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Uncertainty relations using the Jordan-Schwinger map

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Abstract

Following recent applications of the so-called multicopy method in the context of quantum optics, relations between bosonic modes and spin systems have proven to be instrumental to analyze uncertainty relations. Here, we derive several second-order uncertainty relations and establish a general mapping between the number basis of bosonic modes and the eigenbasis of spin operators. More precisely, in the first part of this dissertation, we apply the Jordan-Schwinger map in order to convert simple inequalities on spin (or more general) operators into non-trivial inequalities for the quadratures of multimode bosonic systems. In the second part, we employ Lie algebra theory to set up an algorithm decomposing the Fock space of multimode systems in irreducible submodules of $\mathfrak{su}(2)$; it enables us to deduce the eigenbases of spin operators and highlight the desired relation between bosonic modes and spin systems. We exemplify our algorithm for the non-trivial 3-mode system and show that eigenstates of other spin operators could be deduced by applying unitary operators expressed in terms of Wigner d-matrices.

Keywords: quantum optics, uncertainty relations, Jordan-Schwinger map, bosonic modes, spin operators, spin basis, multicopy method.

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1. Introduction

The uncertainty relations introduced in 1927 by Heisenberg [1] highlight the existence of an inherent tradeoff on the precision of the results when measuring particular pairs of physical quantities in quantum systems, such as the position and momentum of a particle. It is a big divergence between classical and quantum theories, as, in classical theories, there are no theoretical limits on the precision one can obtain from a measurement and the knowledge of a physical property should not affect the measurement of another one. In contrast, in quantum theories, there is a limit on the precision of some measurements, and there are observable pairs for which the knowledge of one gives a lower bound on the accessible knowledge on the other. This principle was formalized by Kennard with inequalities using the variance of the measured distribution [2], while Beckner [3], Bialynicki-Birula, and Mycielski [4] used entropies instead. Even if there is still an epistemological debate about whether uncertainty is an intrinsic property of particles or of their measurements, this feature of quantum theories provides fundamental insights into the limits of measurement precision and bounds on physically allowed operations. For instance, in quantum cryptography, it helps deduce the presence of an eavesdropper [5, 6] and in quantum information it helps witness entanglement [7–9]. Although extensive research on uncertainty relations has already been performed since the inception of quantum mechanics, there remain unexplored aspects and potential extensions of these relations that warrant further investigation.

In particular, an entropic uncertainty relation for continuous-variable systems which is saturated for all Gaussian pure states remains a conjecture and the derivation of stronger inequalities for multimode systems is still an open research problem. In 2019, A. Hertz, O. Oreshkov and N. Cerf showed that the Schrödinger-Robertson inequality (the strongest quadratic inequality for one-mode systems) could be derived from trivial inequalities obeyed by a multimode operator defined via the Jordan-Schwinger map and applied to several copies of the state of interest [10].

Chapter 3 generalizes the results of A. Hertz *et al.* by considering trivial inequalities on more general operators created via the Jordan-Schwinger map and applying them to general bosonic multimode systems. This multicopy approach yields new uncertainty relations for multimode systems. Yet, limitations are met due to the lack of knowledge about the eigenstructure of operators generated by the Jordan-Schwinger map. For instance, we are not able to deduce the saturating states for these new uncertainty relations nor to investigate the entropy of these operators. Therefore, we choose to investigate these eigenstructures in Chapter 4. This research extends the already known relation between the number basis and the spin eigenbasis of operators created through the Jordan-Schwinger map for two-mode states to general n-mode states, and provides several generalizations. This result can hopefully contribute not only to computing the entropy of the uncertainty operators, but also to addressing questions related to witnessing entanglement, nonclassicality, or nonlocality, as it echoes questions raised by several papers at QuIC (Center for Quantum Information and Communication).

2. Quantum mechanics and algebras

2.1 Hilbert spaces in quantum theories

This section focuses on the mathematical foundations of quantum mechanics. Quantum theory describes the states of a system by vectors (and further endomorphisms) lying in a Hilbert space $\mathcal{H}$. We extensively use the properties of this mathematical structure and its relationships with other structures, in particular Lie algebras. First, we define the notion of groups, particularize it into abelian groups, vector spaces, and end with Hilbert spaces. Subsequently, we explore several mathematical tools used in this thesis. The notations, conventions and definitions employed are those from [11–14].

2.1.1 Groups, vector spaces to Hilbert spaces

A Group G is a set[1] of entities g_α, the group elements, with a composition law (for short, we can multiply them together)[14]. The composition law is described as

$$\circ : G \times G \to G : (g_\alpha, g_\beta) \to g_\alpha \circ g_\beta. \tag{2.1}$$

This law must obey several postulates:

- Internality:
$$\forall g_\alpha, g_\beta \in G, g_\gamma = g_\alpha \circ g_\beta \Rightarrow g_\gamma \in G, \tag{2.2}$$
- Associativity:
$$\forall g_\alpha, g_\beta, g_\gamma \in G, (g_\alpha \circ g_\beta) \circ g_\gamma = g_\alpha \circ (g_\beta \circ g_\gamma), \tag{2.3}$$
- Existence of the identity I:
$$\exists I \in G, \forall g \in G : I \circ g = g \circ I = g, \tag{2.4}$$
- Existence of the inverse:
$$\forall g \in G, \exists h : h \circ g = g \circ h = I, \tag{2.5}$$
where $g^{-1} \equiv h$ denotes the inverse of g.

The notation for the group formed by the set G associated with the law $\circ$ is written $G, \circ$.
Numerous mathematical structures are derived from groups, which can be discrete or continuous, with a finite or infinite (countable or not) number of elements, bounded, or not, and many others. Group theory is an important and large field of mathematics, but we will restrict ourseleves to describing the following two particular kinds the abelian and Lie groups, and then give examples of these groups in physics.

Abelian groups are groups with a commutative composition relation, *i.e*:

$$\forall g_\alpha, g_\beta \in G, g_\alpha \circ g_\beta = g_\beta \circ g_\alpha. \tag{2.6}$$

Thus, these are also known as commutative groups.

[1] We could delve deeper and first focus on the notion of sets, but we assume the reader to be familiar with this topic.

A Lie group is a continuous and finite dimensional group such that G corresponds to a differentiable manifold[2]. Both composition and inversion need to be differentiable everywhere (we will say that the group is smooth).
There are many groups in physics: all vector spaces (cf. Section 2.1.1) are groups, symmetry transformations are described by groups ($SO(3)$ for rotations of vectors in space or $\mathbb{R}^3$ for translations, both of which are Lie groups)[14]. Special relativity is the study of the Poincaré group (composition of the Lorentz group $SO(3,1)$ for rotations and boosts, with the translation group), and general relativity is the study of the general group of diffeomorphisms (reversible changes of coordinates on manifolds, differentiable, and whose inverse is also differentiable) [15, 16]. The standard model describes particles based on the group $SU(3) \times SU(2) \times U(1)$ [14]. Many mathematical tools form groups, for instance, on functions: the Fourier transform, the parity, the inverse Fourier transform, together with an identity transformation are a four elements group, which can be extended to the fractional Fourier transform to make a continuous group. Similarly, the differential operators of the functions form a group and can be extended to fractional differentiation[17].

Vector spaces are particular abelian groups. A vector space is defined over a particular field $\mathcal{F}$[3] For notations, we will use the Dirac notation, where the elements of the vector space V, the vectors, are called *kets* and represented by the symbol $|v\rangle$. In quantum mechanics, the system states are described by vectors (or more complex objects, as presented in Section 2.1.5), which motivates our interest in vector spaces. The composition law for vector spaces is called an addition law, written $+$, such that the fundamental abelian group behind the vector space is $V, +$.
Finally, the identity of the addition law is called the null vector written $\vec{0}$.[4]
The second law is a multiplication by a field element

$$\cdot : \mathcal{F} \times V \to V : (\lambda, |\psi\rangle) \to \lambda \cdot |\psi\rangle = \lambda |\psi\rangle . \tag{2.7}$$

This relationship follows five postulates:

- Internality in V:
$$\forall \lambda \in \mathcal{F}, \forall |\psi\rangle \in V : \lambda |\psi\rangle \in V, \tag{2.8}$$
- Mixed associativity:
$$\forall \lambda, \mu \in \mathcal{F}, \forall |\psi\rangle \in V : \lambda(\mu |\psi\rangle) = (\lambda\mu) |\psi\rangle , \tag{2.9}$$
- Mixed distributivity:
$$\forall \lambda, \mu \in \mathcal{F}, \forall |\psi\rangle \in V : (\lambda + \mu) |\psi\rangle = (\lambda |\psi\rangle) + (\mu |\psi\rangle). \tag{2.10}$$
This property combined with the existence (and unicity) of $\vec{0}$ implies
$$\forall \lambda \in \mathcal{F} : \lambda\vec{0} = \vec{0}, \tag{2.11}$$
- Vectorial distributivity:
$$\forall \lambda \in \mathcal{F}, \forall |\psi\rangle , |\phi\rangle \in V : \lambda(|\psi\rangle + |\phi\rangle) = (\lambda |\psi\rangle) + (\lambda |\phi\rangle), \tag{2.12}$$

[2]Once again we will spare the reader the definition of manifolds.

[3]We will not delve very far into the definition of fields, but to summarize it, it is a set of elements with two binary laws: addition and multiplication. An additive and multiplicative inverse exists. The laws of associativity, commutativity and distributivity are the same as for the rationals or the reals.
Moreover, the field does not need to be complete, vector spaces are not always complete.

[4]We do not write $|0\rangle$ as this notation will be used to describe the vacuum in section 2.2.3.

- Existence of an identity element for $\cdot$ in $\mathcal{F}$:

$$\forall |\psi\rangle \in V : 1 \cdot |\psi\rangle = |\psi\rangle , \tag{2.13}$$

  where 1 denotes an identity element.

The Vector space is then written $\mathcal{F}, V, +$.

Hausdorff pre-Hilbert spaces or inner product spaces are vector spaces with an additional law: the inner product (if $\mathcal{F}$ is a scalar field[5], it is called a scalar product).
First we require the field to have a complex conjugate operation[6]

$$* : \mathcal{F} \to \mathcal{F} : \lambda \to \lambda^* . \tag{2.14}$$

The conjugate needs to be distributive over addition and multiplication, and needs to be selfinversive (an involution), *i.e.*:

$$\forall \lambda \in \mathcal{F} : (\lambda^*)^* = \lambda . \tag{2.15}$$

Then the inner product is a positive semi-definite Hermitian form

$$\langle \cdot | \cdot \rangle : V \times V \to \mathcal{F} : |v\rangle , |w\rangle \to \langle v|w\rangle . \tag{2.16}$$

In Dirac notation, this is called a braket. The postulates for this law are as follows:

- Hermitian symmetry or conjugate symmetry:

$$\forall v, w \in V : \langle v|w\rangle = \langle w|v\rangle^* , \tag{2.17}$$

- Linearity at right:

$$\forall |w\rangle \in V, \forall |v\rangle = \sum_i c_i |v_i\rangle \in V, c_i \in \mathcal{F} \text{ with } i \in \{1, 2, ..., n\}, : \langle w|v\rangle = \sum_i \lambda_i \langle w|v_i\rangle . \tag{2.18}$$

  Combined with the first condition, this also implies anti-linearity at left:

$$\forall |w\rangle \in V, \forall |v\rangle = \sum_i c_i |v_i\rangle \in V, c_i \in \mathcal{F} \text{ with } i \in \{1, 2, ..., n\}, : \langle v|w\rangle = \sum_i c_i^* \langle v_i|w\rangle , \tag{2.19}$$

- Positive definiteness,

$$\forall v \in V \, \langle v|v\rangle \geq 0, \quad \langle v|v\rangle = 0 \Leftrightarrow v = \vec{0}. \tag{2.20}$$

Two vectors are considered orthogonal if their scalar product is 0.

Hilbert spaces Finally, if we add a distance law d or a norm $\| \cdot \|$[7] such that, with this law, the vector space becomes a complete metric space, we obtain a Hilbert space. First, we require an absolute value for the field elements $| \cdot |$ and a square root law $\sqrt{\ }$ for the image of $f(v) = \langle v, v\rangle$ [8].
Then we define the norm as

$$\| \cdot \| : V \to \mathcal{F} : |v\rangle , |w\rangle \to \sqrt{\langle v|v\rangle}. \tag{2.21}$$

We use the following abusive notation: $\| |v\rangle \| = \|v\|$. In general, to be a norm, a law requires the following properties:

[5] Each element of the field can be represented by a single number.
[6] For the real field, this operation is the identity.
[7] A distance law and a norm are equivalent and related via $d(|v\rangle , \vec{0}) = \|v\|$ and $d(|v\rangle , |w\rangle) = \| |v\rangle - |w\rangle \|$.
[8] In particular this relations requires $\sqrt{\lambda}^2 = \lambda, \forall \lambda \in \mathcal{F}$.

- Subadditivity or triangular inequality:

$$\forall |v\rangle, |w\rangle \in V : \|v + w\| \leq \|v\| + \|w\|, \tag{2.22}$$

- Absolute homogeneity:

$$\forall v \in V, \forall \lambda \in \mathcal{F} : \|\lambda |v\rangle\| = |\lambda| \|v\|, \tag{2.23}$$

- Positive definiteness

$$\forall v \in V \|v\| \geq 0, \quad \|v\| = 0 \Leftrightarrow v = \vec{0}. \tag{2.24}$$

Fortunately, we can demonstrate that Equation (2.21) ensures all these properties. Finally, $V, \|\cdot\|$ is a complete metric space if any Cauchy sequence converges in it.
For the vocabulary, a vector is normal if its norm is 1. Normalizing a vector reduces to dividing it by its norm to obtain a new vector that is normal. Two vectors are parallel or aligned if their scalar product is the product of their respective norm. In quantum theories, we consider a Hilbert space $\mathcal{H}$, defined on the complex field $\mathbb{C}$ with classical arithmetic laws[9]. The states of the system are described by normalized vectors.
Vector spaces are well represented in physics, for instance, in classical mechanics, phase-space, or space[10] itself are Hilbert spaces, while in special relativity, space-time is a pseudo-Hilbert space[11]. Electromagnetic fields are vector spaces as well, such as the states of the system in quantum field theory or statistical physics.

2.1.2 Dual space and space of the endomorphims

Several spaces can be built on each vector space V, such as the dual space or the endomorphisms space.

The Dual space V^* of a vector space V is the space of all linear forms of V. A linear form is an application $\langle w|$ (this is the Dirac notation, and they are named *bra's*):

$$\langle w| : V \to \mathcal{F} : |v\rangle \to \langle w| (|v\rangle). \tag{2.25}$$

As the name indicates, it is linear:

$$\forall |v_1\rangle, |v_2\rangle \in V : \langle w| (|v_1\rangle + |v_2\rangle) = \langle w| (|v_1\rangle) + \langle w| (|v_2\rangle). \tag{2.26}$$

In space V^*, the addition and multiplication laws are defined based on the application of the linear form to the vectors:

$$\begin{aligned} + :& V^* \times V^* \to V^* : \langle w_1|, \langle w_2| \to \langle w_1| + \langle w_2|, \\ & \text{such that } \forall |v\rangle \in V : (\langle w_1| + \langle w_2|) (|v\rangle) = \langle w_1| (|v\rangle) + \langle w_2| (|v\rangle), \end{aligned} \tag{2.27}$$

$$\begin{aligned} \cdot :& \mathcal{F} \times V^* \to V^* : \lambda, \langle w| \to \lambda \cdot \langle w|, \\ & \text{such that } \forall |v\rangle \in V : (\lambda \langle w|) (|v\rangle) = \lambda (\langle w| (|v\rangle)). \end{aligned} \tag{2.28}$$

Note that the dual of the dual of a space is the space itself. A space and its dual are both isomorphic. If the vector space is now a pre-Hilbert space $\mathcal{H}$, we can create a bijection from vectors to linear forms, owing to the inner product. We say that the dual form $\langle v|$ associated with a vector $|v\rangle$ is the only linear form such that

$$\forall |v\rangle \in \mathcal{H} : \langle v| (|w\rangle) = \langle v|w\rangle. \tag{2.29}$$

[9] In the following we will write $\mathcal{H}$ if we consider Hilbert spaces and V otherwise.
[10] The space as the space continuum in which objects lives.
[11] The scalar product associated with this space is not positive definite.

Space of all linear endomorphisms. A linear endomorphism[12] in V is a map intern in V; *i.e.*,

$$\hat{O} : V \to V : \psi, \tag{2.30}$$

The map must be intern and linear. We will also discuss linear operators, and most of the time, we will not underline the linearity and only say endomorphism or operator. The set of all endomorphisms, $\mathrm{End}(V)$, with the following addition and multiplication laws is itself a vector space:

$$\begin{aligned} &+ :\mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V) : \hat{O}_1, \hat{O}_2 \to \hat{O}_1 + \hat{O}_2 \\ &\quad \text{such that } \forall |v\rangle \in V : \left(\hat{O}_1 + \hat{O}_2\right)(|v\rangle) = \hat{O}_1(|v\rangle) + \hat{O}_2(|v\rangle), \end{aligned} \tag{2.31}$$

$$\begin{aligned} &\cdot :\mathcal{F} \times \mathrm{End}(V) \to \mathrm{End}(V) : \lambda, \hat{O} \to \lambda \cdot \hat{O} \\ &\quad \text{such that } \forall |v\rangle \in V : \left(\lambda \hat{O}\right)(|v\rangle) = \lambda \left(\hat{O}(|v\rangle)\right). \end{aligned} \tag{2.32}$$

In Dirac's notation, the parentheses are removed for simplicity, such that $\hat{O}(|v\rangle) = \hat{O}|v\rangle$. It also forms a group with a composition relation such that

$$\circ : \hat{O}, \hat{P} \to \left[|v\rangle \to \left(\hat{O} \circ \hat{P}\right)|v\rangle = \hat{O}\hat{P}|v\rangle = \hat{O}\left(\hat{P}|v\rangle\right)\right]. \tag{2.33}$$

The neutral element for this operation is called the identity, and is written $\hat{\mathbb{1}}$. This space is isomorphic to the space of all linear forms in $V^* \otimes V$: $(V^* \otimes V)^* = V \otimes V^*$. with the following isomorphism: [13]:

$$\hat{O} \to \left[\langle w|, |v\rangle \to \langle w|\hat{O}|v\rangle\right]. \tag{2.34}$$

These two spaces are not distinguished in the remainder of this thesis. This isomorphism, combined with the idea that $V \otimes V^*$ is self-dual (up to an isomorphism), implies that $\mathrm{End}(V)$ is also self-dual. We can therefore build an operation, intern in $\mathrm{End}(V)$, which plays the role of the dual of operators[14]. This operation is the adjoint operation, and we say that an operator is the adjoint/conjugate transpose of another. The adjoint of $\hat{O}$ is the only operator $\hat{O}^\dagger$ such that

$$\langle\phi|\hat{O}^\dagger|\psi\rangle = \langle\psi|\hat{O}|\phi\rangle^*. \tag{2.35}$$

Owing to the properties of dual and complex conjugation, any proposition in quantum mechanics can be conjugated to obtain another equivalent proposition. To do so, each element is replaced by its dual (a scalar by its complex conjugate, a *ket* by a *bra*, a *bra* by a *ket*, an operator by its adjoint, etc.), and their apparition order is inverted. For instance:

$$\lambda\hat{A}|v\rangle\langle w|\hat{T}|\psi\rangle = |x\rangle + |y\rangle \Leftrightarrow \langle\psi|\hat{T}^\dagger|w\rangle\langle v|\hat{A}^\dagger\lambda^* = \langle x| + \langle y|. \tag{2.36}$$

Several endomorphism subfamilies exist. The self-adjoints aka Hermitian operators or observables[15] are operators such that

$$\hat{A} = \hat{A}^\dagger. \tag{2.37}$$

[12] We denote the endomorphisms using a hat. This is used to distinguish, for instance, matrices and operators of quantum mechanics.

