional output state of the DFT interferometer that contains q photons in the first two output modes (and vacuum for the rest) for the input state (equation (14)), namely

$$\left|\psi_{\text{out}}^{(\star)}\right\rangle = \frac{1}{q^{q/2}}\prod_{j=0}^{q-1}\left(\frac{\hat{a}_{h,0}^\dagger}{\sqrt{2}} + \omega^j \hat{c}_+^\dagger + \omega^{2j}\frac{\hat{a}_{v,1}^\dagger}{\sqrt{2}}\right)|0\rangle. \tag{16}$$

This polarization two-mode state is the counterpart of the NOON state (equation (12)). It comprises many terms, most importantly the one containing q photons in mode $\hat{c}_+^\dagger$, namely

$$\left|\psi_{\text{out}}^{(\star)}\right\rangle = \frac{(-1)^{q-1}}{q^{q/2}}(\hat{c}_+^\dagger)^q|0\rangle + \ldots. \tag{17}$$

In what follows, we show that this term alone is enough to prove that the bunching violation ratio in equation (11) grows at least linearly

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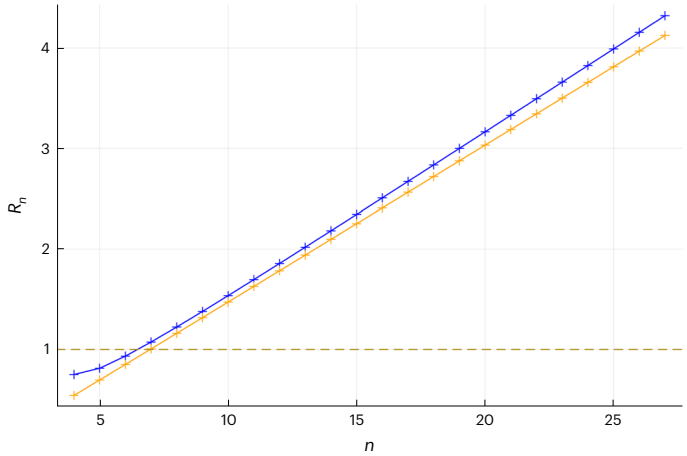

Fig. 6 | Bunching violation ratio R_n as a function of n. The value of the bunching violation ratio R_n is plotted (blue) for the family of optical schemes shown in Fig. 2, with the horizontal dashed line indicating the threshold $R_n = 1$. For $n \geq 7$, it appears that partially distinguishable particles outperform indistinguishable bosons as witnessed by $R_n > 1$. The lower bound given by equation (11) is also shown (orange).

(see Methods for a more detailed derivation). The other components of state (equation (17)) are orthogonal to the state $(\hat{c}_+^\dagger)^q|0\rangle$ and thus can only contribute with additional positive terms to the bunching probability. The probability to have q photons (regardless of their polarization) in the first two output modes of the DFT interferometer is thus lower bounded by $q!/q^q$. The subsequent part of the interferometer is fed with the state

$$\hat{a}_{h,1'}^\dagger\, \hat{a}_{v,0'}^\dagger \left|\psi_{\text{out}}^{(\star)}\right\rangle = \frac{(\hat{c}_+'^\dagger)^2 - (\hat{c}_-'^\dagger)^2}{2} \left|\psi_{\text{out}}^{(\star)}\right\rangle, \tag{18}$$

where we have described the two extra photons with antipodal polarization (H and V) using the spatio-polarization modes

$$\hat{c}_\pm'^\dagger = \frac{\hat{a}_{h,1'}^\dagger \pm \hat{a}_{v,0'}^\dagger}{\sqrt{2}}, \tag{19}$$

defined in analogy with equation (15). Thus, the leading term of the final output state of the interferometer is

$$\frac{(-1)^{q-1}}{2q^{q/2}} \hat{V}\left((\hat{c}_+'^\dagger)^2 - (\hat{c}_-'^\dagger)^2\right)(\hat{c}_+^\dagger)^q|0\rangle + \ldots, \tag{20}$$