[13] The inverse of this relationship requires the notion of bases. If $|v_i\rangle$ is a basis of V and $\langle w_i|$ a basis of V^*, then we have

$$O(\langle w|, |v\rangle) \to \hat{O} = \sum_{i,j} |w_i\rangle O(\langle w_i|, |v_j\rangle)|v_j\rangle.$$

[14] This affirmation can be understood by comparing the norm of a vector $\|v\| = \sqrt{\langle v|v\rangle}$ and the Frobenius norm of operators $\|\hat{O}\|_F = \sqrt{\mathrm{tr}(A^\dagger A)}$. We could properly define the Frobenius inner product and demonstrate that it is related to the classical inner product.

[15] This name is due to the existence of a one to one correspondence between these operators and the physical quantities that can be observed in nature.

The unitary operators follow the property

$$\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \hat{\mathbb{1}}. \tag{2.38}$$

The idempotent operators are defined by

$$\hat{Q}^2 = \hat{Q}, \tag{2.39}$$

and the bounded operators by

$$\exists M \in \mathbb{R}_0^+ : \forall \left|v\right\rangle \in V : \left\|\hat{B}\left|v\right\rangle\right\| \leq M \left\|v\right\|. \tag{2.40}$$

Finally, we define several operations for the operators. The trace of an operator is the only linear form on $\mathrm{End}(V)$, such that

$$\mathrm{tr}\left(\left|v\right\rangle\left\langle w\right|\right) = \left\langle w|v\right\rangle. \tag{2.41}$$

To obtain a more precise definition, we need to develop the notion of bases[16].
The commutator $[\cdot,\cdot]$ and the anticommutator $\{\cdot,\cdot\}$ of two operators are binary operations defined as

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}, \quad \{\hat{A},\hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}. \tag{2.42}$$

If respectively the commutator (anticommutator) is the null operator, then we say that the operators (anti-)commute. Several properties of the commutator are extensively used in the following sections:

- Anticommutativity:
$$[\hat{A},\hat{B}] = -[\hat{B},\hat{A}], \tag{2.43}$$
- Alternativity:
$$[\hat{A},\hat{A}] = 0, \tag{2.44}$$
- Expulsion formulas:
$$[\hat{A},\hat{B}\hat{C}] = \hat{B}[\hat{A},\hat{C}] + [\hat{A},\hat{B}]\hat{C}, \quad [\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}, \tag{2.45}$$
- Jacobi's identity:
$$[\hat{A},[\hat{B},\hat{C}]] + [\hat{B},[\hat{C},\hat{A}]] + [\hat{C},[\hat{A},\hat{B}]] = 0. \tag{2.46}$$

2.1.3 Bases and components

The concepts of linear independence and basis are powerful tools in linear algebra and quantum mechanics. Let us define a set of vectors $\{\left|v_i\right\rangle\}$. Then a vector $\left|\psi\right\rangle$ is a linear combination of this set if and only if there exists coefficients $c_i \in \mathcal{F}$ such that

$$\sum_i c_i \left|v_i\right\rangle = \left|\psi\right\rangle. \tag{2.47}$$

The set of all linear combinations of $\{\left|v_i\right\rangle\}$ is written as $\mathrm{Span}\left(\{\left|v_i\right\rangle\}\right)$ and is called the set spanned by $\{\left|v_i\right\rangle\}$, or the span of $\{\left|v_i\right\rangle\}$. The vectors of a set are linearly independent if and only if no vectors in this set are explicitly a linear combination of the others:

$$\forall \left|v_i\right\rangle \in \{\left|v_i\right\rangle\} : \left|v_i\right\rangle \notin \mathrm{Span}\left(\{\left|v_i\right\rangle, i \neq j\}\right). \tag{2.48}$$

[16] Alternatively we could also precise the isomorphism between $End(V)$ and $V \otimes V^*$.

This set is also said to be a free part of V. Similarly, a set $\{|v_i\rangle\}$ is said to be a generative part of V, or a complete part, if and only if

$$\text{Span}\left(\{|v_i\rangle\}\right) = V. \tag{2.49}$$

If a set $\{|v_i\rangle\}$ is simultaneously a free part, and a generative part of V, then this set is said to be a basis. In this case, the components of a vector $|\psi\rangle$ are the coefficients c_i in Equation (2.47). The components always exist (because a basis is a generative part) and are unique (because a basis is a free part).
Hereafter, we will only work with a Hilbert space $\mathcal{H}$ as we will use inner products. The dual basis of the basis $\{|v_i\rangle\}$ is the set of linear forms $\langle v_i|$, such that $\langle v_i|$ is the dual form associated with the vector $|v_i\rangle$, as they are each other's complex conjugate

$$|\psi\rangle = \sum_i c_i |v_i\rangle \Leftrightarrow \langle\psi| = \sum_i c_i^* \langle v_i| , \tag{2.50}$$

as shown in Appendix A.1.
A basis $\{|n_i\rangle\}$ is orthogonal if all the vectors of the basis are orthogonal to each other, whereas it is normal if they are all normal, or orthonormal if both. In this last case, the components ψ_i of $|\psi\rangle$ on this basis are equals to

$$\psi_i = \langle n_i|\psi\rangle , \tag{2.51}$$

as shown in Appendix A.2. A powerful relation for orthonormal basis is the closure relation. To find it, let us decompose any vector $|\psi\rangle$ on an orthonormal basis:

$$|\psi\rangle = \sum_i \psi^i |n_i\rangle = \sum_i \langle n_i|\psi\rangle |n_i\rangle = \left(\sum_i |n_i\rangle \langle n_i|\right) |\psi\rangle \quad \forall |\psi\rangle . \tag{2.52}$$

As this is true for all $|\psi\rangle$, we find a relation whose name is the closure relation:

$$\sum_i |n_i\rangle \langle n_i| = \hat{\mathbb{1}}. \tag{2.53}$$

Changes of basis in this formalism are represented by invertible matrices. Considering only orthonormal bases, then the changes of basis are described by unitary matrices.The change of basis from $\{|t_i\rangle\}$ to $\{|u_i\rangle\}$ (both orthonormal) is an operator sending any vector of the t basis on the corresponding vector of the u basis:

$$\hat{U} = \sum_i |u_i\rangle \langle t_i| . \tag{2.54}$$

This operator is indeed unitary since

$$\hat{U}\hat{U}^\dagger = \left(\sum_i |u_i\rangle \langle t_i|\right)\left(\sum_j |t_j\rangle \langle u_j|\right) = \sum_{ij} |u_i\rangle \langle t_i|t_j\rangle \langle u_j| = \sum_{ij} |u_i\rangle \delta_{ij} \langle u_j| = \sum_i |u_i\rangle \langle u_i| = \hat{\mathbb{1}}. \tag{2.55}$$

Alternatively, if $\hat{U}$ is unitary and $\{|t_i\rangle\}$ is an orthonormal basis, then $|u_i\rangle = U|t_i\rangle$ is an orthonormal basis:

$$\langle u_i|u_j\rangle = \langle t_i|U^\dagger U|t_j\rangle = \langle t_i|t_j\rangle = \delta_{ij}. \tag{2.56}$$

The change of basis applied to a vector $|\psi\rangle$, gives us a vector $\hat{U}|\psi\rangle$. The vision of changing the basis of the vectors is referred to as the Schrödinger picture. Yet, in Section 2.2.1 we will explain that we can only measure matrix elements of operators $\langle v|\hat{O}|w\rangle$; this is equivalent to considering that the change is not applied to vectors, but to operators. This representation is the Heisenberg picture, and a change of basis leaves the vectors unchanged, but the operators become $\hat{U}^\dagger\hat{O}\hat{U}$.

The trace of an operator can be made explicit thanks to bases. If $\{|n_i\rangle\}$ is an orthonormal basis, then the trace of $\hat{A}$ is defined as

$$\mathrm{tr}\left(\hat{A}\right) = \sum_i \langle n_i|\hat{A}|n_i\rangle \,. \tag{2.57}$$

The closure relation helps showing that the trace is independent of the basis, this is done in Appendix A.3. The trace exhibits several interesting properties. It is linear and independent with respect to the basis, which directly proves that the trace is unitary invariant:

$$\mathrm{tr}\left(\hat{U}^\dagger \hat{O} \hat{U}\right) = \mathrm{tr}\left(\hat{A}\right). \tag{2.58}$$

A stronger property is its cyclic invariance, which can be demonstrated using the closure relation.

Projectors are a particular kind of operators. If $\mathcal{I}$ is a subspace of $\mathcal{H}$, then the operator projecting a vector onto $\mathcal{I}$ is said to be a projector. The projector can be written with an orthonormal basis of $\mathcal{I}$, $\{n_i\}$, as

$$\hat{P}_\mathcal{I} = \sum_i |n_i\rangle \langle n_i| \,. \tag{2.59}$$

This operator is Hermitian, idempotent, and bounded. It is extensively used for measurements. A generalization of the closure relation for the projectors is that for a collection of sets $\{\mathcal{I}_i\}$, orthogonal to each other (all vectors of $\mathcal{I}_i$ are orthogonal to all the vectors of $\mathcal{I}_j$ if $i \neq j$), such that their direct sum is $\mathcal{H}$ itself (all vectors of $\mathcal{H}$ can be written as a linear combination of vectors of $\{\mathcal{I}_i\}$). Then:

$$\sum_i \hat{P}_{\mathcal{I}_i} = \hat{\mathbb{1}}. \tag{2.60}$$

2.1.4 Eigenvalues, eigenspaces and spectral decomposition

Very useful concepts in quantum mechanics are the notions of eigenvalues, eigenvectors, and eigenspaces. Any non-zero vector $|\lambda\rangle$ such that

$$\hat{A}\,|\lambda\rangle = \lambda\,|\lambda\rangle \,, \tag{2.61}$$

is said to be an eigenvector of the operator $\hat{A}$ with an eigenvalue λ. The subspace of all vectors with the same eigenvalue is called eigenspace of the operator $\hat{A}$. The dimension of the eigenspace associated with λ_i is the (geometric) degeneracy g_i of the eigenvalue[17]. We call the spectrum of an operator the set of all its eigenvalues $\sigma\left(\hat{A}\right)$, and spectral decomposition the set of pairs (λ_i, g_i). We call the eigenstructure the spectral decomposition coupled with eigenvectors. All operators do not possess the same number of eigenvectors, and some do not even possess one eigenvector. However, their existence is guaranteed for Hermitian operators by the spectral theorem:

If $\hat{A}$ is an Hermitian operator on $\mathcal{H}$, then there exists an orthonormal basis of $\mathcal{H}$ consisting of the eigenvectors of $\hat{A}$. The eigenvalues are real.

This theorem guarantees the existence of eigenvalues and eigenvectors, and that the eigenspaces associated with different eigenvalues are orthogonal to each other. A basis composed of eigenvectors is called eigenbasis.
Note that for observables, we can use the eigenbasis of an operator to compute the trace, and we find that for observables, the trace is the sum of the eigenvalues of the operator, weighted by their degeneracies. A second important theorem is the compatibility theorem, which states that:

[17] As we will only work with Hermitian operators, our matrices will be diagonalizable and we will not distinguish the notions of algebraic and geometric degeneracies.

If $\hat{A}$ and $\hat{B}$ are observables, those three affirmations are equivalent:

1. *$\hat{A}$ and $\hat{B}$ are compatible,*
2. *$\hat{A}$ and $\hat{B}$ share a common eigenbasis,*
3. *$\hat{A}$ and $\hat{B}$ commute.*

If $\{\hat{A}_i\}$ is a set of observables, then if they all commute with each other, it is said to be a commuting set; at least one common basis exists for all of them. If the basis vectors are unequivocally determined by the knowledge of the eigenvalues or quantum numbers[18], then the set of commuting operators is said to be complete. These sets are complete sets of commuting observables (CSCO). Choosing a CSCO fixes the basis in which we work. We say that a CSCO is minimal if removing any of the observables removes the completeness of this set.
Several generalizations of the concepts of eigenvalues exist, such as the singular values decomposition, whose applications include the Schmidt decomposition for any state of a multipartite system.

2.1.5 State and density operator

First, the states of quantum systems were described by normalized vectors; however, this formalism cannot consider classical uncertainty. To solve this problem, states of a system are in full generality described by density operators $\hat{\rho}$ (see Section 2.2.1). These are Hermitian endomorphisms, positive semi-definite and normalized; we will write $\mathcal{T}(\mathcal{H}) \subseteq \mathrm{End}(\mathcal{H})$ the set of all the endomorphisms with those properties. The positive semi-definiteness is defined as

$$\langle\psi|\hat{\rho}|\psi\rangle \geq 0 \quad \forall\, |\psi\rangle \in \mathcal{H} \Leftrightarrow \lambda \geq 0 \quad \forall \lambda \in \sigma(\hat{\rho})\,. \tag{2.62}$$

The normalization of the operator is defined as

$$\mathrm{tr}(\hat{\rho}) = 1 \Leftrightarrow \sum_{\lambda_i \in \sigma(\hat{\rho})} \lambda_i = 1. \tag{2.63}$$

This representation is more general than simply considering vectors. The vectors only represent a part of the states of the system, which we will call pure states, as they do not contain *classical uncertainty*. The density operator considers the stochastic distribution of the states and is called a (statistical) mixture. Any pure state $|\psi\rangle$ matches with a density operator $\hat{\rho}_\psi$ by the following relation

$$\hat{\rho}_\psi = |\psi\rangle\langle\psi|\,. \tag{2.64}$$

Because this application is not surjective, not all elements of $\mathcal{T}(\mathcal{H})$ are images of a pure state. This shows that the density operators are a more general representation of the state of a system. If we consider that the state of the system is one of the states $\{|\psi_i\rangle\}$, with a probability p_i respectively (we impose that $\sum p_i = 1$), then the system will be described by the statistical mixture

$$\hat{\rho} = \sum_i p_i\, |\psi_i\rangle\langle\psi_i|\,. \tag{2.65}$$

This relationship from $\mathcal{H}^n \otimes \Delta^{n-1} \to \mathcal{T}(\mathcal{H})$ is surjective if n (the number of vectors considered in the mixture) is sufficiently large. We can interpret any density operator as a mixture of $\dim(\mathcal{H})$ vectors[19]; indeed, if we consider the spectral decomposition of $\hat{\rho}$ (it exists as $\hat{\rho}$ is Hermitian), we get

$$\hat{\rho} = \sum_i \lambda_i\, |\lambda_i\rangle\langle\lambda_i|\,. \tag{2.66}$$

[18]The quantum numbers are a set of numbers in bijection with the set of eigenvalues.
[19]That is, sufficiently large n.

Based on Equations (2.62) and (2.63), we see that the eigenvalues are indeed probabilities (those equations match the first two axioms of probabilities, and the third axiom will be obtained owing to the measurement postulate, presented in Section 2.2.1). This proves that any density matrix corresponds to a mixture of its eigenstates, with the probabilities being the eigenvalues. We can add two additional comments.
First, the application presented in Equation (2.65) is not injective, implying that several mixtures correspond to the same state. This confirms that the global phase of a state is irrelevant. Indeed let us consider two mixtures $\{|\psi_i\rangle\}$ with probabilities p_i and $\{e^{i\phi_i}|\psi_i\rangle\}$ with the same probabilities, then the density matrix for the second mixture is

$$\hat{\rho} = \sum_i p_i \left(e^{i\phi_i}|\psi_i\rangle\right)\left(e^{-i\phi_i}\langle\psi_i|\right) = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{2.67}$$

Therefore, the mixtures are identical. Or, working with $\mathcal{H}$ of dimension 2, with a basis $\{|0\rangle, |1\rangle\}$, then the mixtures of $\{|0\rangle, |1\rangle\}$ with a probability of 1/2 each, of $\{\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\}$, and a continuous distribution evenly distributed, *i.e.* $\{\cos\theta|0\rangle + \sin\theta|1\rangle\}$ give rises to the same density operator, since

$$\begin{aligned} \frac{1}{2}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\frac{\langle 0|+\langle 1|}{\sqrt{2}} + \frac{|0\rangle-|1\rangle}{\sqrt{2}}\frac{\langle 0|-\langle 1|}{\sqrt{2}}\right) &= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right), \\ \int_0^{2\pi}\frac{1}{2\pi}\left(\cos\theta|0\rangle + \sin\theta|1\rangle\right)\left(\cos\theta\langle 0| + \sin\theta\langle 1|\right)d\theta &= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right). \end{aligned} \tag{2.68}$$

Second, Equation (2.64) proves that any pure state gives a density operator with an eigenvalue of one and all the other 0; and putting such a distribution in Equation (2.66) shows that such a distribution gives a pure state. Therefore, we identify the pure state with the subset of $\mathcal{T}(\mathcal{H})$ with one eigenvalue 1, the others 0.

2.2 Elements of quantum mechanics

The great novelty of quantum mechanics lies in the use of mathematical tools of linear algebra to represent physics with simple postulates, which we discuss below using [12, 18, 19].

2.2.1 Postulate of quantum mechanics

The State of the system is describe thanks to a density operator $\hat{\rho} \in \mathcal{F}(\mathcal{H})$, as explained in Section 2.1.5.

The measurement on the system in quantum mechanics will be described through Hermitian operators $\hat{A}$. This justifies the name of observable for Hermitian operators as they represent physical observable quantities. The equivalence relation between operators and physical quantities is known as correspondence rule. As observable are Hermitian operators, it is possible to describe them through their spectral decomposition:

$$\hat{A} = \sum_\lambda \lambda \hat{P}_\lambda. \tag{2.69}$$

The outcomes of a measurement can only be the eigenvalues λ of the observable, and the corresponding probabilities are given by

$$P(\lambda) = \mathrm{tr}\left(\hat{\rho}\hat{P}_\lambda\right). \tag{2.70}$$

Given this postulate, we can compute the mean value of the measured observable as

$$\sum_\lambda \lambda \mathcal{P}(\lambda) = \sum_\lambda \lambda \mathrm{tr}\left(\hat{\rho}\hat{P}_\lambda\right) = \mathrm{tr}\left(\sum_\lambda \lambda \hat{\rho}\hat{P}_\lambda\right) = \mathrm{tr}\left(\hat{\rho}\hat{A}\right). \tag{2.71}$$

We will abbreviated this by $\langle \hat{A} \rangle_{\hat{\rho}}$ or $\langle \hat{A} \rangle$. After the measurement, if the outcome of the measurement is λ, the state of the system will be given by

$$\hat{\rho}' = \frac{\hat{P}_\lambda \hat{\rho} \hat{P}_\lambda}{\mathrm{tr}\left(\hat{\rho}\hat{P}_\lambda\right)}, \tag{2.72}$$

where the denominator ensures normalization. We may note that this kind of measurement is in fact a particular kind of measurement, the projective (von Neumann) measurements. But there are several generalizations such as the generalized measurements, or the positive operator valued measure (POVM), but those will not be used in this thesis.

The postulate on evolution of the system states that the state of the system at time t_1, $\hat{\rho}_{t_1}$, and at time t_2, $\hat{\rho}_{t_2}$ are related by a unitary $U(t_1, t_2)$ depending only on t_1 and t_2. This unitary can be related to the Hamiltonian in order to satisfy the Liouville–von Neumann equation (the generalization of the Schrödinger equation for mixtures)

$$i\frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] \Rightarrow \hat{\rho}_{t_2} = \hat{U}\hat{\rho}_{t_1}\hat{U}^\dagger \text{ with } \hat{U} = \exp\left(-i\int_{t_1}^{t_2} \hat{H} dt\right). \tag{2.73}$$

2.2.2 Composite quantum systems

In quantum theories, the state of composite quantum systems lies in the tensor product of the vector spaces $\mathcal{E}_i$ associated with each subsystem[20]

$$\mathcal{H} = \bigotimes_i \mathcal{E}_i. \tag{2.74}$$

Then, if $\{|v_j^i\rangle, \forall j\}$ is a basis of $\mathcal{E}_i$, a basis of $\mathcal{H}$ is the collection of vectors $\{\otimes_i |v_{j_i}^i\rangle, \forall j_i\}$, abbreviated as $\{|v_{j_1}^1, v_{j_2}^2, ...\rangle\}$. Note that we will also sometimes drop the commas.
An important difference between quantum and classical theories is that this representation allows the appearance of new phenomena, with nonclassical correlations in the results of measurements. This phenomenon is called entanglement, and presents several behaviours that classical systems could not have, such as non-locality.
Let us describe particular kinds of states. First, if we describe a system where each subsystem is in a state $\hat{\rho}_i$, then the global state is called a product state and is given by

$$\hat{\rho} = \otimes_i \hat{\rho}_i. \tag{2.75}$$

A more general situation is considering statistical mixtures of product states, a separable state

$$\hat{\rho} = \sum_j p_j \otimes_i \hat{\rho}_i^j, \tag{2.76}$$

that is, a non-entangled state.

[20] It also describes systems with several independent degrees of freedom.

Now let us consider a particular family of observable, that are the product of observable applied on each subsystem independently

$$\hat{O} = \otimes_i \hat{O}_i. \tag{2.77}$$

Then generally the mean value of this operator cannot be simplified,

$$\langle \hat{O} \rangle = \mathrm{tr}\left(\hat{\rho}\hat{O}\right) = \mathrm{tr}\left(\hat{\rho}\left(\otimes_i \hat{O}_i\right)\right), \tag{2.78}$$

but for separable states, we can separate the mean values of each subsystem

$$\langle \hat{O} \rangle = \mathrm{tr}\left(\hat{\rho}\hat{O}\right) = \mathrm{tr}\left(\sum_j p_j (\otimes_i \hat{\rho}_i^j)(\otimes_k \hat{O}_k)\right) = \sum_j p_j \mathrm{tr}\left(\otimes_i \hat{\rho}_i \hat{O}_i\right) = \sum_j p_j \prod_i \langle \hat{O}_i \rangle^j. \tag{2.79}$$

Even further, for product states, it becomes

$$\langle \hat{O} \rangle = \prod_i \langle \hat{O}_i \rangle.$$

In this thesis we will also work with a special family of product states which are the copy states. If $\mathcal{E}_i = \mathcal{E}, \forall i$, a product state is a state such that $\hat{\rho}_i = \hat{\rho}_0, \forall i$:

$$\hat{\rho} = \otimes_i \hat{\rho}_0. \tag{2.80}$$

2.2.3 Bosonic quantum systems

A boson is a particle (or more generally a system, for example a pair of electrons) symmetric with respect to the exchange of any two bosons. In particular, it is not constrained by Pauli's exclusion principle, as several bosons can be in the same state. If $\mathcal{H}$ is the Hilbert space for a single boson, and $\hat{S}$ is the symmetrization operator; then N bosons lie in the symmetric space $W_N = \hat{S}\mathcal{H}^{\otimes N}$. As we do not fix the total number of particles, the system is described by the direct sum of each space with a fixed number of particles W_N:

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} \hat{S}\mathcal{H}^{\otimes N}. \tag{2.81}$$

This space is called a Fock space. We call mode the n different degrees of freedom of the system. A simple basis of the Fock space is the Fock basis, aka occupation number basis, or shortly, the number basis. The vectors forming this basis are the Fock states, and they are described by the number of particles in each mode, *i.e.* the occupation number of each mode. Therefore, we will write $|n_1, n_2, ..., n_n\rangle$ for the state with $n_1 \geq 0$ particles in mode $|\psi_1\rangle$, $n_2 \geq 0$ in mode $|\psi_2\rangle$, etc. For instance, with 3 modes

$$\hat{S}\left(|\psi_1\rangle \otimes |\psi_1\rangle \otimes |\psi_3\rangle \otimes |\psi_2\rangle \otimes |\psi_1\rangle \otimes |\psi_3\rangle\right) = |3, 1, 2\rangle. \tag{2.82}$$

The orthonormality relation reads

$$\langle n_1, n_2, ..., n_n | n_1', n_2', ..., n_n' \rangle = \delta_{n_1,n_1'} \delta_{n_2,n_2'} ... \delta_{n_n,n_n'}. \tag{2.83}$$

The vacuum state $|0, 0, 0, ...\rangle$ will be abbreviated by $|0\rangle$.
We can define the annihilation $\hat{a}_i$ and creation $\hat{a}_i^\dagger$ operators as

$$\hat{a}_i |n_1, n_2, ..., n_i, ...n_n\rangle = \sqrt{n_i} |n_1, n_2, ..., n_i - 1, ...n_n\rangle \tag{2.84}$$

$$\hat{a}_i^\dagger |n_1, n_2, ..., n_i, ...n_n\rangle = \sqrt{n_i + 1} |n_1, n_2, ..., n_i, ...n_n\rangle \tag{2.85}$$

One can verify that they are each other's conjugate, and that they follow bosonic commutation relations:

$$[\hat{a}_i, \hat{a}_j] = 0, \quad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \quad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}.$$

Those operators allow us to relate any Fock state to the vacuum via the following relation

$$|n_1, n_2, ..., n_i, ...n_n\rangle = \left(\prod_i \sqrt{n_i!}^{-1} \left(\hat{a}_i^\dagger\right)^{n_i} \right) |0\rangle \,. \tag{2.86}$$

We can also define the number operators as

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i, \tag{2.87}$$

which when applied on a state gives the number of particles in the mode i:

$$\begin{aligned} \hat{n}_i \, |n_1, n_2, ..., n_i, ...n_n\rangle &= \hat{a}_i^\dagger \hat{a}_i \, |n_1, n_2, ..., n_i, ...n_n\rangle \,, \\ &= \sqrt{n_i} \hat{a}_i^\dagger \, |n_1, n_2, ..., n_i - 1, ...n_n\rangle \\ &= n_i \, |n_1, n_2, ..., n_i, ...n_n\rangle \,. \end{aligned} \tag{2.88}$$

The total number of particles is then defined as

$$\hat{N} = \sum_i \hat{n}_i. \tag{2.89}$$

For a set of free bosons, each mode can be interpreted as an independent harmonic oscillator with a total Hamiltonian

$$H = \sum_i \left(\omega_i \hat{n}_i + \frac{1}{2} \right) \tag{2.90}$$

where ω_i is the frequency of the i^{th} oscillator. We can also define the position and the momentum operators

$$\hat{x}_i = \frac{\hat{a}_i + \hat{a}_i^\dagger}{\sqrt{2\omega_i}}, \quad \hat{p}_i = i\sqrt{\omega_i} \frac{\hat{a}_i^\dagger - \hat{a}_i}{\sqrt{2}}, \Longleftrightarrow \hat{a}_i = \frac{\omega_i \hat{x}_i + i\hat{p}_i}{\sqrt{2\omega_i}}, \quad \hat{a}_i^\dagger = \frac{\omega_i \hat{x}_i - i\hat{p}_i}{\sqrt{2\omega_i}}. \tag{2.91}$$

Using these relations for Equation (2.90) leads to

$$H = \sum_i \frac{\omega_i \hat{x}_i^2}{2} + \frac{\hat{p}_i^2}{2\omega_i} + \frac{1}{2}. \tag{2.92}$$

For convenience, we set $w_i = 1$ for now on.
In this formalism, a particular class of operators are the single-body observables. It is the class of operators that acts on each particle separately (for instance the kinetic energy, or an external potential are single body observable, while the potential due to the interaction of two particles is a two-body observable). This class of observable can be expressed as

$$\hat{T} = \sum_{ij} \hat{a}_i^\dagger T_{ij} \hat{a}_j. \tag{2.93}$$

This formalism describes for instance photons in a black box, or more generally quanta of the electro-magnetic field [20]. In classical theories the electromagnetic field is described as

$$\vec{E}(\vec{r}, t) = \sum_{\vec{k},\lambda} E_k \vec{e}_k^{(\lambda)} \left[\alpha_{\vec{k},\lambda} e^{i(\vec{k}\vec{r} - \omega_k t)} + \alpha_{\vec{k},\lambda}^* e^{-i(\vec{k}\vec{r} - \omega_k t)} \right]. \tag{2.94}$$

The quantized vesion of it is obtained by replacing the amplitude factor $\alpha_{\vec{k},\lambda}$, by an annihilation operator $\hat{a}_{\vec{k},\lambda}$ on a mode characterized by its wavenumber $\vec{k}$ and a polarisation λ. The complex conjugate $\alpha_{\vec{k},\lambda}^*$ is replaced by the creation operator $\hat{a}_{\vec{k},\lambda}^\dagger$. The quantum field theory, in general, is then an extension of this formalism to infinitely many modes.

2.3 Algebras and Lie algebras

In this thesis we will use the Jordan-Schwinger map extensively. This map, developed in Section 3.1, binds matrices to operators, and preserves the commutation relations. As a vector space with commutation relations is a Lie algebra, both the space of matrices and the space of operators created thanks to the Jordan-Schwinger map are Lie algebras, and we will need several of their properties. All the definitions and notations we will used are adopted from [21].

2.3.1 Definition:

An algebra $\mathfrak{g}$ is a vector space defined over some field K equipped with of some binary bilinear operation $f : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} : (A, B) \to C = f(A, B)$.
A specific kind of algebra is the Lie Algebra. In this case, the bilinear map f is a Lie bracket, written, $[\cdot, \cdot]$ obeying the following postulates:

- Alternativity:
$$\forall g \in \mathfrak{g} : [g, g] = 0, \tag{2.95}$$
- Jacobi's identity:
$$\forall g, h, i \in \mathfrak{g} : [g, [h, i]] + [h, [i, g]] + [i, [g, h]] = 0, \tag{2.96}$$
- Anticommutativity:
$$\forall g, h \in \mathfrak{g} : [g, h] = -[h, g]. \tag{2.97}$$

Using bilinearity, one can show that the knowledge of the Lie bracket in some basis on the vector space g_i is sufficient to totally determine the action of that given operation (cf. Appendix B.1).
An important example of a Lie algebra is $\mathrm{End}(V)$ with the commutator as Lie bracket for any vector space V over a field F of finite dimension n. It is of dimension n^2, and it will be denoted by $\mathfrak{gl}(V)$. Indeed, the set of all the endomorphisms of V, $\mathrm{End}(V) = \{f : V \to V | f \text{ linear}\}$ is a vector space, and the commutator defined with the classical composition law for endomorphisms satisfies all the required postulates for Lie brackets (cf. Equations (2.43), (2.44) and (2.46)). This algebra, and any of its subalgebras (a subspace of $\mathfrak{gl}(V)$ where $[\cdot, \cdot]$ is intern) are called linear Lie algebras.

2.3.2 Representations of Lie algebras

A representation of an algebra $\mathfrak{g}$ is a homomorphism[21] $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$. Later in this thesis, if such a homomorphism ϕ exists, by abuse of language, we will refer to $\mathfrak{gl}(V)$ as the representation itself. The Jordan-Schwinger map being such a representation, the study of propertiesof representations will be useful in Chapter 4.
First, as any linear operation, it can be defined by its action on a basis. Second, if the Lie bracket is defined by its action on a basis, then the homomorphism conserves its definition. Explicitly, for the particular homomorphism ϕ that directly maps the basis $\{g_i\}$ of $\mathfrak{g}$ on a set $\{G_i\} \in \mathfrak{gl}(V)$ (with $\phi_0(g_i) = G_i \forall i \in 1, ..., k$), the defining properties of the Lie bracket are conserved by the commutator:

$$[g_i, g_j] = \sum_k f_{ij}^k g_k \Rightarrow [G_i, G_j] = \sum_k f_{ij}^k G_k \quad \forall i, j \in 1, ..., k \quad \text{cf. appendix B.2.} \tag{2.98}$$

An example of a representation is the identification of endomorphisms in V with matrices by their action on a basis. The homomorphism used is the canonical homomorphism between $\mathfrak{gl}(V)$ and

[21] A homomorphism ϕ is a linear transformation from $\phi : L \to L'$ such that $\phi([x, y]) = [\phi(x), \phi(y)]$.

the algebra constituted by the set of $n \times n$ matrices with entries in F and the Lie bracket $\mathfrak{gl}(n, F)$ is the commutator.
Additionally, a representation will be said faithful if the homomorphism is injective.

$$g_i \neq g_j \Rightarrow \phi(g_i) \neq \phi(g_j). \tag{2.99}$$

An example of not faithful homomorphism is the application sending any operator $\hat{A}$ to 0. Indeed, if $[\hat{A}, \hat{B}] = \hat{C}$, then $[\phi\left(\hat{A}\right), \phi\left(\hat{B}\right)] = \phi\left(\hat{C}\right)$ is automatically verified as $[0, 0] = 0$ is always true.

2.3.3 Modules

The concept of modules is equivalent to representations. If V is a vector space with an operation $\mathfrak{g} \times V \to V : (x, v) \to x \cdot v$ with the following postulates:

- Linearity on Lie algebras elements:

$$(ax + by) \cdot v = a(x \cdot v) + b(y \cdot V), \tag{2.100}$$

- Linearity on vectors:

$$x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w), \tag{2.101}$$

- Lie bracket acts as the commutator:

$$[x, y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v. \tag{2.102}$$

Then V is a $\mathfrak{g}$-module.
The connection with representations is immediate: in the representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$, $\mathfrak{gl}(V)$ is a $\mathfrak{g}$-module where $x \cdot v = \phi(x)(v)$. Alternatively, the $\mathfrak{g}$-module $\mathfrak{gl}(V)$ defines the representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$.
Defining modules helps extending the ideas behind representations by adding several concepts. A submodule W of V is a subspace of V that is a $\mathfrak{g}$-module. This is possible if and only if W is an invariant subspace of V, *i.e.*

$$\forall x \in \mathfrak{g}, \forall w \in W \subseteq V : x \cdot w \in W. \tag{2.103}$$

A $\mathfrak{g}$-module is irreducible if it has only two submodules, $\vec{0}$ and itself. Those notions will be helpful to construct a basis of the elements of the Jordan-Schwinger map. An important results on irreducible modules, is Schur's lemma. It has several formulations, but we will just note one of its corrolaries. On irreducible submodules, the only operators commuting with all elements of the algebra should act proportionally to the identity.

2.3.4 Casimir element of a representation

Let L be any semisimple algebra[22], and β any nondegenerate symmetric associative bilinear form on L. If $\{x_i\}$ is a basis of L, then one can show that there exists one unique other basis $\{y_i\}$ such that $\beta(x_i, y_j) = \delta_{ij}$. This basis is called the dual basis relative to β.
For a given representation $\phi : L \to \mathfrak{gl}(V)$, one can study the following endomorphism of V

$$c_\phi(\beta) = \sum_i \phi(x_i)\phi(y_i). \tag{2.104}$$

[22]The semisimplicity is here only to ensure the existence of a non-degenerate trace later. To be semismple, the algebra should be non-abelian, with no non-zero proper ideals. An ideal $\mathfrak{i}$ is a subalgebra of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$. It is proper if $\mathfrak{i} \neq \mathfrak{g}$ and non-zero if $\mathfrak{i} \neq \{\vec{0}\}$.

As shown in Appendix B.3 this endomorphism of V has commutes with $\phi(L)$, or explicitly

$$[\phi(v), c_\phi(\beta)] = 0 \quad \forall v \in L \tag{2.105}$$

In the particular case where $\beta(x,y) = tr\left(\phi(x)\phi(y)\right)$ then $c_\phi = c_\phi(\beta)$ is the Casimir element of ϕ. An important remark is that c_ϕ is not an element of the algebra L or its representation

$$\neg\left\{\forall L, V, \quad \forall \phi : L \to V, \exists v \in L | \phi(v) = c_\phi\right\}. \tag{2.106}$$

And as explained before c_ϕ commutes with all elements of the representation of L.
One can examine the dual basis of an algebra for $\beta(x,y) = \mathrm{Tr}\left(\phi(x)\phi(y)\right)$. If ${M_i}^j$ is the basis change matrix such as $y_i = \sum_j {M_i}^j x_j$, and $\beta_{ij} = \beta(x_i, x_j)$

$$\delta_{ij} = \beta(x_i, y_j) = \mathrm{tr}\left(\phi(x_i)\phi(y_j)\right) = \sum_k {M_j}^k \mathrm{tr}\left(\phi(x_i)\phi(x_k)\right) = \sum_k {M_j}^k \beta_{ik} = \sum_k {M_j}^k \beta_{ki} \tag{2.107}$$

If β_{ij} is invertible, then one finds that $M = \beta^{-1}$. The condition for β_{ij} to be invertible, or even to exist will be important later, as we will work with Fock space, *i.e.* non-finite dimensional space where traces of some operators will diverge; for instance, for a 1-mode oscillator $\mathrm{tr}(\hat{n}) = \infty$.

2.3.5 Relation between Lie algebras and Lie groups

There is a correspondence between Lie groups and Lie algebras; the following section will only give an idea of this correspondence, but one should be really careful with applying it.
To deduce the algebra associated with a Lie group, one can say that the neighbourhood of the identity generates the Lie algebra. The Lie algebra is the tangent space to the Lie group at the identity. As the Lie groups are manifolds, there exists a coordinate system[23] $\{x^\mu\}$ such that to each vector x^μ corresponds one and only one operator $J(x^\mu)$; it is possible to fix the coordinates in such a way that $J(0)$ is the identity. As the manifold is differentiable one can define a notion of a derivative ∇_μ, and define the operators $A_\mu = (\nabla_\mu J)|_{x^\mu=0}$. Those operators will serve as a basis for the vector space of the Lie algebra, and as they are operators the Lie bracket will simply be defined as their commutator. Let us note that the derivative should be defined properly such that A_μ behave as a vector under a change of coordinates. For instance, the groups $U(1)$ or $SO(2)$ are both equivalent topologically to a circle, therefore their Lie algebra are both 1-dimensional, as represented in Figure 2.1 a). If there existed some Lie group diffeomorphic[24] to a sphere, see b) (it is not the case), then their algebra would be a plane, and it is the same for toroïdal groups, see c). $SU(2)$ is diffeomorphic to a 3-sphere, therefore its algebra is 3-dimensional. For flat groups such as the translation group (which is diffeomorphic to $\mathbb{R}^3$), the tangeant space is isomorphic to the group. Physically, this means that for instance both $SU(2)$ and the translation group have three generators.
To derive the Lie group from the algebra, one should use the exponential map, which in most cases in physics can be interpreted as a matrix exponentiation:

$$\exp : \mathfrak{g} \to G : w = \sum_\mu w^\mu A_\mu \to exp(w) = exp(\sum_\mu w^\mu A_\mu), \Longrightarrow J(w_\mu) = \sum_{i=0}^{\infty} \frac{(\sum_\mu w^\mu A_\mu)^i}{i!}. \tag{2.108}$$

[23] An implicit assumption made here is that the manifold can be covered by an unique chart, but this is not always the case and one should be careful with the topology of manifolds.

[24] Two topological objects are diffeomorphic to one another if there exists a continuous differentiable isomorphism with a differentiable inverse between the two.