where we have defined $\hat{V} = \hat{U}_{\text{BS}}^{0,0'}\, \hat{U}_{\text{BS}}^{1,1'}$ as the operator describing the joint operation of the two beamsplitters in both polarization (Methods). Using the fact that the beamsplitters have the same transmittance η, it appears that $\hat{V}$ also acts as a beamsplitter of transmittance η that couples modes $\hat{c}_+^\dagger$ and $\hat{c}_+'^\dagger$. Note that it also couples $\hat{c}_-^\dagger$ and $\hat{c}_-'^\dagger$ but we disregard the corresponding term here as its contribution is much smaller and we seek a lower bound on the probability (Methods). We thus have interference between q photons in mode $\hat{c}_+^\dagger$ (occurring with probability $q!/q^q$) and two photons in mode $\hat{c}_+'^\dagger$ (occurring with probability 1/2). Hence, the resulting probability of obtaining $n = q + 2$ photons in output mode $\hat{c}_+'^\dagger$ (which leads to the detection of n photons in the output bin $\mathcal{K} = \{0', 1'\}$) is governed by the double-bunching mechanism sketched in the lower part of Fig. 5, associated with probability $\binom{q+2}{2}\eta^2(1-\eta)^q$. As a result, we obtain the following bound on the two-mode bunching probability in the case of partially distinguishable photons:

$$P_n^{(\star)} \geq \frac{(q+2)!}{4q^q}\eta^2(1-\eta)^q. \tag{21}$$

This probability is asymptotically larger than its counterpart for fully indistinguishable photons (equation (13)). Using equations (13) and (21), we indeed obtain a lower bound on the bunching violation ratio

$$R_n = \frac{P_n^{(\star)}}{P_n^{(\text{bos})}} \geq \frac{q+2}{8} = \frac{n}{8}, \tag{22}$$

which confirms the dominant term in equation (11) and shows that it grows at least linearly with n. A more detailed calculation given in Methods leads to the second term in equation (11). As seen in Fig. 6, this bound seems to describe well enough the behaviour of R_n up to $n = 30$. In the special case where $n = 7$, we may compute exactly all terms in equation (17), which gives $R_7 = 1{,}237/1{,}152$, in perfect agreement with equation (9). Note also that equation (11) shows no dependence on the transmittance η. However, the absolute probability of bunching events can be maximized by maximizing the term $\eta^2(1-\eta)^q$ over η, which yields $\eta = 2/n$, as mentioned in the caption of Fig. 2.

Discussion and outlook

The complex behaviour of interferometric experiments with multiple partially distinguishable photons has been explored in several theoretical and experimental works[33–41], revealing that many-body interference does not reduce to a simple dichotomy between distinguishable and indistinguishable photons. This is evident, for example, from the fact that certain outcome probabilities do not behave monotonically as one makes photons more distinguishable[33,34]. However, the scheme of Fig. 2 is the first explicit set-up showing that boson bunching can be boosted by partial distinguishability to the point where it actually beats ideal (fully indistinguishable) bosons. This disproves the common belief that bunching effects are necessarily maximized in this ideal scenario.

It is intriguing to observe that the state of partially distinguishable photons we have found to exhibit boosted bunching is, in a sense, far from the state of fully indistinguishable photons. This can be seen by computing the relative contribution of the fully (permutation-) symmetric component of the internal wavefunction[17] $|\Phi\rangle = |\phi_1\rangle|\phi_2\rangle \ldots |\phi_n\rangle$:

$$\mathrm{d}(S) = \langle\Phi|\hat{\mathcal{S}}_n|\Phi\rangle = \frac{\mathrm{perm}\,(S)}{n!}, \tag{23}$$

where $\hat{\mathcal{S}}_n = (1/n!)\sum_{\sigma\in S_n}\hat{P}_\sigma$ is the symmetrizer in n dimensions. This measure is 1 for fully indistinguishable photons ($S = \mathbb{E}$) and $1/n!$ for fully distinguishable ones ($S = \mathbb{1}$). For the simplest case of seven photons, we have that