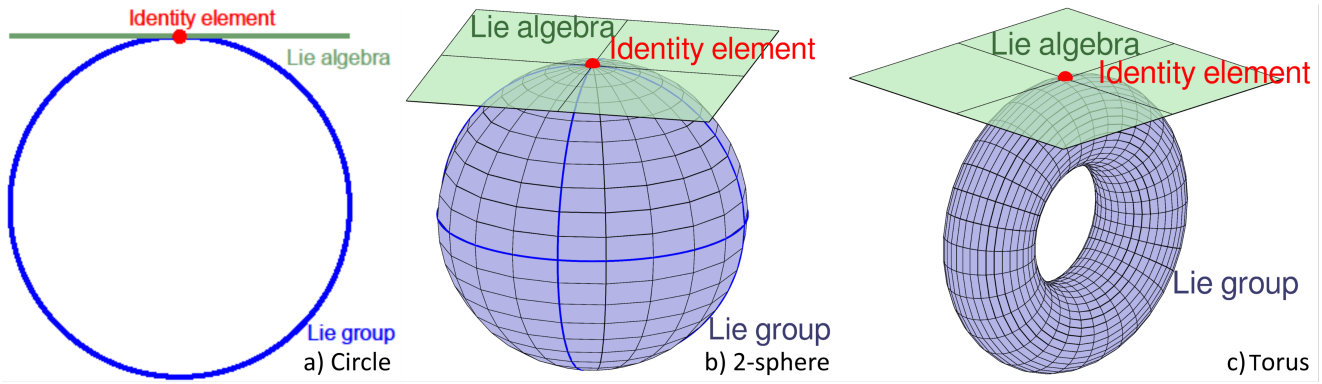

Figure 2.1: Topological representations of Lie groups and the related algebras.

2.3.6 Examples of Lie Algebra in physics

Physics is full of Lie algebras. First, as the symmetry groups are Lie groups, the vector space generated by their generators with the commutation relations deduced from the commutation of the elements of the groups are Lie algebras. $\mathfrak{so}(3)$ corresponds to the rotation group $SO(3)$ and $\mathfrak{su}(2)$ to the rotation of spin 1/2 particles $SU(2)$. Those are not the only algebras considered in physics; in each discipline using vector spaces, then, the space of endomorphisms is an algebra and in quantum mechanics, the spaces of all the observables, of all the projectors or the one of all the POVMs are subalgebras of that space. If we consider $\mathrm{Span}(\hat{a}, \hat{a}^\dagger)$, the Heisenberg-Weyl algebra is obtained where the Lie bracket is deduced from $[\hat{a}, \hat{a}^\dagger] = 1$. In general relativity theory, the set of generators of the group of changes of coordinates (the symmetry group considered is the group of all the coordinate changes, as the equations of general relativity are covariant under any diffeomorphism), constitutes a Lie algebra.

2.3.7 Properties of $\mathfrak{su}(2)$

A particularly important Lie algebra in this thesis and in general in physics is $\mathfrak{su}(2)$. This algebra is of finite dimension 3 and for a basis $\{g_i\}$, the Lie bracket is defined by:

$$[g_i, g_j] = \sum_k i \cdot \epsilon_{ijk} g_k \tag{2.109}$$

In this section we will first present some properties of $\mathfrak{su}(2)$ and then, we will define the complexified algebra, and delve in the structure of irreducible modules of this algebra.
Let ϕ be a representation of $\mathfrak{su}(2)$ such that $\phi(g_i) = G_i$. First we show that in any representation, the operators G_i are traceless. Using the classical commutator as Lie bracket, the linearity of the trace and its invariance under cyclic permutations.

$$\sum_k i\epsilon_{ijk} \mathrm{tr}\left(G_k\right) = \mathrm{tr}\left([G_i, G_j]\right) = 0 \tag{2.110}$$

The Casimir operator of $\mathfrak{su}(2)$ is $G^2 = \sum G_i^2$ (up to a normalization)[25]. We can easily verify that

[25] It is not easy to show in all generality. It would require to find the dual basis of $\{g_i\}$ according to Section (2.3.4), but it can be verified for 2-dimensional representations, using the Pauli matrices as a basis of the algebra.

this operator commutes with each basis element of the representation,

$$\begin{aligned}[G^2, G_j] &= \sum_i [G_i G_i, G_j] = \sum_i G_i[G_i, G_j] + [G_i, G_j]G_i = \sum_{i,k} \left(i\epsilon_{ijk} G_i G_k + i\epsilon_{ijk} G_k G_i\right) \\ &= \sum_{i,k} i(\epsilon_{ijk} + \epsilon_{kji}) G_i G_k = 0.\end{aligned} \tag{2.111}$$

Further calculations will be made using another basis of $\mathfrak{su}(2)$, which is called the complexified algebra $\mathfrak{su}(2)$,

$$\{G_+ = G_1 + iG_2, G_- = G_1 - iG_2, G_3\}. \tag{2.112}$$

The Lie bracket in this basis is then given by the following relations

$$\left\{ \begin{array}{rclcrcl} [G_3, G_+] &=& [G_3, G_1] + i[G_3, G_2] &=& iG_2 + i(-i)G_1 &=& G_+, \\ [G_3, G_-] &=& [G_3, G_1] - i[G_3, G_2] &=& iG_2 - i(-i)G_1 &=& -G_-, \\ [G_+, G_-] &=& i[G_2, G_1] - i[G_1, G_2] &=& i(-i)G_3 - i(+i)G_3 &=& 2G_3. \end{array} \right. \tag{2.113}$$

We will not describe the structure of irreducible modules of $\mathfrak{su}(2)$ precisely. If V is a module of the complexified algebra, then the definition of an eigenvector is similar to the one for operators : $v \in V$ is an eigenvector of $G = \phi(g)$ of eigenvalue $\lambda \in \mathcal{F}$ ($\mathbb{R}$ or $\mathbb{C}$ for the complexified algebra), if and only if

$$Gv = \lambda v. \tag{2.114}$$

Then, let g_3 be an eigenvector of G_3 of eigenvalue λ. From this one, other eigenvectors can be deduced easily, yielding

$$G_3 G_+ g_3 = (G_+ G_3 + [G_3, G_+])g_3 = (\lambda + 1)G_+ g_3, \tag{2.115}$$

$$G_3 G_- g_3 = (G_- G_3 + [G_3, G_-])g_3 = (\lambda - 1)G_+ g_3. \tag{2.116}$$

This shows that both $G_+ g_3$ and $G_- g_3$ are eigenvectors of G_3, and as they have different eigenvalues they cannot be the same vectors. Further, this explains why G_+ and G_- will be called rising and lowering operators, respectively.
Now if there exists some W as submodule of V, irreducible and finite dimensional, there is only a finite number of eigenvalues for any $g \in \mathfrak{gl}(W)$ (at most the dimension of W). By contradiction there must exist some eigenvector w_0 of eigenvalue λ such that $G_- w_0 = 0$. Indeed, by construction if $\lambda - 1$ is a non-zero eigenvalue, nor $\lambda - n \quad \forall n \in \mathbb{N}$, there exists an infinite number of different eigenvalues. We will note $w_k = G_+^k w_k$. By induction, we can deduce

$$G_- w_k = -k(2\lambda + k - 1)w_{k-1}, \tag{2.117}$$

as long as $w_{k-1} \neq \vec{0}$. Once again there is only a finite number of non-zero eigenvectors, therefore

$$\exists l \in \mathbb{N} : w_l = \vec{0}. \tag{2.118}$$

Lets note w_m the last non-zero eigenvector, then:

$$\vec{0} = w_{m+1} = G_+ w_{m+1} = -(m+1)(2\lambda + m)w_m. \tag{2.119}$$

As $w_m \neq \vec{0}$ and $m \geq 0$, we have:

$$\lambda = -\frac{m}{2}. \tag{2.120}$$

With this method, we have constructed $m + 1$ vectors $w_0, ...w_m$:

$$\begin{gathered} G_- w_0 = 0 \quad \text{and} \quad G_- w_k = -k(k - m - 1)w_{k-1} \quad \forall k > 0, \\ G_+ w_m = 0 \quad \text{and} \quad G_+ w_k = w_{k+1} \forall k < m, \\ G_3 w_k = (k - \frac{m}{2})w_k. \end{gathered} \tag{2.121}$$

The vector space generated by the w_k is of dimension $d = m + 1$, and is a submodule of W as G_3, G_+, G_- are intern. As W is irreducible this submodule which is of dimension $d > 0$ can only be W itself.
W being an irreducible module of $\mathfrak{su}(2)$, Schur's lemma gives that all operators commuting with each operator of the representation should be proportional to the identity on W. As G^2 is proportional to the Casimir element, it should then behave as the identity:

$$\begin{aligned} G^2 = G_1^2 + G_2^2 + G_3^2 &= \left(\frac{G_+ + G_-}{2}\right)^2 - \left(\frac{G_+ - G_-}{2}\right)^2 + G_3^2 = \frac{G_+G_- + G_-G_+}{2} + G_3^2, \\ &= G_-G_+ + G_3 + G_3^2 \\ \Rightarrow G^2 w_k = (-(k+1)(k-m) + (k - \frac{m}{2}) + (k - \frac{m}{2})^2)w_k &= \left(\frac{m}{2} + \left(\frac{m}{2}\right)^2\right) w_k = \frac{d-1}{2}\frac{d+1}{2} w_k, \\ \Rightarrow G^2 = \frac{d-1}{2}\frac{d+1}{2}\mathbb{1} & \end{aligned} \tag{2.122}$$

Interestingly, the proportionality coefficient is only a function of the dimension of the irreducible module considered.

2.4 Uncertainty principle

2.4.1 Uncertainty relations

Uncertainty relations, and the uncertainty principle in general, are key findings in quantum theories. In classical theories, there are no theoretical limitations on the precision one can get for the measurement of observable quantities, and overall the knowledge of one does not impact the precision obtainable on another. By opposition, in quantum theories, there exist uncompatible observables, such that the precision of the measurement of one gives a lower bound on the precision achievable with the measurement of the other.
The uncertainty principle was first introduced in 1927 by Heisenberg as a tought experiment consisting of projecting photons on an electron to deduce its position and its momentum [1]. However this method of measurement introduces a little uncertainty on both the position of the electron (as the photon comports as a wave its position is not precisely defined), and its momentum (similarly, the direction of the photon is not precisely defined). In this procedure, Heisenberg showed there was a trade off between both such that

$$\delta x \delta p \approx 2\pi, \tag{2.123}$$

where we set $\hbar = 1$. Later Kennard formalized this idea in terms of the variance of both probability distributions over the two quadratures $\hat{x}$, and $\hat{p}$ [2],

$$\Delta^2 x \Delta^2 p \geq \frac{1}{4}. \tag{2.124}$$

Where $\Delta^2 x$ and $\Delta^2 p$ are respectively the variance of the probability distributions for the particle with respect to the quadratures. Formally, he considered that both distributions are Fourier transform from one another, and deduced this relation as a fundamental property of Fourier transforms.
Physically, there are two interpretations of uncertainty, Kennard considered that it was an intrinsic property of states, saying that any physical state has a variance in its position and its momentum. In opposition, Heisenberg considerd that uncertainty relations describe an error disturbance only due to the measurement of the system, and Ozawa derived formally his inequality in this sense [22].

There have been many formulations of this principle. First in 1928 Robertson extended this principle for any observables $\hat{A}$ and $\hat{B}$ [23], as

$$\Delta^2\hat{A}\Delta^2\hat{B} \geq \frac{1}{4}|\langle\psi|\left[\hat{A},\hat{B}\right]|\psi\rangle|^2, \tag{2.125}$$

where if we replace $\hat{A}$ and $\hat{B}$ by $\hat{x}$ and $\hat{p}$, with $[\hat{x},\hat{p}] = i$ we get back Kennard's result. In 1930, Schrödinger [24] and Robertson [25] both independently improved this relation, to give the Schrödinger-Robertson inequality

$$\Delta^2\hat{A}\Delta^2\hat{B} \geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2 + \frac{1}{4}|\langle\{\hat{A}-\langle\hat{A}\rangle, \hat{B}-\langle\hat{B}\rangle\}\rangle|^2. \tag{2.126}$$

If the covariance of two observables is defined as $\mathrm{cov}_{\hat{A},\hat{B}} = \frac{1}{2}\langle\{\hat{A}-\langle\hat{A}\rangle, \hat{B}-\langle\hat{B}\rangle\}\rangle$, with naturally $\Delta^2\hat{A} = \mathrm{cov}_{\hat{A},\hat{A}}$, the covariance matrix is the matrix

$$\gamma_{\hat{A},\hat{B}} = \begin{pmatrix} \mathrm{cov}_{\hat{A},\hat{A}} & \mathrm{cov}_{\hat{A},\hat{B}} \\ \mathrm{cov}_{\hat{B},\hat{A}} & \mathrm{cov}_{\hat{B},\hat{B}} \end{pmatrix} \tag{2.127}$$

and the Scrhödinger-Robertson inequality becomes

$$\det\left(\gamma_{\hat{A},\hat{B}}\right) \geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2. \tag{2.128}$$

Amongst others, a notable family of uncertainty relations are the entropic uncertainty relations. These are expressed in terms of the entropies of the distributions[26, 27]. For continuous variables such as $\hat{x}$ and $\hat{p}$, the entropy is expressed as

$$h(f) = -\int_{-\infty}^{\infty} f(x)\log f(x)dx. \tag{2.129}$$

Then Beckner [28], Bialynicki-Birula and Mycielski [4] showed that

$$h(x) + h(p) \geq 1 + \log\pi. \tag{2.130}$$

Entropic uncertainty relations are particularly important as they establish the link with quantum information theories. Often, they are stronger than the classical Heisenberg relation that can be deduced from it. Let us note that this formulation has, however, several weaknesses. Schrödinger-Robertson's inequality was saturated for all Gaussian pure states, and was invariant under symplectic transformations[26], which is not true for Equation (2.130).Several recent works such as the one from Hertz *et al.* try to tackle those weaknesses [10, 30].
Uncertainty relations play a fundamental role in several parts of quantum applications. In quantum cryptography, many protocols rely on the idea that the sender and the emitter can deduce from the receiver's error, the error on an eavesdropper's side, and deduce the presence of one eavesdropper or not [5, 6]. There also exists entanglement criteria that use uncertainty relations to check if a state is entangled [7–9].

2.4.2 Multicopy uncertainty relations

In the article [10], Hertz *et al.* showed that the Scrödinger-Robertson uncertainty relation could be deduced from simple relations on multicopy observables (by this we mean multi-mode operators acting on a copy state). For instance, if we consider the operator

$$\hat{L}_y = \hat{x}_1 \otimes \hat{p}_2 - \hat{p}_1 \otimes \hat{x}_2 \tag{2.131}$$

[26]For more details on this, have a look at [29].

The positivity of the mean value of $\hat{L}_y^2$ on a copy state $\hat{\rho} \otimes \hat{\rho}$ directly implies the Schrödinger-Robertson inequality for centered states[27]. This can be generalized to 3-mode observables, allowing to drop the centered assumption. These results motivate the use of such observables to deduce uncertainty relations from simple inequalities. For instance, considering their entropy could lead to stronger entropic uncertainty relations, or considering other states than copy states could lead to new multi-mode uncertainty relations.

[27]States for which the mean-value of $\hat{x}$ and $\hat{p}$ are null.

3. Uncertainty relations from the Jordan-Schwinger map

3.1 Jordan-Schwinger map

The Jordan-Schwinger map is a homomorphism $\phi : \mathfrak{h} \subseteq \mathfrak{gl}(n, \mathbb{C}) \rightarrow \mathfrak{gl}(\mathcal{F}(n))$ between two representations of a Lie algebra $\mathfrak{g}$ of dimension N. More on this mapping can be found in the initial article of Jordan [31], or in the article explaining how Schwinger used it [32]. Complete development for $\mathfrak{su}(2)$ can also be found in [13, 33]. These sources only consider the case of $\mathfrak{su}(2)$, but here we will define it for the general case.
$\mathfrak{h}$ is a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$, the vector space of n-dimensional square matrices with complex entries. By definition of a representation, $\mathfrak{h}$ is of dimension N as $\mathfrak{g}$, so if the set of N vectors $\{g_i\}$ is a basis of $\mathfrak{g}$, there exists some set of N matrices $\{G_i\}$ making a basis of $\mathfrak{h}$ and obeying the same commutation rule as the g_i's, as already seen with Equation (2.98):

$$[g_i, g_j] = \sum_k f_{ij}^k g_k \Rightarrow [G_i, G_j] = \sum_k f_{ij}^k G_k \forall i, j \in 1, ..., N. \tag{3.1}$$

We note that for $\mathcal{F}(n)$, *i.e.* the Fock space of a n-mode harmonic oscillator, $\mathfrak{gl}(\mathcal{F}(n))$ is the vector space $\mathrm{End}(\mathcal{F}(n))$, the set of all the operators in $F(n)$, with the commutator as Lie bracket. A basis of this algebra are the n creation and the n annihilation operators $\{\hat{a}_1^\dagger, ..., \hat{a}_n^\dagger\}$ and $\{\hat{a}_1, ..., \hat{a}_n\}$ with the bosonic commutation relations, *i.e.*

$$[\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger] = 0, \quad [\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}. \tag{3.2}$$

The Jordan-Schwinger map can be defined by its action on the basis of $\mathfrak{h} : \{G_i\}$ as:

$$\hat{G}_i \equiv \sum_{\alpha\beta} \hat{a}_\alpha^\dagger \left(G_i\right)_{\alpha,\beta} \hat{a}_\beta \quad \forall i \in [1, ..., N]. \tag{3.3}$$

One can then show that the set of the operators $\hat{G}_i$ is a representation of $\mathfrak{g}$ by proving that the commutation relation are preserved:

$$[\hat{G}_i, \hat{G}_j] = \sum_k f_{ij}^k \hat{G}_k \quad \forall i, j \in [1, ..., N]. \tag{3.4}$$

This is done in Appendix C.1. This also proves that the Jordan-Schwinger map is a homomorphism as the Lie bracket is conserved. If one wants the operators $\hat{G}_i$ to be observables, the matrices G_i should be Hermitian, as

$$\hat{G}_i^\dagger = \sum_{\alpha\beta} \left(\hat{a}_\alpha^\dagger \left(G_i\right)_{\alpha,\beta} \hat{a}_\beta\right)^\dagger = \sum_{\alpha\beta} \hat{a}_\beta^\dagger \left(G_i\right)_{\alpha,\beta}^* \hat{a}_\alpha = \sum_{\alpha\beta} \hat{a}_\alpha^\dagger \left(G_i\right)_{\beta,\alpha}^* \hat{a}_\beta. \tag{3.5}$$

The definition of the Jordan-Schwinger map directly shows that the operators generated thanks to the Jordan-Schwinger map are only single body observables, as they are all of the form of Equation (2.93).

3.2 General strategy

The aim of the first part of this thesis is to derive uncertainty relations for a multimode Hilbert space using the Jordan-Schwinger map. The idea is to take a matrix, pass it through the Jordan-Schwinger map, and then use inequalities already known for any operator (more precisely on the results of their measurements in particular states) to deduce new inequalities on the quadratures $\hat{x}$ and $\hat{p}$ (on their measurements in these states). For instance A. Hertz *et al.* used the image of the Pauli matrix σ_y by the Jordan-Schwinger map and considered that its variance should be positive in a 2-copy state to deduce the classical Schrödinger-Robertson relation [10].
Several inequalities can be identified. For instance, for any observable, the mean square value and variance of the measurement results should be positive.

$$\langle \hat{G}^2 \rangle \geq 0 \quad \text{and} \quad \Delta^2 \hat{G} \geq 0. \tag{3.6}$$

Further the entropy over the distribution of the outcomes of the measurement should also be positive

$$S(\hat{G}) \geq 0. \tag{3.7}$$

As discussed already, a big advantage of the Jordan-Schwinger map is that it preserves the commutation relations. Let us not consider one matrix, but a set of matrices $\{L_i\}$ such that

$$[L_i, L_j] = \sum_k f_{ijk} L_k, \tag{3.8}$$

then the commutation relations are the same for the operators $\{\hat{L}_i\}$ associated to them, *i.e.*

$$[\hat{L}_i, \hat{L}_j] = \sum_k f_{ijk} \hat{L}_k. \tag{3.9}$$

We can then use the Schrödinger-Robertson on this set of operators as a new suitable inequality:

$$\Delta^2 \hat{L}_i \Delta^2 \hat{L}_j \geq \sum_k \frac{1}{4} \left| f_{ijk} \langle \hat{L}_k \rangle \right|^2 + \frac{1}{4} \left| \langle \left\{ \hat{L}_i - \langle \hat{L}_i \rangle , \hat{L}_j - \langle \hat{L}_j \rangle \right\} \rangle \right|^2. \tag{3.10}$$

For the particular case of $\mathfrak{su}(2)$[28] this can be simplified as

$$\Delta^2 \hat{L}_i \Delta^2 \hat{L}_j \geq \sum_k \frac{|\epsilon_{ijk}|}{4} \langle \hat{L}_k \rangle^2 + \frac{1}{4} \left| \langle \left\{ \hat{L}_i - \langle \hat{L}_i \rangle , \hat{L}_j - \langle \hat{L}_j \rangle \right\} \rangle \right|^2. \tag{3.11}$$

Alternatively we can use simpler of those inequalities such as for the mean square value

$$\langle \left(\hat{G}_1 \pm \hat{G}_2 \right)^2 \rangle \geq 0 \iff \langle \hat{G}^2 + \hat{H}^2 \rangle \geq \left| \langle \{ \hat{G}, \hat{H} \} \rangle \right|, \tag{3.12}$$

or for Schrödinger-Robertson

$$\left(\Delta^2 \hat{G} + \Delta^2 \hat{H} \right) \geq \left(\Delta^2 \hat{G} \right)^2 + \left(\Delta^2 \hat{H} \right)^2 + \frac{1}{2} \left| \langle [\hat{G}, \hat{H}] \rangle \right|^2 + \frac{1}{4} \left| \langle \left\{ \hat{G} - \langle \hat{G} \rangle , \hat{H} - \langle \hat{H} \rangle \right\} \rangle \right|^2. \tag{3.13}$$

All operators we use are derived from the Jordan-Schwinger map; therefore, they are expressed in terms of creation and annihilation operators. As Equation (2.91) establishes a clear link between these operators and the quatratures on each mode, all inequalities can be converted into inequalities in terms of position and momentum in each mode.
In the following we consider a general state and then particularize to several kinds of state, including separable, product, and copy states.
In Section 3.3 we demonstrate that this approach does not lead to anything for 1-mode matrices. In Section 3.4 we derive several novel relations for 2-mode representations of $\mathfrak{su}(2)$. In Section 3.5 we deduce general inequalities by considering general matrices.

[28] We consider only representations where $\hat{L}_i$ are observables. Their measurement outcomes are then always real.

3.3 1-mode inequality

1-mode inequalities will not give interesting results; indeed, the only operators that can be created with the Jordan-Schwinger map are proportional to the occupation number operator:

$$\hat{G} = \hat{a}^\dagger g \hat{a} = g\hat{n} = g(\hat{x}^2 + \hat{p}^2). \tag{3.14}$$

As there is only one operator that we can create, all inequalities with more than one operator are irrelevant. The variance and mean square values of $\hat{G}$ are trivially positive.

3.4 2-mode representation of $\mathfrak{su}(2)$

We consider Hermitian matrices[29] in $\mathfrak{gl}(2,\mathbb{C})$ such that

$$L_i \equiv \frac{\sigma_i}{2}, \tag{3.15}$$

where σ_i are the three Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.16}$$

Using the Jordan-Schwinger map (*cf.* Equation (3.3)), the spin matrices become spin operators given by

$$\begin{aligned} \hat{L}_1 = \hat{L}_x &\equiv \sum_{\alpha,\beta} \hat{a}^\dagger_\alpha \left(L_x\right)_{\alpha,\beta} \hat{a}_\beta = \frac{\hat{a}^\dagger_1 \hat{a}_2 + \hat{a}^\dagger_2 \hat{a}_1}{2}, \\ \hat{L}_2 = \hat{L}_y &\equiv \sum_{\alpha,\beta} \hat{a}^\dagger_\alpha \left(L_y\right)_{\alpha,\beta} \hat{a}_\beta = i \cdot \frac{\hat{a}^\dagger_2 \hat{a}_1 - \hat{a}^\dagger_1 \hat{a}_2}{2}, \\ \hat{L}_3 = \hat{L}_z &\equiv \sum_{\alpha,\beta} \hat{a}^\dagger_\alpha \left(L_z\right)_{\alpha,\beta} \hat{a}_\beta = \frac{\hat{a}^\dagger_1 \hat{a}_1 - \hat{a}^\dagger_2 \hat{a}_2}{2}. \end{aligned} \tag{3.17}$$

Using the relations between annihilation and creation operators, and the position, momentum (Equations (2.91)) and the number operators (Equation (2.87)), the spin observables can be re-expressed as

$$\hat{L}_x = \frac{\hat{x}_1 \otimes \hat{x}_2 + \hat{p}_1 \otimes \hat{p}_2}{2}, \quad \hat{L}_y = \frac{\hat{x}_1 \otimes \hat{p}_2 - \hat{p}_1 \otimes \hat{x}_2}{2}, \quad \hat{L}_z = \frac{\hat{n}_1 - \hat{n}_2}{2} = \frac{\hat{x}_1^2 + \hat{p}_1^2 - \hat{x}_2^2 - \hat{p}_2^2}{4}. \tag{3.18}$$

Each operator provides access to information about the position and momentum of the states. In particular, $\hat{L}_z$ is simple to measure with optical circuits as it requires only counting the photon number in each modes.
Using these expressions we can develop the inequalities given in Section 3.2.

[29] It is not necessary to consider other 2×2 representations of $\mathfrak{su}(2)$, the matrices used n those representations are linear combinations of the Pauli matrices, therefore the inequalities generated by those new matrices are bilinear combinations of the one generated by the Paui matrices

3.4.1 Inequalities from $\langle \hat{L}_y^2 \rangle \geq 0$

A. Hertz *et al.* showed that this inequality applied to a 2-copy state is equivalent to the Schrödinger-Robertson inequality for a centered state. Here, we develop $\langle \hat{L}_y^2 \rangle \geq 0$ for a general 2-mode state:

$$0 \leq \langle \hat{L}_y^2 \rangle \, 0 \leq \frac{1}{4} \left\langle (\hat{x}_1 \hat{p}_2 - \hat{p}_1 \hat{x}_2)^2 \right\rangle, 0 \leq \frac{1}{4} \left[\langle \hat{x}_1^2 \hat{p}_2^2 \rangle + \langle \hat{p}_1^2 \hat{x}_2^2 \rangle - \langle \{ \hat{x}_1 \hat{p}_2, \hat{p}_1 \hat{x}_2 \} \rangle \right],$$
$$\Rightarrow 0 \leq \frac{1}{4} \left[\langle \hat{x}_1^2 \hat{p}_2^2 \rangle + \langle \hat{p}_1^2 \hat{x}_2^2 \rangle - \frac{1}{2} \langle \{ \hat{x}_1, \hat{p}_1 \} \{ \hat{x}_2, \hat{p}_2 \} \rangle - \frac{1}{2} \right]. \tag{3.19}$$

Equation (3.19) is an explicit uncertainty relation for a two-mode system. We can particularize this result for several types of states using the properties of their means presented in Section 2.2.2. For separable states we find:

$$0 \leq \frac{1}{4} \sum_i p_i \left[\langle \hat{x}_1^2 \rangle^i \langle \hat{p}_2^2 \rangle^i + \langle \hat{x}_2^2 \rangle^i \langle \hat{p}_1^2 \rangle^i - \frac{1}{2} \langle \{ \hat{x}_1, \hat{p}_1 \} \rangle^i \langle \{ \hat{x}_2, \hat{p}_2 \} \rangle^i - \frac{1}{2} \right], \tag{3.20}$$

while for product states we get

$$0 \leq \frac{1}{4} \left[\langle \hat{x}_1^2 \rangle \langle \hat{p}_2^2 \rangle + \langle \hat{x}_2^2 \rangle \langle \hat{p}_1^2 \rangle - \frac{1}{2} \langle \{ \hat{x}_1, \hat{p}_1 \} \rangle \langle \{ \hat{x}_2, \hat{p}_2 \} \rangle - \frac{1}{2} \right], \tag{3.21}$$

Finally, for copy-states we obtain

$$0 \leq \frac{1}{4} \left[2 \langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle - \frac{1}{2} \langle \{ \hat{x}, \hat{p} \} \rangle^2 - \frac{1}{2} \right]. \tag{3.22}$$

Which can be rewritten as the classical Schrödinger-Robertson relation

$$\det(\gamma) \geq \frac{1}{4}, \tag{3.23}$$

where γ is the covariance matrix for the centered states[30],

$$\gamma = \begin{pmatrix} \langle \hat{x}^2 \rangle & \frac{1}{2} \langle \{ \hat{x}, \hat{p} \} \rangle \\ \frac{1}{2} \langle \{ \hat{x}, \hat{p} \} \rangle & \langle \hat{p}^2 \rangle \end{pmatrix}. \tag{3.24}$$

This result was already obtained by A. Hertz *et al.* in [10]. We proved that Equation (3.19) implies the classical Schrödinger-Robertson inequality, (3.23). Similarly, for the particular case of centered separable states, we can show that the mean system is indeed submitted to the Schrödinger-Robertson inequality. If $\hat{\bar{x}} = \frac{\hat{x}_1 + \hat{x}_2}{2}$ and $\hat{\bar{p}} = \frac{\hat{p}_1 + \hat{p}_2}{2}$, and $\bar{\gamma}$ their covariance matrix we expect that

$$\det \bar{\gamma} \geq \frac{1}{16}, \tag{3.25}$$

using $[\hat{\bar{x}}, \hat{\bar{p}}] = \frac{i}{2}$. If we note γ_i the covariance matrix on mode i, then

$$16 \det \bar{\gamma} = \det \gamma_1 + \det \gamma_2 + \langle \hat{x}_1^2 \rangle \langle \hat{p}_2^2 \rangle + \langle \hat{x}_2^2 \rangle \langle \hat{p}_1^2 \rangle - \frac{1}{2} \langle \{ \hat{x}_1, \hat{p}_1 \} \rangle \langle \{ \hat{x}_2, \hat{p}_2 \} \rangle \geq \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1. \tag{3.26}$$

This should be true for any state, but we only succeeded in demonstrating it this way for separable states.

Other way around, even if the Robertson inequality (2.125) trivially implies (3.19) if you consider $\hat{A} = \hat{B} = \hat{L}_y$, indeed,

$$\left(\Delta^2 \hat{L}_y \right)^2 \geq \langle [\hat{L}_y, \hat{L}_y] \rangle = 0 \Rightarrow \langle \hat{L}_y^2 \rangle \geq 0. \tag{3.27}$$

[30] A centered state is a state such that $\langle \hat{x} \rangle = \langle \hat{p} \rangle = 0$

However, we were not able to derive Equation (3.19) in full generality from combinations of Schrödinger-Robertson inequalities applied to the single-mode quadratures or a combination of both (for instance, the mean momentum/position of the two modes or the half-difference). If this was the case, then Equation (3.19) would not give any new insights. In contrast, if we show that it is not possible to derive Equation (3.19) from simple inequalities, then there is a real novelty in this equation, giving a coupling between the pairs $\hat{x}_1, \hat{p}_2$ and $\hat{x}_2, \hat{p}_1$.
Equation (3.20) can be used as an entanglement criterion, as if this inequality is violated, then the state is not a product state, it is entangled. However, it requires the knowledge of the distribution p_i we do not know in general. Although, Equation (3.21) can be used as an entanglement criterion for pure states, if this inequality is violated, then the state is not a product state, it is entangled[31]. Future research should examine the states saturating Equations (3.19), (3.20), and (3.21). The classical Schrödinger-Robertson and its symplectic generalization are saturated by all Gaussian pure states, whereas ours is saturated by eigenstates of $\hat{L}_y$ with eigenvalue $l_y = 0$, (which can be derived using the results in Sections 4.1 and 4.7).

3.4.2 Inequalities from $\Delta^2 L_y \geq 0$

This inequality gives very similar results to those obtained in the previous section, and for centered states, they are exactly the same. Similarly for the centered state, we get back Scrödinger-Robertson's inequality.The only difference between the equations in the previous section is an additional term, non-zero only for non-centered states, such that

$$\langle \hat{x}_1^2 \hat{p}_2^2 + \hat{p}_1^2 \hat{x}_2^2 \rangle - \langle \hat{x}_1 \hat{p}_2 + \hat{p}_1 \hat{x}_2 \rangle^2 - \frac{1}{2} \langle \{\hat{x}_1, \hat{p}_1\} \{\hat{x}_2, \hat{p}_2\} \rangle \geq \frac{1}{2}. \tag{3.28}$$

In terms of variance, it becomes

$$\Delta^2 (\hat{x}_1 \hat{p}_2) + \Delta^2 (\hat{p}_1 \hat{x}_2) - \frac{1}{2} \langle \{\hat{x}_1, \hat{p}_1\} \{\hat{x}_2, \hat{p}_2\} \rangle - 2 \langle \hat{x}_1 \hat{p}_2 \rangle \langle \hat{p}_1 \hat{x}_2 \rangle \geq \frac{1}{2}. \tag{3.29}$$

This equation and those derived for separable and product states are saturated for all eigenstates of $\hat{L}_z$.

3.4.3 Inequalities from $\langle \hat{L}_x^2 \rangle \geq 0$

We now consider $\hat{L}_x^2$. For a general state, the inequality $\langle \hat{L}_x^2 \rangle \geq 0$ is equivalent to:

$$\frac{1}{4} \left[\langle \hat{x}_1^2 \hat{x}_2^2 \rangle + \langle \hat{p}_1^2 \hat{p}_2^2 \rangle + \langle \{\hat{x}_1, \hat{p}_1\} \{\hat{x}_2, \hat{p}_2\} \rangle - \frac{1}{2} \right] \geq 0. \tag{3.30}$$

For separable states we get

$$\frac{1}{4} \sum_i P_i \left[\langle \hat{x}_1^2 \rangle^i \langle \hat{x}_2^2 \rangle^i + \langle \hat{p}_1^2 \rangle^i \langle \hat{p}_2^2 \rangle^i + \frac{1}{2} \langle \{\hat{x}_1, \hat{p}_1\} \rangle^i \langle \{\hat{x}_2, \hat{p}_2\} \rangle^i - \frac{1}{2} \right] \geq 0, \tag{3.31}$$

for product states

$$\frac{1}{4} \left[\langle \hat{x}_1^2 \rangle \langle \hat{x}_2^2 \rangle + \langle \hat{p}_1^2 \rangle \langle \hat{p}_2^2 \rangle + \frac{1}{2} \langle \{\hat{x}_1, \hat{p}_1\} \rangle \langle \{\hat{x}_2, \hat{p}_2\} \rangle - \frac{1}{2} \right] \geq 0, \tag{3.32}$$

and finally for 2-copy states

$$\frac{1}{4} \left[\langle \hat{x}^2 \rangle^2 + \langle \hat{p}^2 \rangle^2 + \frac{1}{2} \langle \{\hat{x}, \hat{p}\} \rangle^2 - \frac{1}{2} \right] \geq 0. \tag{3.33}$$

[31] It is not a product state as we considered only pure states.

The latter inequality is a weaker inequality than the Schrödinger-Robertson inequality. Plugging

$$\left(\langle\hat{x}^2\rangle - \langle\hat{p}^2\rangle\right)^2 \geq 0 \Rightarrow 2\langle\hat{x}^2\rangle\langle\hat{p}^2\rangle \leq \langle\hat{x}^2\rangle^2 + \langle\hat{p}^2\rangle^2, \tag{3.34}$$

and $\langle\{\hat{x},\hat{p}\}\rangle^2 \geq 0$ in the Schrödinger-Robertson inequality, we find

$$\frac{1}{2} \leq 2\langle\hat{x}^2\rangle\langle\hat{p}^2\rangle - \frac{1}{2}\langle\{\hat{x},\hat{p}\}\rangle^2 \leq \langle\hat{x}^2\rangle^2 + \langle\hat{p}^2\rangle^2 + \frac{1}{2}\langle\{\hat{x},\hat{p}\}\rangle^2. \tag{3.35}$$

These inequalities are saturated for the eigenstates of $\hat{L}_x$ with $l_x = 0$.
Similarly as for $\hat{L}_y$ we had derived the Schrödinger-Robertson for the mean system by adding inequalities on each mode separately (here Equation (3.33)) and inequality on the global 2-mode system (Equation (3.31)) gives for separable states

$$\langle\frac{\hat{x}_1^2+\hat{x}_2^2}{2}\rangle^2 + \langle\frac{\hat{p}_1^2+\hat{p}_2^2}{2}\rangle^2 + \frac{1}{2}\langle\frac{\{\hat{x}_1,\hat{p}_1\}+\{\hat{x}_2,\hat{p}_2\}}{2}\rangle^2 \geq \frac{1}{2}. \tag{3.36}$$

And for centered states, we obtain an inequality on the mean system

$$\langle\hat{\bar{x}}^2\rangle^2 + \langle\hat{\bar{p}}^2\rangle^2 + \frac{\langle\{\hat{\bar{x}},\hat{\bar{p}}\}\rangle}{2} \geq \frac{1}{8}. \tag{3.37}$$

The same way we have proven that (3.33) could be derived from the classical Schrödinger-Robertson inequality, this relation on the mean system can be derived from (3.25). With this new inequality, we have derived a new equation for 2-mode systems, which indicates correlations between $\hat{x}_1\hat{x}_2$ and $\hat{p}_1\hat{p}_2$, but all its consequences are weaker than those of $\langle\hat{L}_y^2\rangle \geq 0$ therefore, we conjecture that Equation (3.30) is weaker than (3.19).
Finally, the results for $\Delta^2\hat{L}_x$ are analogous to those for $\langle\hat{L}_x\rangle$; there is only an additional term that is nonzero for the centered states.