$$\begin{aligned} \mathrm{d}(\mathbb{1}) &= \frac{1}{7!} \approx 1.98 \times 10^{-4} \\ \mathrm{d}(S^{(\star)}) &= \frac{45}{7!} \approx 8.93 \times 10^{-3} \ll 1 \end{aligned} \tag{24}$$

where $S^{(\star)}$ is the Gram matrix of the partially distinguishable polarization state shown in Fig. 2. This state is thus somehow closer to fully distinguishable photons. It is also natural to ask whether there may exist states violating the generalized bunching conjecture already in the vicinity of a fully indistinguishable state. We show in the 'Stability around the bosonic case' section in Supplementary Information that first-order perturbations around $S = \mathbb{E}$ leave the multimode bunching probability constant, which suggests that it is a local extremum. However, the question remains open whether this probability may still increase near $S = \mathbb{E}$ if second-order terms are taken into account, which is an interesting possibility to investigate.

Furthermore, it must be noted that we have modelled distinguishable photons with a two-dimensional internal degree of freedom, namely polarization. In contrast, in the fully distinguishable setting, each photon occupies a different internal state, forming an orthonormal basis of an n-dimensional space. Is it possible to find counter-examples where the internal states live in a larger space, going beyond small perturbations around our counter-examples? This could model realistic situations with photons occupying partly overlapping time bins, possibly leading to even higher bunching violation ratios. We leave this question for future work.

On a final note, we stress that our findings corroborate the deep connection between bosonic interferences in quantum physics on the one hand, and the algebra of matrix permanents on the other hand. The transposition of Drury's matrix counter-example into a quantum interferometric experiment has inspired us to find a new family of n-dimensional matrices that not only violate the Bapat–Sunder conjecture but also exhibit a relative violation increasing with n. We anticipate that other mathematical conjectures on permanents may be addressed by exploiting this fruitful interplay with physics-inspired mechanisms such as those shown in Fig. 5. This may even help solve other questions on the Bapat–Sunder conjecture[42]. For example, the smallest known counter-example is a 7×7 matrix, so it would be interesting to find a simpler counter-example if it exists, or show that this is not possible. Another open question in matrix theory is whether there is a counter-example involving a real matrix of dimension smaller than 16 (ref. 43), which may be resolved by considering interferometry within real quantum mechanics.

Overall, we hope that this work will open new paths to explore the connection between distinguishability and boson bunching, leading not only to a better understanding of multiparticle quantum interference but perhaps also to novel applications of partially distinguishable photons to quantum technology.

Online content

Any methods, additional references, Nature Portfolio reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at https://doi.org/10.1038/s41566-023-01213-0.

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Methods

Bunching probability

In this section we summarize the main steps needed to derive equation (4). Following the colloquial conventions of ref. 45, we consider n photons sent through an (m, m) linear interferometer described by a unitary matrix U. We limit ourselves to at most one photon per input mode. Without loss of generality, we consider that the photons occupy the first n input modes. We denote the vector of occupation numbers of the output modes as $\mathbf{s} = (s_i)$ where $0 \le s_i \le n$ is the number of photons in output mode i. Naturally $\sum s_i = n$. We define the mode assignment list $\mathbf{d} = \mathbf{d}(\mathbf{s}) = \oplus_{i=1}^{m} \oplus_{k=1}^{s_i} (i)$. For example, if $\mathbf{s} = (2, 0, 1)$ then $\mathbf{d} = (1, 1, 3)$.

Consider the probability $P(\mathbf{d})$ that the photons give an output configuration $\mathbf{s}$ with a mode assignment list $\mathbf{d} = \mathbf{d}(\mathbf{s})$. Tichy shows that this probability can be expanded as a multi-dimensional tensor permanent[13]

$$P(\mathbf{d}) = \frac{1}{\mu(\mathbf{s})} \sum_{\sigma,\rho \in S_n} \prod_{j=1}^{n} \left(U_{\sigma_j, d_j} U^{*}_{\rho_j, d_j} S_{\rho_j, \sigma_j} \right) \tag{25}$$

with $\mu(\mathbf{s}) = \prod_{i=1}^{m} s_i!$ and S being the distinguishability matrix.