3.4.4 Inequalities from $\langle\hat{L}_z^2\rangle \geq 0$

We can treat $\hat{L}_z$ in the same way as we treated $\hat{L}_x$ and $\hat{L}_y$:

$$\begin{aligned}
0 \geq \langle\hat{L}_z^2\rangle &= \frac{1}{16}\langle\left(\hat{x}_1^2+\hat{p}_1^2-\hat{x}_2^2-\hat{p}_2^2\right)^2\rangle \\
0 \geq &\frac{1}{16}(\langle\hat{x}_1^4\rangle + \langle\hat{p}_1^4\rangle + \langle\hat{x}_2^4\rangle + \langle\hat{p}_2^4\rangle - 2\langle\hat{x}_1^2\hat{x}_2^2\rangle - 2\langle\hat{x}_1^2\hat{p}_2^2\rangle \\
&\quad - 2\langle\hat{p}_1^2\hat{x}_2^2\rangle - 2\langle\hat{p}_1^2\hat{p}_2^2\rangle + \langle\{\hat{x}_1^2,\hat{p}_1^2\}\rangle + \langle\{\hat{x}_2^2,\hat{p}_2^2\}\rangle) \\
0 \geq &\frac{1}{16}(\langle\hat{x}_1^4\rangle + \langle\hat{p}_1^4\rangle + \langle\hat{x}_2^4\rangle + \langle\hat{p}_2^4\rangle - 2\langle\hat{x}_1^2\hat{x}_2^2\rangle - 2\langle\hat{x}_1^2\hat{p}_2^2\rangle \\
&\quad - 2\langle\hat{p}_1^2\hat{x}_2^2\rangle - 2\langle\hat{p}_1^2\hat{p}_2^2\rangle + \frac{\langle\{\hat{x}_1,\hat{p}_1\}\rangle^2 + \langle\{\hat{x}_2,\hat{p}_2\}\rangle^2}{2} - 3)
\end{aligned} \tag{3.38}$$

If we particularize this equation to copy states, it gives $0 \geq 0$. This shows that not all multi-mode inequalities obtained by our method will give interesting when applying them to copy states.

3.4.5 Inequalities from $\langle\hat{L}_x^2 + \hat{L}_y^2\rangle \geq |\langle\{\hat{L}_x,\hat{L}_y\}\rangle| \geq 0$

Let us consider a last equation, which has more than one operator. This equation can be developed as the sum of those developed for $\hat{L}_x^2$ and $\hat{L}_y^2$. This leads to the following results:

$$\langle(\hat{x}_1^2+\hat{p}_1^2)(\hat{x}_2^2+\hat{p}_2^2)\rangle \geq 1 \tag{3.39}$$

This simplifies to

$$\langle(\hat{x}_1^2+\hat{p}_1^2)\rangle\,\langle(\hat{x}_2^2+\hat{p}_2^2)\rangle \geq 1, \tag{3.40}$$

for product states, and for 2-copy states to

$$\langle(\hat{x}^2+\hat{p}^2)\rangle \geq 1. \tag{3.41}$$

This inequality is another consequence of the Schrödinger-Robertson inequality. None of the inequalities we considered yielded stronger results than the Schrödinger-Robertson inequality. However, several multimode inequalities were found.

3.5 General case

Let us now consider the general case of a Hermitian matrix G. To simply the expressions, we use the index notation, which implies summation over repeated indices. Applying the Jordan-Schwinger map to this matrix gives the operator

$$\hat{G} = \hat{a}_\alpha^\dagger G_{\alpha\beta}\hat{a}_\beta, \tag{3.42}$$

and using Equation (2.91), we can directly express this operator in terms of the single-mode quadratures, namely

$$\begin{aligned}\hat{G} &= \frac{1}{2}(\hat{x}_\alpha - i\hat{p}_\alpha)G_{\alpha\beta}(\hat{x}_\beta + i\hat{p}_\beta)\\ &= \frac{G_{\alpha\beta}}{2}\left(\hat{x}_\alpha\hat{x}_\beta + \hat{p}_\alpha\hat{p}_\beta + i\hat{x}_\alpha\hat{p}_\beta - i\hat{p}_\alpha\hat{x}_\beta\right)\\ &= \mathcal{R}\left(G_{\alpha\beta}\right)\left(\hat{x}_\alpha\hat{x}_\beta + \hat{p}_\alpha\hat{p}_\beta\right) - \frac{\mathcal{I}\left(G_{\alpha\beta}\right)}{2}\left\{\hat{x}_\alpha, \hat{p}_\beta\right\},\end{aligned} \tag{3.43}$$

where $\mathcal{R}$ and $\mathcal{I}$ denote the real and imaginary parts, respectively. As we consider Hermitian matrices, $\mathcal{R}\left(G_{\alpha\beta}\right) \equiv R_{\alpha\beta}$ is symmetric and $\mathcal{I}\left(G_{\alpha\beta}\right) \equiv I_{\alpha\beta}$ is antisymmetric, and they are independent. Because I is zero for diagonal terms, we can remove the anticommutator and only keep the product $\hat{x}_\alpha\hat{p}_\beta$. For any operator $\hat{G}$, we have $\langle\hat{G}^\dagger\hat{G}\rangle \geq 0$; in particular, for observables, this becomes $\langle\hat{G}^2\rangle \geq 0$. Using this inequality on equation (3.43), we get

$$\langle\left(R_{\alpha\beta}\left(\hat{x}_\alpha\hat{x}_\beta + \hat{p}_\alpha\hat{p}_\beta\right) - I_{\alpha\beta}\hat{x}_\alpha\hat{p}_\beta\right)\left(R_{\gamma\delta}\left(\hat{x}_\gamma\hat{x}_\delta + \hat{p}_\gamma\hat{p}_\delta\right) - I_{\gamma\delta}\hat{x}_\gamma\hat{p}_\delta\right)\rangle \geq 0 \tag{3.44}$$

Developing the product leads to the following inequality

$$\begin{aligned}&R_{\alpha\beta}R_{\gamma\delta}\left(\langle\hat{x}_\alpha\hat{x}_\beta\hat{x}_\gamma\hat{x}_\delta\rangle + \langle\hat{x}_\alpha\hat{x}_\beta\hat{p}_\gamma\hat{p}_\delta\rangle + \langle\hat{p}_\alpha\hat{p}_\beta\hat{x}_\gamma\hat{x}_\delta\rangle + \langle\hat{p}_\alpha\hat{p}_\beta\hat{p}_\gamma\hat{p}_\delta\rangle\right)\\ -&R_{\alpha\beta}I_{\gamma\delta}\left(\langle\hat{x}_\alpha\hat{x}_\beta\hat{x}_\gamma\hat{p}_\delta\rangle + \langle\hat{p}_\alpha\hat{p}_\beta\hat{x}_\gamma\hat{p}_\delta\rangle + \langle\hat{x}_\gamma\hat{p}_\delta\hat{x}_\alpha\hat{x}_\beta\rangle + \langle\hat{x}_\gamma\hat{p}_\delta\hat{p}_\alpha\hat{p}_\beta\rangle\right)\\ +&I_{\alpha\beta}I_{\gamma\delta}\langle\hat{x}_\alpha\hat{p}_\beta\hat{x}_\gamma\hat{p}_\delta\rangle \geq 0\end{aligned} \tag{3.45}$$

The last term can be simplified to

$$\begin{aligned}I_{\alpha\beta}I_{\gamma\delta}\langle\hat{x}_\alpha\hat{p}_\beta\hat{x}_\gamma\hat{p}_\delta\rangle &= I_{\alpha\beta}I_{\gamma\delta}\langle(\hat{x}_\alpha\hat{p}_\delta)(\hat{p}_\beta\hat{x}_\gamma)\rangle\\ &= \frac{I_{\alpha\beta}I_{\gamma\delta}}{4}\langle([\hat{x}_\alpha\hat{p}_\delta] + \{\hat{x}_\alpha\hat{p}_\delta\})([\hat{p}_\beta\hat{x}_\gamma] + \{\hat{p}_\beta\hat{x}_\gamma\})\rangle\\ &= \frac{I_{\alpha\beta}I_{\gamma\delta}}{4}\langle(i\delta_{\alpha,\delta} + \{\hat{x}_\alpha\hat{p}_\delta\})(-i\delta_{\beta,\gamma} + \{\hat{p}_\beta\hat{x}_\gamma\})\rangle\\ &= \frac{I_{\alpha\beta}I_{\gamma\delta}}{4}\langle\delta_{\alpha,\delta}\delta_{\beta,\gamma} - i\{\hat{x}_\alpha\hat{p}_\delta\}\delta_{\beta,\gamma} + i\delta_{\alpha,\delta}\{\hat{p}_\beta\hat{x}_\gamma\} + \{\hat{x}_\alpha\hat{p}_\delta\}\{\hat{p}_\beta\hat{x}_\gamma\}\rangle\\ &= \mathrm{tr}\left(I^2\right) + \frac{I_{\alpha\beta}I_{\gamma\delta}}{4}\langle\{\hat{x}_\alpha\hat{p}_\delta\}\{\hat{x}_\gamma\hat{p}_\beta\}\rangle,\end{aligned} \tag{3.46}$$

In the last step, we used the antisymmetry of I to cancel the two imaginary terms with coefficient i. If we insert the imaginary part of the matrix $\hat{L}_y$ and develop the sum, we obtain back all the terms in equation 3.19.
Now, we focus on the terms with two matrices R. We are not able to simplify the terms with four $\hat{x}$'s or $\hat{p}$'s, but we can simplify the two other terms:

$$R_{\alpha\beta}R_{\gamma\delta}\left(\langle\hat{p}_\alpha\hat{p}_\beta\hat{x}_\gamma\hat{x}_\delta\rangle + \langle\hat{x}_\alpha\hat{x}_\beta\hat{p}_\gamma\hat{p}_\delta\rangle\right) = R_{\alpha\beta}R_{\gamma\delta}\frac{1}{2}\langle\{\hat{x}_\alpha,\hat{p}_\gamma\}\{\hat{x}_\beta,\hat{p}_\delta\}\rangle - \frac{3}{2}\mathrm{tr}\left(R^2\right). \tag{3.47}$$

The development is written in Appendix C.2. By inserting L_z into this equation, we obtain the terms in Equation (3.38). However, this expression does not directly yield the expression we obtained for $\hat{L}_x$, as there are still constants hidden in the couple of commutators at the end of the equation. Therefore, this result is not yet satisfactory, but we were unable to find a better simplification.
For terms with both R and I, we were not able to find simplifications either. Putting everything together, we have the general inequality

$$\begin{aligned}\frac{3}{2}\mathrm{tr}\left(R^2\right) - \mathrm{tr}\left(I^2\right) \leq & R_{\alpha\beta}R_{\gamma\delta}\left(\langle\hat{x}_\alpha\hat{x}_\beta\hat{x}_\gamma\hat{x}_\delta\rangle + \langle\hat{x}_\alpha\hat{x}_\beta\hat{p}_\gamma\hat{p}_\delta\rangle + \frac{1}{2}\langle\{\hat{x}_\alpha,\hat{p}_\gamma\}\{\hat{x}_\beta,\hat{p}_\delta\}\rangle\right) \\ & - R_{\alpha\beta}I_{\gamma\delta}\langle\{\hat{x}_\alpha\hat{x}_\beta + \hat{p}_\alpha\hat{p}_\beta, \hat{x}_\gamma\hat{p}_\delta\}\rangle \quad + \frac{I_{\alpha\beta}I_{\gamma\delta}}{4}\langle\{\hat{x}_\alpha\hat{p}_\delta\}\{\hat{x}_\gamma\hat{p}_\beta\}\rangle,\end{aligned} \tag{3.48}$$

with some terms being not simplified. Note that the trace of the square of an antisymmetric operator is always negative as the eigenvalues of I are all imaginary.

3.6 Outlook

In this chapter, in Section 3.4 we developed several equations for the two mode states, mainly using the positivity of the mean square value and variance. We did not present inequalities derived from the Schrödinger-Robertson as we did not succeed in obtaining satisfactory expressions. In future studies, one should continue to investigate these inequalities and try to apply them to more modes. We also left open the question of saturating states because we needed the tools developed in Chapter 4. These tools will also be useful for developing inequalities in the entropy of measurement outcomes.
We then investigated more general uncertainty relations, where we performed calculations for general matrices. However, we were not successful in simplifying all terms.

4. Relating number and spin eigenstates

Several questions raised in the last chapter require a connection between the number basis and the spin basis. Indeed, the states saturating the inequalities for the spin-like observables are easy to find in the eigenbasis of those operators, but not in the number basis. Similarly, computing the entropy of one of these spin operators requires knowledge of their eigenbasis. Further, as there are many well-known results on finite dimensional, and particularly spin systems, whose analogs for continuous variable systems have not yet been found, this research could provide new insights on how to transfer those properties and help understand the behavior of continuous variable systems. Beyond the entropic uncertainty relations, M. Arnhem, C. Griffet and N. Cerf raised the need for such relationships since 2022 [34–36]. This motivates the investigation of the relation between the spin basis and the number basis. We will develop the basis corresponding to the CSCO containing (among others) the operators $\hat{L}^2$ and $\hat{L}_z$, and the change of basis in relation to the CSCO containing the number operators of each mode. In Section 4.1 particular cases of one and two modes are considered, and a more general procedure is described for n-mode systems in Section 4.4. In this chapter the representation of $\mathfrak{su}(2)$ considered is always the one with the matrix L_z being diagonal, shortly this will be called as the diagonal representation of $\mathfrak{su}(2)$. A more general treatment for $\hat{L}_x, \hat{L}_y$, or other bases of the representation of $\mathfrak{su}(2)$ obtained by Jordan-Schwinger can be derived from Section 4.7, where the effects of unitaries and rotations in the spin space are discussed.

4.1 Low dimension cases

For particular cases of 1- and 2-mode systems, the relations between the number and spin basis are trivial or already described in the literature. In this section, we review these relations and provide several remarks to foreshadow their generalization.

4.1.1 1-mode systems

The particular case of 1-mode systems is trivial because $\hat{L}_i = \hat{L}^2 = \hat{0}$[32]. All states are eigenstates of these operators, and they all share the same eigenvalue (or quantum number) 0. The entire Hilbert space is an eigenspace for these operators. Neither $\hat{L}_i$ nor $\hat{L}^2$ helps distinguish different eigenvectors; they will not be useful in the construction of the basis. We arrive at the same conclusion, considering that observables $\hat{L}_i$ and $\hat{L}^2$ commute with all operators and do not add any information on a common basis. Consequently, adding or removing those operators from a set of observables does not affect the general commutation or its completeness. $\hat{L}_i$ and $\hat{L}^2$ are irrelevant to any CSCO.

4.1.2 2-mode systems

2-mode systems are also a particular case, as we show at the end of this subsection that $\{\hat{L}^2, \hat{L}_z\}$ forms a CSCO. In Section 2.3.7, we already showed that for any representation of $\mathfrak{su}(2)$, $\hat{L}^2$ and $\hat{L}_z$ commute, we only need to show that their respective quantum numbers are sufficient to fully

[32] In Section 3.3, we said that all the 1-mode operators generated with the Jordan-Schwinger map are proportional to the number operator. As the one dimensional matrix representation of spin is given by $\hat{L}_i = (0)$, the proportionality coefficient is 0.

describe a basis. To establish the common eigenbasis of $\{\hat{L}^2, \hat{L}_i\}$ we examine their eigenspaces and intersections, see *e.g.* [13].
Let us define $|v\rangle = \sum_{n_1,n_2} v_{n_1 n_2} |n_1, n_2\rangle$ as a common eigenvector of $\hat{L}^2$ and $\hat{L}_z$, associated with the eigenvalues λ and λ_z, respectively. First, we compute λ by computing the matrix elements of $\langle n_1, n_2|\hat{L}^2|v\rangle$ for any vector $|n_1, n_2\rangle$ of the number basis.

$$\begin{aligned}
\langle n_1, n_2|\hat{L}^2|v\rangle &= \langle v|\hat{L}^2|n_1 n_2\rangle^*, \\
\lambda \langle n_1, n_2|v\rangle &= \langle v|\frac{\hat{n}_1 + \hat{n}_2}{2}\left(\frac{\hat{n}_1 + \hat{n}_2}{2} + 1\right)|n_1, n_2\rangle^* \\
&= \frac{n_1 + n_2}{2}\left(\frac{n_1 + n_2}{2} + 1\right)\langle v|n_1, n_2\rangle^* \\
&= \frac{n_1 + n_2}{2}\left(\frac{n_1 + n_2}{2} + 1\right)\langle n_1, n_2|v\rangle \\
\Rightarrow \lambda &= \frac{n_1 + n_2}{2}\left(\frac{n_1 + n_2}{2} + 1\right), \text{ or } \langle n_1, n_2|v\rangle = 0, \quad \forall n_1, n_2 \in \mathbb{N}.
\end{aligned} \tag{4.1}$$

Defining $l = \frac{n_1+n_2}{2}$, we obtain the expression of the eigenvalues of L^2 analog to the common descriptions of spin systems:

$$\lambda = l \cdot (l + 1). \tag{4.2}$$

We use l as a quantum number associated with the observable $\hat{L}^2$. Equation 4.1 indicates that the subspace associated with a given eigenvalue $\lambda = l \cdot (l + 1)$, $\mathcal{E}^l_{\hat{L}^2}$, is a subspace with $n_1 + n_2$ fixed. In other words, it is an eigenspace of the total particle number operator of particles $\hat{N} = \sum_i \hat{n}_i$ (this illustrates the importance of $\hat{N}$ in the discussion). We can reformulate the condition for a vector $|v\rangle$ to belong to $\mathcal{E}^l_{\hat{L}^2}$ as

$$|v\rangle \in \mathcal{E}^l_{\hat{L}^2} \Leftrightarrow v_{n_1 n_2} \propto \delta^l_{(n_1+n_2)/2}. \tag{4.3}$$

At this point, we introduce a representation that will be helpful for understanding what happens in higher dimensions. In Figure 4.1, we present all vectors of the number basis in a lattice, where the coordinates of each point are the occupation numbers of each mode. In this representation, the vectors spanning $\mathcal{E}^l_{\hat{L}^2}$ are given by the isolines of the function $l = \frac{N}{2} = \frac{n_1+n_2}{2}$, as shown in the left plot in Figure 4.2. Physically, this implies that the spin of the system is equal to the mean number of particles.
Similar to the computation of the eigenvalues of $\hat{L^2}$, we now compute λ_z based on the matrix element $\langle n_1 n_2|\hat{L}_z|v\rangle$.

$$\begin{aligned}
\langle n_1, n_2|\hat{L}_z|v\rangle\,, &= \langle v|\hat{L}_z|n_1, n_2\rangle^* \\
\lambda_z \langle n_1, n_2|v\rangle &= \langle v|\frac{\hat{n}_1 - \hat{n}_2}{2}|n_1, n_2\rangle^* \\
&= \frac{n_1 - n_2}{2}\langle v|n_1, n_2\rangle^* \\
&= \frac{n_1 - n_2}{2}\langle n_1, n_2|v\rangle \\
\Rightarrow \lambda_z &= \frac{n_1 - n_2}{2}\,, \text{ or } \langle n_1, n_2|v\rangle = 0, \quad \forall n_1, n_2 \in \mathbb{N}.
\end{aligned} \tag{4.4}$$

Further, we will use indiscriminately λ_z and l_z, the first is the eigenvalue of $\hat{L}_z$, and the second is the quantum number associated, but in this particular case, they are equal. The eigenvalues of $\hat{L}_z$ are then $\frac{n_1-n_2}{2}$ and the eigenspaces are subspaces with $n_1 - n_2$ fixed. The condition to belong to a specific eigenspace $\mathcal{E}^{l_z}_{\hat{L}_z}$ becomes

$$v \in \mathcal{E}^{l_z}_{\hat{L}_z} \Leftrightarrow v_{n_1 n_2} \propto \delta^{l_z}_{(n_1-n_2)/2}. \tag{4.5}$$

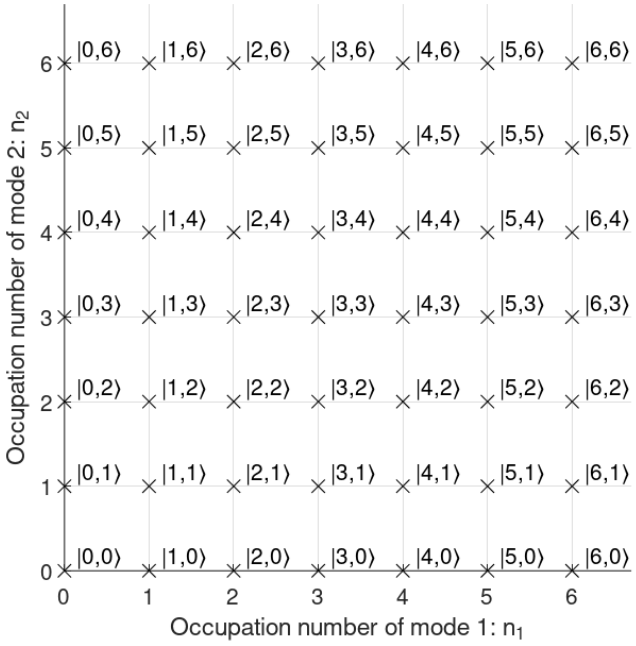

Figure 4.1: Lattice representation of the vectors in the number basis for 2-mode systems.

We again represent the eigenspaces in the lattice representation. $\mathcal{E}^{l_z}_{\hat{L}_z}$ is now represented by the isolines of the function $l_z = \frac{n_1 - n_2}{2}$, as shown in the right plot of Figure 4.2. The projection of the spin is given by half the difference between occupation numbers.
Finally, the intersections of the eigenspace of L^2 corresponding to a given value l and the eigenspace of L_z of eigenvalue l_z are given by

$$\begin{aligned} v \in \mathcal{E}^{l_z}_{\hat{L}_z} \cap \mathcal{E}^{l}_{\hat{L}^2} &\Leftrightarrow v_{n_1 n_2} \propto \delta^{l_z}_{(n_1-n_2)/2} \delta^{l}_{(n_1+n_2)/2} \\ &\Leftrightarrow v_{n_1 n_2} \propto \delta^{l+l_z}_{n_1} \delta^{l-l_z}_{n_2}. \end{aligned} \tag{4.6}$$

We can use this to express $|v\rangle$ with the two quantum numbers l and l_z:

$$\begin{aligned} |v\rangle &= \sum_{n_1,n_2} v_{n_1 n_2} |n_1 n_2\rangle \\ &\propto \sum_{n_1,n_2} \delta^{l+l_z}_{n_1} \delta^{l-l_z}_{n_2} |n_1 n_2\rangle \\ &\propto |l + l_z, l - l_z\rangle . \end{aligned} \tag{4.7}$$

This relation allows us to fully describe a vector $|v\rangle$ using the knowledge of l and l_z. All vectors of the common basis of $\hat{L}^2$ and $\hat{L}_z$ are fully determined by knowledge of these quantum numbers. Similarly, given that $|l_z| \leq l$, any pair of l and l_z describes only one vector of the Hilbert space. This demonstrates the completeness of the set of commuting observables, $\{\hat{L}_z, \hat{L}^2\}$. We can also establish an explicit relation between what we call the spin basis (denoted by the index s) and the number basis (which will, from now on, be noted by the index n):

$$|l, l_z\rangle_s = |l + l_z, l - l_z\rangle_n \quad \Longleftrightarrow \quad |n_1, n_2\rangle_n = |\frac{n_1 + n_2}{2}, \frac{n_1 - n_2}{2}\rangle_s . \tag{4.8}$$

This relation is illustrated in Figure 4.3, where two intersections are shown: the intersection of $\mathcal{E}^{3}_{\hat{L}^2}$ (red line) and $\mathcal{E}^{-1}_{\hat{L}_z}$ (blue line) corresponds to the state $|2, 4\rangle_n$ as expected. The intersections of $\mathcal{E}^{4}_{\hat{L}^2}$ and $\mathcal{E}^{3/2}_{\hat{L}_z}$ determine the state $|5.5, 2.5\rangle$ according to the rules given by Equation (4.8); however, this

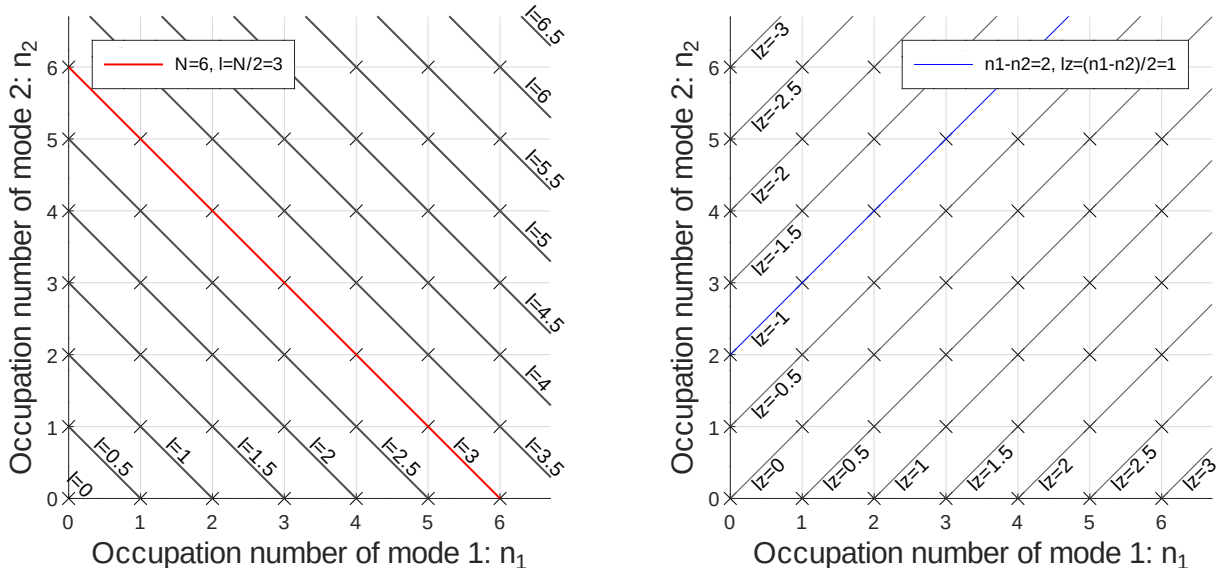

Figure 4.2: Lattice representation of the elements of the number basis spanning $\mathcal{E}^l_{\hat{L}^2}$ on the left, and $\mathcal{E}^{l_z}_{\hat{L}_z}$ on the right

is not a real state. Thus, we need to add the habitual selection rule stating that if the spin is an integer, then the projection is also an integer, and similarly, for half integers. This completes the construction of the spin basis for 2-mode systems. Before moving to the general case, we provide some remarks. The creation of an appropriate CSCO containing the operators $\hat{L}^2$ and $\hat{L}_z$ is related to the decomposition of the Hilbert space $\mathcal{H}$, which is viewed as a module of $\mathfrak{su}(2)$ in submodules and, more precisely, irreducible submodules in the particular case of the 2-mode system.
The eigenspace $\mathcal{E}^l_{\hat{L}^2}$ is indeed a submodule of $\mathfrak{su}(2)$. In this case, for any $|v\rangle \in \mathcal{E}^l_{\hat{L}^2}$ and for any operator $\hat{G}$ in the Jordan-Schwinger representation of $\mathfrak{su}(2)$, the application of $\hat{G}$ on $|v\rangle$ is intern in $\mathcal{E}^l_{\hat{L}^2}$. To demonstrate this, we use that $\hat{L}^2$ is the Casimir element of the representation and, therefore, commutes with all the elements of the algebra:

$$\hat{L}^2\hat{G}\,|v\rangle = \hat{G}\hat{L}^2\,|v\rangle = l(l+1)\hat{G}\,|v\rangle \Rightarrow \hat{G}\,|v\rangle \in \mathcal{E}^l_{\hat{L}^2}. \tag{4.9}$$

The application of $\hat{G}$ is intern to $\mathcal{E}^l_{\hat{L}^2}$, which proves that the eigenspace of $\hat{L}^2$ is a $\mathfrak{su}(2)$-submodule. As we only used the property that $\hat{L}^2$ commutes with each operator in the algebra, we can show in the same way that the eigenspaces of any operator commuting with all the elements of the algebra are submodules of the algebra.
Moreover, we can show that $\mathcal{E}^l_{\hat{L}^2}$ is an irreducible submodule. In Section 2.3.7, we have proven that for an irreducible submodule of dimension d, we get

$$\hat{L}^2 = \frac{d-1}{2}\frac{d+1}{2}\hat{\mathbb{1}}, \tag{4.10}$$

and the eigenvalues of $\hat{L}_z$ are the d values from $\frac{-d+1}{2}$ to $\frac{d-1}{2}$ in steps of one. In the last paragraph, we showed that $\mathcal{E}^l_{\hat{L}^2}$ is a submodule and we only have to prove its irreducibility. On the subspace $\mathcal{E}^l_{\hat{L}^2}$, the action of $\hat{L}^2$ is proportional to the identity

$$\hat{L}^2\,|v\rangle = l(l+1)\hat{\mathbb{1}}\,|v\rangle\,, \tag{4.11}$$

with $l = \frac{n_1+n_2}{2}$. By comparing Equations (4.10) and (4.11), we find that for all irreducible submodules included in $\mathcal{E}^l_{\hat{L}^2}$, their dimension is $d = 2l+1 = n_1+n_2+1$. However, the dimension

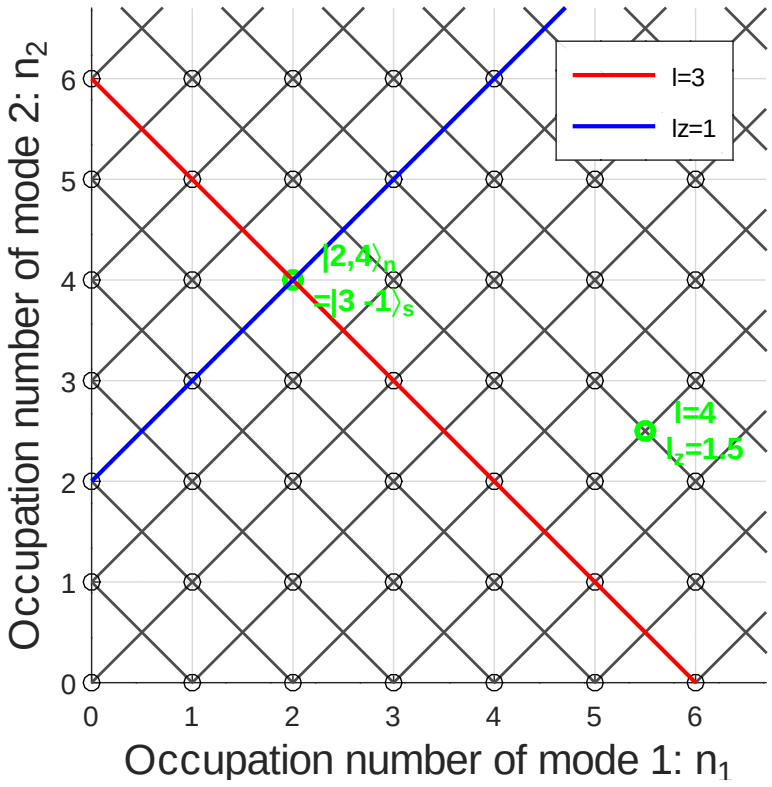

Figure 4.3: Intersections of the eigenspaces of $\mathcal{E}^{l}_{\hat{L}^2}$ and $\mathcal{E}^{l_z}_{\hat{L}_z}$

of $\mathcal{E}^{l}_{\hat{L}^2}$ itself is $d = 2l + 1 = n_1 + n_2 + 1$ (as it is spanned by $2l + 1$ vectors with l_z running from $-l$ to $+l$). Comparing these two dimensions, we see that the submodule $\mathcal{E}^{l}_{\hat{L}^2}$ can be decomposed into only one (nontrivial) submodule: itself. This is sufficient for proving that $\mathcal{E}^{l}_{\hat{L}^2}$ is irreducible.
In the following sections, to create a basis, we first decompose the Hilbert space $\mathcal{H}$ into irreducible submodules of the algebra and use the quantum number associated with $\hat{L}_z$ to determine the basis for those submodules. This procedure has several advantages. First, the operators discriminating the submodules will not only commute with $\hat{L}_z$ but also with all $\hat{G} \in \mathfrak{g}$. This will simplify the expressions of the actions of these operators by concentrating on their action on the submodules. For instance, this construction will also facilitate the change of basis into a basis related to $\hat{L}_x$ or $\hat{L}_y$ as we can restrict ourselves to the base change inside the submodules only.
The last remark is the particular role of the operator $\hat{N}$. Indeed Equation (4.1) shows that the eigenspaces of $\hat{N}$ and $\hat{L}^2$ coincide, and that the eigenvalues are related to each other by the relation

$$\lambda = \frac{N}{2}\left(\frac{N}{2} + 1\right). \tag{4.12}$$

This is enough to conclude that

$$\hat{L^2} = \frac{\hat{N}}{2}\frac{\hat{N} + 2 \cdot \hat{\mathbb{1}}}{2}, \tag{4.13}$$

but this equation can also be explicitly demonstrated (cf. Appendix D.1). This result implies that $\hat{N}^2$ commutes with $\hat{L}^2$ and $\hat{L}_i$ $\forall i$ for 2-mode systems. Hence, the eigenspaces of $\hat{N}$ are submodules of the algebra. In the following section, we demonstrate amongst other that this is also the case for n-mode systems.

4.2 Prerequisite discussion on basis relations

Before describing the general algorithm for deducing the spin basis from the number basis, several prerequisites must be discussed.

4.2.1 Counting quantum numbers

Intuitively, we can assume that for an n-mode system, n quantum numbers are necessary to fully describe the state of the Hilbert space. Indeed, in the number basis, the n occupation numbers n_i giving the number of particles in mode i each describe one basis vector. Nevertheless, this statement is generally false, and the general algorithm uses at most four quantum numbers. To understand how this is possible, we present a small example with the 2-mode Hilbert space and the number basis. Each basis vector can be represented on a lattice as presented in Figure 4.4 on the left. We can number the states as shown on the right-hand side of the figure; in such a case,

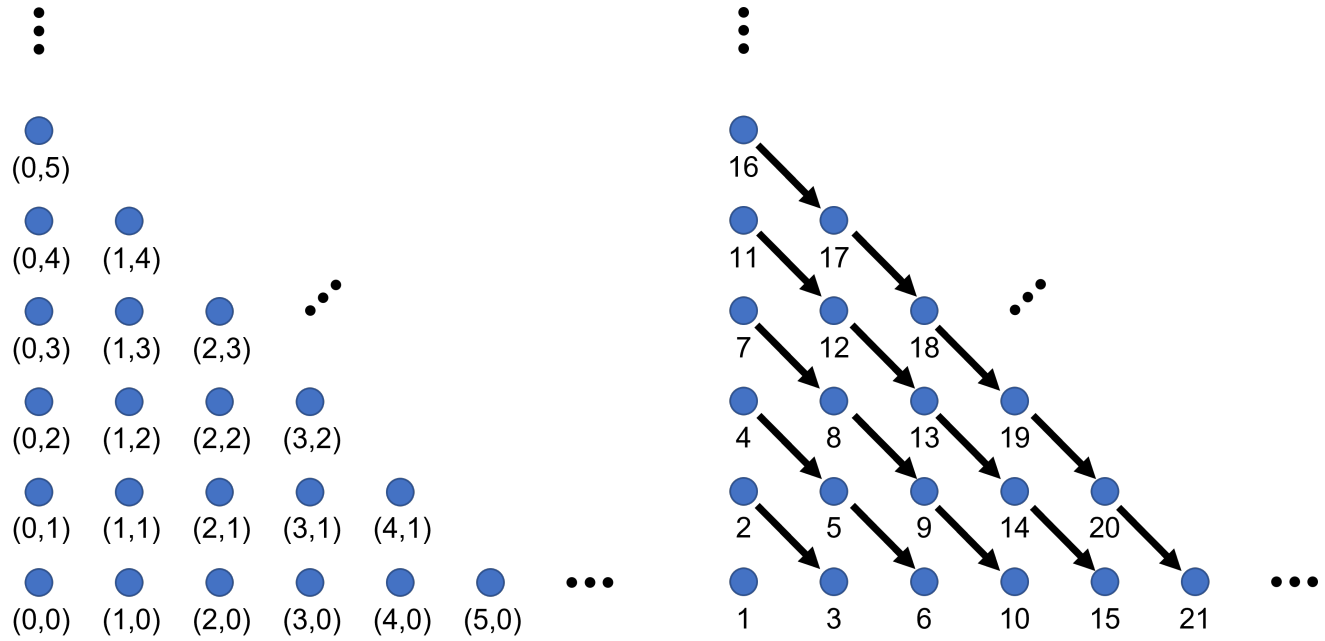

Figure 4.4: Schematic representation of all basis state of a 2-mode Hilbert space on a lattice using the quantum number associated with the number basis (left) and with a simple consecutive numbering (right).

each state is perfectly defined by the new number associated with it and hence, this number can be considered a quantum number describing the state. The map used is explicitly given by

$$\phi : \mathbb{N}^2 \to \mathbb{N} : (n_1, n_2) \to i = \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} + n_1 + 1. \tag{4.14}$$

Now, we rename the state $|n_1, n_2\rangle$ as $|i\rangle$ where i is the number associated with couple $|n_1, n_2\rangle$ by the map presented in Figure 4.4. Finally, we define the operator $\hat{O}$

$$\hat{O} = \sum_i i \, |i\rangle \langle i| \, , \tag{4.15}$$

such that the set $\{\hat{O}\}$ is a CSCO on its own, equivalent to $\{\hat{n}_1, \hat{n}_2\}$ because of the one-to-one connection between quantum numbers i and couples (n_1, n_2). Note that this is possible because

the eigenvalues of $\hat{n}_1$ and $\hat{n}_2$ are infinitely degenerated. By contrast, for $\{\hat{O}\}$ no eigenvalues are degenerated. This is because the cardinals of $\mathbb{N}$ and $\mathbb{N}^2$ are equal and are denoted by $\aleph_0$, and the renumbering map in this section is only a bijection between $\mathbb{N}$ and $\mathbb{N}^2$ which was proposed by Cantor in his first work on cardinality[37]. He also proved the existence of such bijections for complete sets, using the decimal or the binary notations of the numbers, or space-filling curves [38]. These bijections are, however, not bicontinuous. A further look at set theory considerations can be enriching. For instance, we can show that not all continuous-variable states[33] could be represented by a countable basis, as the cardinal of the set of such functions is $\aleph_2 = 2^{\aleph_1} = 2^{2^{\aleph_0}} \neq \aleph_0$. Only adding the condition that the state is normalizable will not work either, indeed the cardinality of the set of all these spaces (it corresponds to the set of the square-summable functions) is the cardinality of the continuum $\aleph_1 = 2^{\aleph_0}$ and not $\aleph_0$ the cardinality of countable sets. Finally, adding the restriction that the state should be continuous (the wave function is a square-summable continuous function) provides the right cardinality. This is in line with the classical considerations of quantum mechanics, which impose the continuity of the wave function.

4.2.2 Eigenspaces of $\hat{N}$

As mentioned in Section 4.1.2, the eigenspaces of any observable $\hat{O}$ that commutes with all elements of the algebra are submodules of the algebra. Indeed, let $|v\rangle \in \mathcal{E}^o_{\hat{O}}$, then $\forall \hat{G} \in \phi(\mathfrak{su}(2))$

$$\hat{O}\hat{G}\,|v\rangle = \hat{G}\hat{O}\,|v\rangle = o\hat{G}\,|v\rangle\,. \tag{4.16}$$

Because the application of $\hat{G}$ is intern to $\mathcal{E}^o_{\hat{O}}$ for all $\hat{G}$, $\mathcal{E}^o_{\hat{O}}$ is a submodule. Now, we can demonstrate that $\hat{N}$ commutes with all the operators generated through the Jordan-Schwinger map. The proof is provided in Appendix D.2, but we present here a more elegant way to demonstrate this. $\hat{N}$ is the application of the Jordan-Schwinger map to the identity, namely:

$$\phi(\mathbb{1}) = \sum_{\alpha\beta} \hat{a}^\dagger_\alpha \delta_{\alpha\beta} \hat{a}_\beta = \sum_\alpha \hat{a}^\dagger_\alpha \hat{a}_\alpha = \hat{N}.$$

As shown in Appendix C.1 the Jordan-Schwinger map preserves commutation relations. However, the identity commutes with any matrix. Therefore, the total number of particles operator commutes with all operators obtained using the Jordan-Schwinger map. This implies that the eigenspaces of the total number operator are submodules of the algebra.