Let us now compute the probability $P_n(S)$ that all n photons bunch into a subset $\mathcal{K}$ of the output modes, for a set interferometer and a Gram matrix S, following a derivation of Shchesnovich[8]. Without loss of generality, consider that the subset $\mathcal{K}$ is the first $K = |\mathcal{K}|$ output modes. The bunching probability is the sum over all event probabilities $P(\mathbf{d})$ with $s_i = 0$ for all $i > K$:

$$P_n(S) = \frac{1}{n!} \sum_{d_1=1}^{K} \cdots \sum_{d_n=1}^{K} \sum_{\sigma,\rho \in S_n} \prod_{j=1}^{n} \left(U_{\sigma_j, d_j} U^{*}_{\rho_j, d_j} S_{\rho_j, \sigma_j} \right). \tag{26}$$

Now, calling

$$H_{a,b} = \sum_{k=1}^{K} U^{*}_{k,a} U_{k,b}, \tag{27}$$

we can rewrite

$$P_n(S) = \sum_{\sigma' \in S_n} \prod_{j=1}^{n} \left(H_{j,\sigma'_j} S^{\mathrm{T}}_{j,\sigma'_j} \right) = \mathrm{perm}\,(H \odot S^{\mathrm{T}}), \tag{28}$$

where the last quantity is the permanent of the Hadamard (or element-wise) product $(H \odot S^{\mathrm{T}})_{ij} \equiv H_{ij} S_{ji}$. Note that, for indistinguishable particles, $S_{ij} = 1, \forall\, i,j$, so that $P_n(\mathbb{E}) = \mathrm{perm}(H)$.

Physical realization of violating matrices

It is possible to show that any counter-example to the Bapat–Sunder conjecture can be used to construct a physical interferometer U and a set of internal states of the photons $\{|\phi_i\rangle\}$ that provide a counter-example to the generalized bunching conjecture. We assume, without loss of generality[46], a simplified form of the Bapat–Sunder conjecture, where A and B are Gram matrices and so $a_{ii} = b_{ii} = 1, \forall\, i \in \{1, \ldots, n\}$. In this case, Conjecture 2 takes the form $\mathrm{perm}\,(A \odot B) \le \mathrm{perm}\,(A)$. As the distinguishability matrix S is a Gram matrix, we can choose $S^{\mathrm{T}} = B$. The set of quantum states realizing any given distinguishability matrix can be obtained from its Cholesky decomposition:

$$B = M'^{\dagger} M'. \tag{29}$$

The matrix M' is of size $r' \times n$, where r' is the rank of B. The n internal photon states $\{|\phi_i\rangle\}$ that realize this Gram matrix can be read out from the columns of M'. Thus, the rank of B determines the dimension of the Hilbert space spanned by the states $\{|\phi_i\rangle\}$. For the physical realization of Drury's counter-example we chose $M' = M^{*}$, with M given in equation (10), hence $S = A = M^{\dagger} M$. This implies that the internal states of the photons live in a two-dimensional space, where each state is obtained from each column of M.

In addition, it is always possible to construct an interferometer U such that $H = \alpha A$, where α is a positive rescaling factor such that $\alpha \le 1$. Note that this rescaling is not important when it comes to showing a violation of the generalized bunching conjecture because if $\mathrm{perm}\,(A \odot B) > \mathrm{perm}\,(A)$, then $\mathrm{perm}\,(\alpha A \odot B) > \mathrm{perm}\,(\alpha A)$. We can write the Cholesky decomposition of αA as

$$H_{a,b} = \alpha \sum_{k=1}^{r} M^{\dagger}_{a,k} M_{k,b} = \sum_{k=1}^{r} \left(\sqrt{\alpha}\, M^{*}_{k,a} \right) \left(\sqrt{\alpha}\, M_{k,b} \right), \tag{30}$$

where r is the rank of A. By comparing with the definition of the matrix H in equation (3) or equation (27), it is possible to see that we obtain $H = \alpha A$ if we appropriately incorporate the matrix $\sqrt{\alpha} M$ as a submatrix of U, for example, in the upper part. This choice determines that the subset $\mathcal{K}$ is given by the first r output modes. Note also that it is always possible to incorporate an arbitrary complex matrix, up to renormalization, into a bigger unitary matrix, using arguments similar to lemma 29 of ref. 47.

In the case discussed in the main text, the aim is to construct an interferometer U that contains a rescaled version of the 2×7 matrix $\sqrt{\alpha} M$, where M is given in equation (10). Here the procedure to construct U is simplified by the fact that the rows of M are already orthogonal vectors. Hence, we can choose $\alpha = 2/7$ to normalize these rows and find five other orthonormal vectors to construct a 7×7 unitary matrix. The unitary built from the circuit presented in Fig. 2 gives one possibility to construct such a unitary, which was chosen for its simplicity.

Bound on the bunching violation ratio

In this section we give the detailed derivation leading to the bound on the bunching violation ratio of equation (11). This also allows for a better physical understanding of the reason behind the enhanced bunching using partially distinguishable photons.

We consider the circuit described in the caption Fig. 2, which is a generalization of the seven-mode optical circuit depicted in this figure to a circuit of n modes. The circuit is composed of a DFT circuit of size $q = n - 2$ applied to input modes $\{0, 1, \ldots, q-1\}$ followed by two beamsplitters of equal transmittance applied between modes $0'$ and 0 as well as between modes $1'$ and 1.

Our quantity of interest is the probability of observing all the n photons in output modes $0'$ and $1'$. Let us first compute this quantity when the input photons are fully indistinguishable. The quantum state at the output of the DFT is given by

$$\hat{U}_{\mathrm{DFT}} |\psi_{\mathrm{in}}\rangle = \hat{U}_{\mathrm{DFT}} \prod_{j=0}^{q-1} \hat{a}^{\dagger}_j |0\rangle \tag{31}$$

$$= \frac{1}{q^{q/2}} \prod_{j=0}^{q-1} \left(\sum_{k=0}^{q-1} \omega^{jk} \hat{a}^{\dagger}_k \right) |0\rangle, \tag{32}$$

where $\omega = \exp(2i\pi/q)$. The only possibility for all of the n photons to be observed in modes $0'$ or $1'$ at the output of the full circuit is if at the output of the DFT interferometer there are $q = n - 2$ photons in modes 0 or 1 and vacuum elsewhere. Hence, we only consider the subnormalized component of the wave function in these output modes, given by

$$\left| \psi^{(\mathrm{bos})}_{\mathrm{out}} \right\rangle = \frac{1}{q^{q/2}} \prod_{j=0}^{q-1} \left(\hat{a}^{\dagger}_0 + \omega^j \hat{a}^{\dagger}_1 \right) |0\rangle. \tag{33}$$

To expand this expression, one can think of $\hat{a}^{\dagger}_0$ and $\hat{a}^{\dagger}_1$ as complex numbers because these two operators commute. In this sense, following ref. 32, we can consider that each term $\hat{a}^{\dagger}_0 + \omega^j \hat{a}^{\dagger}_1$ is an eigenvalue

of the circulant matrix of dimension q given by $\mathrm{circ}(\hat{a}_0^\dagger, \hat{a}_1^\dagger, 0, \ldots, 0)$. Hence, the previous equation can be rewritten as

$$\left|\psi_{\mathrm{out}}^{(\mathrm{bos})}\right\rangle = \frac{1}{q^{q/2}} \det(\mathrm{circ}\,(\hat{a}_0^\dagger, \hat{a}_1^\dagger, 0, \ldots, 0))\,|0\rangle \tag{34}$$

$$= \frac{1}{q^{q/2}} \left((\hat{a}_0^\dagger)^q + (-1)^q (\hat{a}_1^\dagger)^q\right)|0\rangle, \tag{35}$$

which is a NOON state. The subsequent part of the interferometer couples this state with the ancillary modes $0'$ and $1'$, each containing a single photon. We denote these beamsplitters by $\hat{U}_{\mathrm{BS}}^{0,0'}$ and $\hat{U}_{\mathrm{BS}}^{1,1'}$ and their transmittance by η. We use the following convention for the unitary representing the action of the beamsplitter:

$$U_{\mathrm{BS}} = \begin{pmatrix} \sqrt{\eta} & \sqrt{1-\eta} \\ -\sqrt{1-\eta} & \sqrt{\eta} \end{pmatrix}. \tag{36}$$

The joint application of $\hat{U}_{\mathrm{BS}}^{0,0'}$ and $\hat{U}_{\mathrm{BS}}^{1,1'}$ results in the state

$$\hat{U}_{\mathrm{BS}}^{0,0'}\,\hat{U}_{\mathrm{BS}}^{1,1'}\,\hat{a}_{0'}^\dagger \hat{a}_{1'}^\dagger \left|\psi_{\mathrm{out}}^{(\mathrm{bos})}\right\rangle \tag{37}$$

$$= \frac{1}{q^{q/2}} \left(\hat{U}_{\mathrm{BS}}^{0,0'} (\hat{a}_0^\dagger)^q \hat{a}_{0'}^\dagger \hat{U}_{\mathrm{BS}}^{1,1'} \hat{a}_{1'}^\dagger + (-1)^q \hat{U}_{\mathrm{BS}}^{1,1'} (\hat{a}_1^\dagger)^q \hat{a}_{1'}^\dagger \hat{U}_{\mathrm{BS}}^{0,0'} \hat{a}_{0'}^\dagger\right)|0\rangle. \tag{38}$$

The postselection on the component where all photons occupy output modes $\{0', 1'\}$ yields

$$\left|\psi_{\mathrm{post}}^{(\mathrm{bos})}\right\rangle = \frac{(1-\eta)^{q/2}\eta}{q^{q/2}} \left((\hat{a}_{0'}^\dagger)^{q+1} \hat{a}_{1'}^\dagger + (-1)^q (\hat{a}_{1'}^\dagger)^{q+1} \hat{a}_{0'}^\dagger\right)|0\rangle. \tag{39}$$

Finally, the bunching probability in modes $0'$ and $1'$ is given by

$$P_n^{(\mathrm{bos})} = \left\langle \psi_{\mathrm{post}}^{(\mathrm{bos})} \middle| \psi_{\mathrm{post}}^{(\mathrm{bos})} \right\rangle = \frac{2(q+1)!}{q^q} (1-\eta)^q \eta^2. \tag{40}$$

Consider now the analogous calculation for the specially chosen state of partially distinguishable photons, described in the caption of Fig. 2. In this case, as discussed in the main text, the counterpart of the NOON state obtained in equation (35) is given by

$$\left|\psi_{\mathrm{out}}^{(\star)}\right\rangle = \frac{1}{q^{q/2}} \prod_{j=0}^{q-1} \left(\frac{\hat{a}_{h,0}^\dagger}{\sqrt{2}} + \omega^j \hat{c}_+^\dagger + \omega^{2j} \frac{\hat{a}_{v,1}^\dagger}{\sqrt{2}}\right)|0\rangle. \tag{41}$$

After the DFT interferometer, one ancillary photon is introduced in mode $0'$ with vertical polarization and another one in mode $1'$ with horizontal polarization. At this point, the state of the system is given by

$$\hat{a}_{h,1'}^\dagger\, \hat{a}_{v,0'}^\dagger \left|\psi_{\mathrm{out}}^{(\star)}\right\rangle = \frac{(\hat{c}_+'^\dagger)^2 - (\hat{c}_-'^\dagger)^2}{2} \left|\psi_{\mathrm{out}}^{(\star)}\right\rangle, \tag{42}$$

with $\hat{c}_\pm'^\dagger$ defined in equation (19). To analyse the action of the subsequent part of the interferometer, it is useful to define the joint action of the beamsplitter operators $\hat{U}_{\mathrm{BS}}^{0,0'}$ and $\hat{U}_{\mathrm{BS}}^{1,1'}$ as

$$\hat{V} = \hat{U}_{\mathrm{BS}}^{0,0'}\, \hat{U}_{\mathrm{BS}}^{1,1'}. \tag{43}$$

Using the fact that both beamsplitters have equal transmittance, it can be seen that the action of $\hat{V}$ on the delocalized modes $\hat{c}_\pm^\dagger$ and $\hat{c}_\pm'^\dagger$ is given by

$$\hat{V}\hat{c}_\pm'^\dagger \hat{V}^\dagger = \sqrt{\eta}\,\hat{c}_\pm'^\dagger + \sqrt{1-\eta}\,\hat{c}_\pm^\dagger, \tag{44}$$

$$\hat{V}\hat{c}_\pm^\dagger \hat{V}^\dagger = \sqrt{\eta}\,\hat{c}_\pm^\dagger - \sqrt{1-\eta}\,\hat{c}_\pm'^\dagger. \tag{45}$$

We will see that the interference between mode $\hat{c}_+^\dagger$, which is occupied in state $\left|\psi_{\mathrm{out}}^{(\star)}\right\rangle$, and mode $\hat{c}_+'^\dagger$, which is occupied in the ancillary photon state, leads to bosonic bunching effects that are responsible for the largest asymptotic contributions to the bunching probability in modes $\{0', 1'\}$. In contrast, the other modes occupied in state $\left|\psi_{\mathrm{out}}^{(\star)}\right\rangle$, that is, $\hat{a}_{h,0}^\dagger$ and $\hat{a}_{v,1}^\dagger$, do not undergo any enhanced bunching effects as they do not couple either to $\hat{c}_+'^\dagger$ or to $\hat{c}_-'^\dagger$. For completeness, we also write the action of $\hat{V}$ on these modes as

$$\hat{V}\hat{a}_{h,0}^\dagger \hat{V}^\dagger = \sqrt{\eta}\,\hat{a}_{h,0}^\dagger + \sqrt{1-\eta}\,\hat{a}_{h,0'}^\dagger, \tag{46}$$

$$\hat{V}\hat{a}_{v,1}^\dagger \hat{V}^\dagger = \sqrt{\eta}\,\hat{a}_{v,1}^\dagger + \sqrt{1-\eta}\,\hat{a}_{v,1'}^\dagger. \tag{47}$$

We are now ready to analyse the action of $\hat{V}$ on the state given in equation (42), with the aim of computing a bound for the bunching probability. For the aforementioned reasons, it will be useful to expand state $\left|\psi_{\mathrm{out}}^{(\star)}\right\rangle$ as a superposition of states with different occupation numbers in mode $\hat{c}_+^\dagger$. To do so, it is useful to define

$$\hat{B}_j^\dagger = \frac{1}{\sqrt{2}}(\omega^{-j}\,\hat{a}_{h,0}^\dagger + \omega^j\,\hat{a}_{v,1}^\dagger). \tag{48}$$

With this definition, we can write

$$\left|\psi_{\mathrm{out}}^{(\star)}\right\rangle = \frac{(-1)^{q-1}}{q^{q/2}} \prod_{j=0}^{q-1} \left(\hat{c}_+^\dagger + \hat{B}_j^\dagger\right)|0\rangle. \tag{49}$$

Because all operators involved in this expression commute with each other, we can expand it as if $\hat{c}_+^\dagger$ and $\hat{B}_j^\dagger$ were complex numbers. Precisely, we have the following expansion:

$$\prod_{j=0}^{q-1} \left(\hat{c}_+^\dagger + \hat{B}_j^\dagger\right) = \sum_{k=0}^{q} (\hat{c}_+^\dagger)^{q-k}\, \hat{e}_k(\hat{B}_0^\dagger, \ldots, \hat{B}_{q-1}^\dagger), \tag{50}$$

where we have defined

$$\hat{e}_0(\hat{B}_0^\dagger, \ldots, \hat{B}_{q-1}^\dagger) = 1, \tag{51}$$

$$\hat{e}_k(\hat{B}_0^\dagger, \ldots, \hat{B}_{q-1}^\dagger) = \sum_{0 \le j_1 < \ldots < j_k \le q-1} \hat{B}_{j_1}^\dagger \ldots \hat{B}_{j_k}^\dagger. \tag{52}$$

To simplify the notation we denote $\hat{e}_k(\hat{B}_0^\dagger, \ldots, \hat{B}_{q-1}^\dagger)$ simply as $\hat{e}_k$. Newton's identities give us the following recursion relation

$$k\,\hat{e}_k = \sum_{i=1}^{k} (-1)^{i-1}\, \hat{e}_{k-i}\, \hat{p}_i, \tag{53}$$

where $\hat{p}_k$ is the kth power sum:

$$\hat{p}_k = \sum_{j=0}^{q-1} (\hat{B}_j^\dagger)^k. \tag{54}$$

These sums take a simple form for any $k \ge 1$, with

$$\hat{p}_k = \begin{cases} 0, & \text{for } k \text{ odd}, \\ \dfrac{q}{2^{k/2}} \binom{k}{k/2} \left(\hat{a}_{h,0}^\dagger\, \hat{a}_{v,1}^\dagger\right)^{k/2}, & \text{for } k \text{ even}. \end{cases} \tag{55}$$

Using this expression for $\hat{p}_k$ together with equation (53), it can be seen that all the terms with odd k in the expansion given in equation (50)

are suppressed. Moreover, these equations provide a simple way to calculate the first few terms of the expansion in equation (50) and obtain

$$\left|\psi_{\text{out}}^{(\star)}\right\rangle = \frac{(-1)^{q-1}}{q^{q/2}}\left(\left(\hat{c}_{+}^{\dagger}\right)^{q} - \frac{q}{2}\left(\hat{c}_{+}^{\dagger}\right)^{q-2}\hat{a}_{h,0}^{\dagger}\,\hat{a}_{v,1}^{\dagger} + \ldots\right)|0\rangle. \tag{56}$$

The other terms of the expansion are orthogonal to the first two terms and can only contribute with additional positive terms to the bunching probability. In fact, these two terms are enough to obtain the lower bound for the bunching violation ratio presented in the main text (equation (11)). After the action of interferometer $\hat{V}$ it can be shown that component of the wave function containing all the n photons in modes $0'$ and $1'$ is given by

$$\left|\psi_{\text{post}}^{(\star)}\right\rangle = \frac{(1-\eta)^{q/2}\eta}{2q^{q/2}}\,(-1)^{q-1}\left(\left(\hat{c}_{+}^{\prime\dagger}\right)^{q+2} - \frac{q}{2}\left(\hat{c}_{+}^{\prime\dagger}\right)^{q}\hat{a}_{h,0'}^{\dagger}\,\hat{a}_{v,1'}^{\dagger} + \ldots\right)|0\rangle\,. \tag{57}$$

The omitted terms in the previous equation are orthogonal to the first two terms. Hence, we obtain the following lower bound for the bunching probability:

$$P_{n}^{(\star)} = \left\langle\psi_{\text{post}}^{(\star)}\right|\left.\psi_{\text{post}}^{(\star)}\right\rangle \geq \frac{(1-\eta)^{q}\eta^{2}}{4\,q^{q}}\left((q+2)! + \frac{q^{2}}{4}q!\right). \tag{58}$$

Finally, we can use equations (58) and (40) to obtain

$$R_{n} = \frac{P_{n}^{(\star)}}{P_{n}^{(\text{bos})}} \geq \frac{q+2}{8} + \frac{1}{32}\frac{q^{2}}{q+1} \tag{59}$$

$$\geq \frac{n}{8} + \frac{1}{32}\frac{(n-2)^{2}}{n-1}, \tag{60}$$

demonstrating the bound on the bunching violation ratio given in equation (11). This bound exceeds one with seven photons or more, implying enhanced bunching.

Data availability

The data supporting the study and figures are available upon request. We acknowledge A. Franzen's ComponentLibrary for use in making the figures.

Code availability

The project relies on the packages PERMANENTS.JL (https://github.com/benoitseron/Permanents.jl) and BOSONSAMPLING.JL (https://github.com/benoitseron/BosonSampling.jl)[48]. The source code used for generating the figures and the data of this paper is available on GitHub and the data from OSF (https://osf.io/ex63z/).

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Author contributions

All authors developed the original concepts, derived the formulae, discussed the results and wrote the paper. B.S. implemented the numerical simulations.

Competing interests

The authors declare no competing interests.

Additional information

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