4.2.3 Sufficiency of the CSCO $\{\hat{L}_z, \hat{N}\}$

One could expect that there exists a representation obtained using the Jordan-Schwinger map that behaves similarly for n-mode states as for the 2-mode states. We look for a specific matrix representation such that the subspace $W_N = \{|\psi\rangle : \hat{N}\,|\psi\rangle = N\,|\psi\rangle\}$ is an irreducible submodule of the algebra. If this is the case, then the value of l is fixed by the value of N (as Schur's lemma indicates that on an irreducible submodule, $\hat{L}^2$ acts proportionally to the identity, we then have $\hat{L} = f(\hat{N})$), and each state in the submodules can be directly distinguished by their eigenvalue $\hat{L}_z$. In Section 2.3.7, we proved that the action of $\hat{L}^2$ on an irreducible submodule of dimension d is proportional to the identity and that the coefficient of proportionality is $\frac{d-1}{2}\frac{d+1}{2}$. The dimension of W_N is actually a combinatorial question: how many distributions of N balls in n bins do exist? It is a problem similar to the number of combinations with repetition, for which the formula is

$$d = \left(\begin{array}{c} n+N-1 \\ N \end{array} \right) = \frac{(n+N-1)!}{N!(n-1)!} = O(N^{n-1}). \tag{4.17}$$

[33]The state is not continuous, it is expressed in terms of continuous variables, but the wave function does not need to be continuous

This means that if W_N is irreducible, then $\hat{L^2}$ should be given by

$$\hat{L}^2 = \sum_N \frac{1}{4}\left(\frac{(n+N-1)!}{N!(n-1)!}+1\right)\left(\frac{(n+N-1)!}{N!(n-1)!}-1\right)\hat{P}_N, \tag{4.18}$$

where $\hat{P}_N$ denotes the projector onto W_N. When replacing n by 2 for the 2-mode system, it becomes

$$\hat{L}^2 = \sum_N \frac{1}{4}(N+2)(N)\,\hat{P}_N = \frac{\hat{N}}{2}\left(\frac{\hat{N}}{2}+1\right). \tag{4.19}$$

We obtained the same result by explicitly developing it. More generally, Equation (4.18) gives the order of $\hat{L}^2$ with respect to $\hat{N}$, explicitly, we have[34]

$$\hat{L}^2 = O(\hat{N}^{2(n-1)}), \tag{4.20}$$

If we consider the expression of $\hat{L}^2$ in the Jordan-Schwinger map

$$\hat{L}^2 = \sum_{\alpha,\beta,\gamma,\delta}\left(\sum_i (L_i)_{\alpha\beta}(L_i)_{\gamma\delta}\right)\hat{a}^\dagger_\alpha\hat{a}_\beta\hat{a}^\dagger_\gamma\hat{a}_\delta, \tag{4.21}$$

only two pairs of $\hat{a}^\dagger\hat{a}$ appear, which creates operators only in $O(\hat{N}^2)$. This shows that, in the Jordan-Schwinger map, W_N can be an irreducible module only if $n \leq 2$. This being said, two options exist, decomposing W_N in submodules (see Section 4.4) or finding a generalization of the Jordan-Schwinger map (discussed in Appendix D.3).

4.3 Dimension of the eigenspaces of $\hat{L}_z$

Before establishing a connection between the number basis and the spin basis for the general case, we need to examine the eigenstructure of the operator $\hat{L}_z$.

4.3.1 Eigenvalues of $\hat{L}_z$

In this section, we consider the particular case where L_z is diagonal. Other representations can be deduced using unitaries, see Section 4.7. In this matrix representation[35], $(L_z)_{\alpha\beta}$ can be expressed as

$$(L_z)_{\alpha\beta} = \delta_{\alpha\beta}\left(\alpha - \frac{(n+1)}{2}\right) \quad \forall\alpha,\beta \in \{1,..,n\}. \tag{4.22}$$

Through the Jordan-Schwinger map this gives directly

$$\hat{L}_z = \sum_{i=1}^n (i - \frac{(n+1)}{2})\hat{n}_i = \sum_{i=1}^n i\hat{n}_i - \frac{(n+1)}{2}\sum_{i=1}^n \hat{n}_i = \sum_{i=1}^n i\hat{n}_i - \frac{(n+1)}{2}\hat{N}. \tag{4.23}$$

Applying this operator to any vector of the number basis shows that the number basis is an eigenbasis of $\hat{L}_z$

$$\hat{L}_z\,|k_1,...,k_n\rangle = \left(\sum_{i=1}^n i\cdot\hat{n}_i - \frac{(n+1)}{2}\hat{N}\right)|k_1,...,k_n\rangle\,,$$

[34] More precisely, one should say $\lambda = O(N^{2(n-1)})$.

[35] The convention used in this chapter places the eigenvalues of l_z in ascending order in order to facilitate the calculations and interpretation of the combinatorial results we will be using. This convention is the opposite of that used in the literature.

$$
\begin{aligned}
&= \sum_{i=1}^{n} i \cdot \hat{n}_i \left|k_1, ..., k_n\right\rangle - \frac{(n+1)}{2} \hat{N} \left|k_1, ..., k_n\right\rangle, \\
&= \sum_{i=1}^{n} i \cdot k_i \left|k_1, ..., k_n\right\rangle - \frac{(n+1)}{2} \sum_{i=1}^{n} k_i \left|k_1, ..., k_n\right\rangle, \\
&= \left(\sum_{i=1}^{n} i \cdot k_i - \frac{(n+1)}{2} \sum_{i=1}^{n} k_i \right) \left|k_1, ..., k_n\right\rangle. \qquad (4.24)
\end{aligned}
$$

If N is the total number of particules, then the eigenvalue associated with each vector of the number basis is

$$
l_z = \sum_{i}^{n} i \cdot k_i - \frac{(n+1)N}{2}. \qquad (4.25)
$$

This is the sum of the number of particles in each mode weighted by the number associated with the mode, up to an additive constant that depends only on the number of modes and the total number of particles.

4.3.2 Analogous combinatorials problems

The next question that arises in the study of this operator's eigenstructure is the degeneracy degree of each value. Let us define $\tilde{g}(n, N, l_z)$ as the degeneracy of the eigenvalue l_z of the operator $\hat{L}_z$ acting on W_N, which is the subspace of the Hilbert space $\mathcal{H}_n$ with a fixed total number of particles N. In other words, this is the dimension of the intersection of the eigenspace of $\hat{L}_z$ with eigenvalue l_z and W_N. The problem of determining $\tilde{g}(n, N, l_z)$ can be related to a combinatorial problem. First, let us rewrite Equation (4.25) as

$$
l_z = \sum_{i=1}^{n} \left(i - \frac{n+1}{2} \right) n_i. \qquad (4.26)
$$

This expression can be seen as N indistinguishable (in a quantum sense) balls distributed in n boxes labelled by a score $(i - \frac{n+1}{2})$, l_z is the total score computed by adding the number of balls n_i in each box, weighted by the score of the box. This is presented on the left of Figure 4.5. Each box contains a certain number of balls. In the figure on the right, this is particularized for the case of 5 modes (5 boxes) and 6 particles (6 balls). With this particular distribution of the balls, the scored obtained is

$$
(-2) \cdot 0 + (-1) \cdot 3 + 0 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 = 0. \qquad (4.27)
$$

Hence, it corresponds to the eigenvalue $l_z = 0$ for the vector associated to this distribution, which is equivalent to saying that

$$
\hat{L}_z \left|0, 3, 1, 1, 1\right\rangle_n = 0. \qquad (4.28)
$$

As $g(n, \tilde{N}, l_z)$ is the number of distributions of the N balls in the n boxes that gives a score l_z, we can write that

$$
\tilde{g}(n, N, l_z) = \# \left\{ \vec{k} \in \mathbb{N}^n \middle| \sum_{i=1}^{n} k_i = N, \sum_{i=1}^{n} \left(i - \frac{n+1}{2} \right) \cdot k_i = l_z \right\}, \qquad (4.29)
$$

where a constant factor $\left(-\frac{n+1}{2}\right)$ appears. For the remainder of this section, we can make this disappear by renumbering the boxes for simplicity: we define $U(n, N, u)$ as the number of distributions of N balls in n boxes numbered from 1 to n giving a score u, and $T(n, N, t)$ as the number of distributions of N balls in n boxes numbered from 0 to $n-1$ giving a score t.

$$
\left\{ \begin{array}{rcl}
U(n, N, u) & = & \#\{\vec{k} \in \mathbb{N}^n | \sum_{i=1}^{n} k_i = N, \sum_{i=1}^{n} i \cdot k_i = u\} \\
T(n, N, t) & = & \#\{\vec{k} \in \mathbb{N}^n | \sum_{i=1}^{n} k_i = N, \sum_{i=1}^{n} (i-1) \cdot k_i = t\}
\end{array} \right. \qquad (4.30)
$$

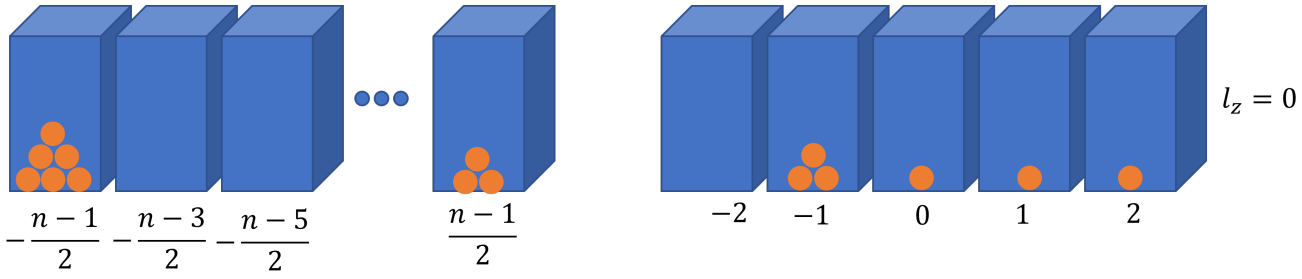

Figure 4.5: Illustration of the combinatorial problem equivalent to the computation of the degeneracy of l_z, in the general case (left) and for the particular case of 5 modes and 6 particles (right).

This is schematically presented for the general and a particular case in Figure 4.6.

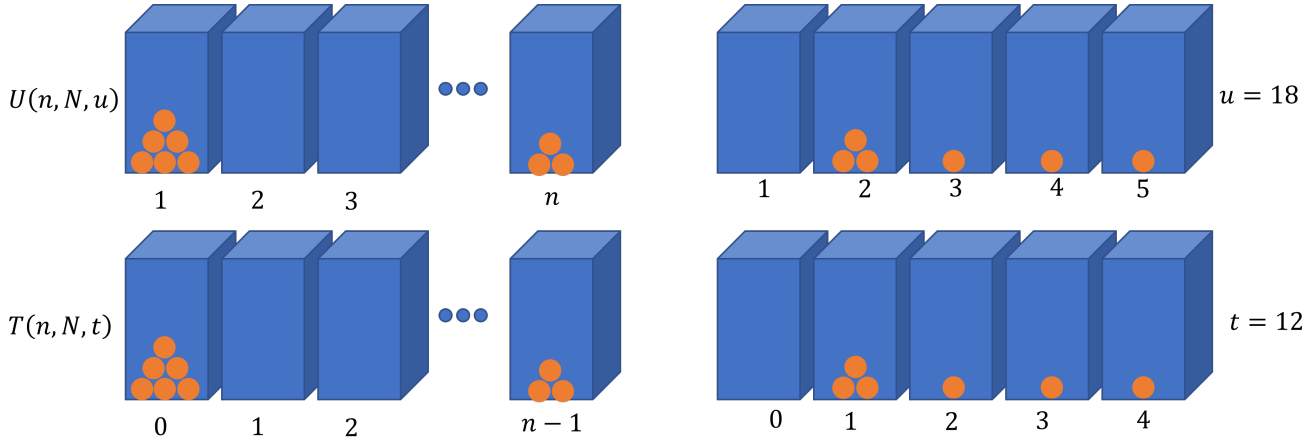

Figure 4.6: Schematic representation of $U(n,N,u)$ and $T(n,N,t)$ in a general case, and the particular case of 5 boxes and 6 balls.

As we only renumbered the boxes, it is easy to relate these three quantities by modifying the total score, which gives the following relations:

$$\begin{cases} \tilde{g}(n,N,l_z) &= U(n,N,l_z+\frac{(n+1)N}{2}) \\ T(n,N,t) &= U(n,N,t+N) \end{cases} \tag{4.31}$$

In Figure 4.7, we present all combinations of 6 balls in 5 boxes, giving a score of $u = 13$, $t = 7$ or $l_z = -5$, to exemplify Equations (4.31). It is important to emphasize that this is the only possible combination, and that exchanging two balls does not count for a new combination. In contrast to classical combinatorial problems, we find here the statistics of bosons that were considered for the Jordan-Schwinger map. To compute these quantities, we attempted to find several recurrence relations. For instance, we can express $U(n+1,N,u)$ using induction. Equation 4.31 can be written as

$$U(n+1,N,u) = T(n+1,N,u-N). \tag{4.32}$$

$T(n+1,N,u-N)$ is the number of ball distributions in the $(n+1)$ boxes numbered from 0 to n giving the total score $u-N$. These distributions can be separated into $N+1$ sets according to the number of balls in the first box (numbered 0). In the k^{th} set, there will only be distributions such that there are k balls in box 0 and $N-k$ balls in the others (then numbered from 1 to n), with k going from 0 to N. That is, the number of distributions of N balls in $n+1$ boxes is the number

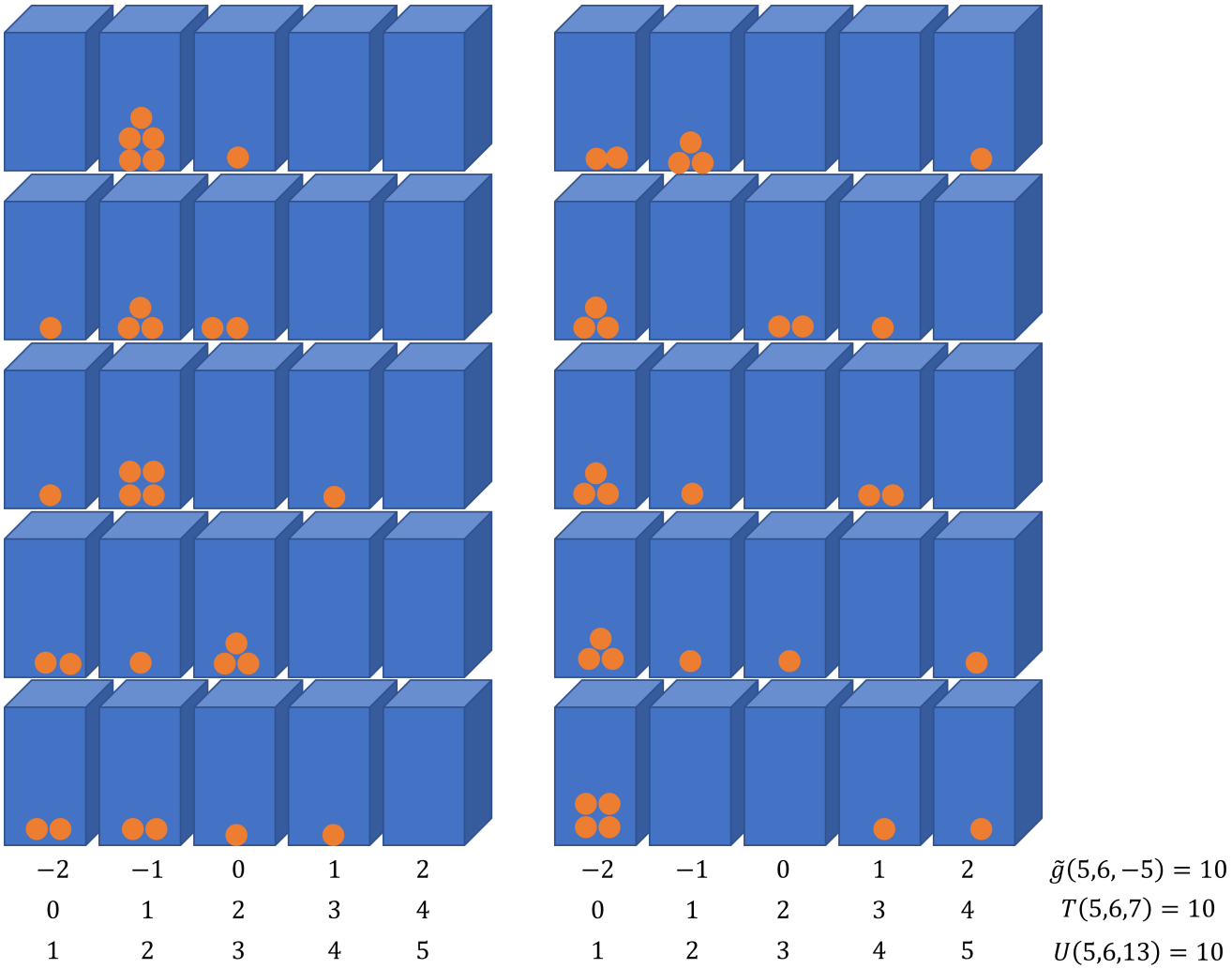

Figure 4.7: All combinations of 6 balls in 5 boxes giving a score of $u = 13$, $t = 7$ or $l_z = -5$.

of distributions with all the balls in the n last boxes (and the right score), plus the number of distributions with 1 balls in the first box and $N-1$ in the others, plus the number of distributions with 2 balls in the first box, *etc.* until all the balls are in the first box (always with the constraint to have the right score). The cardinal of the set of distributions giving the score $u - N$ is the sum of the cardinal of each of those subsets (giving the score $u - N$ and with k balls in the first box)

$$T(n+1, N, u-N) = \sum_{k=0}^{N} \#\{\vec{k} \in \mathbb{N}^{n+1} | \sum_{i=1}^{n+1} k_i = N, \sum_{i=1}^{n+1} (i-1) \cdot k_i = u - N, \quad k_1 = k\}. \tag{4.33}$$

The number of balls in the first box (numbered by 0) does not impact the score in this configuration; therefore, we can forget this box and only consider the $N - k$ balls in the other boxes. The total score should stay equal to $u - N$

$$T(n+1, N, u-N) = \sum_{k=0}^{N} \#\{\vec{k} \in \mathbb{N}^{n} | \sum_{i=2}^{n+1} k_i = N - k, \sum_{i=2}^{n+1} (i-1) \cdot k_i = u - N\}. \tag{4.34}$$

To be more precise, there is only one way to place k balls in box 0, owing to the indistinguishability of the balls; if it was not the case we should multiply each terms in the summation by the number of possibilities to place k balls in this one box. Now, we count the number of possibilities of

placing $N-k$ balls in n boxes numbered from 1 to n realizing a score $u-N$. This quantity is $U(n, N-k, u-N)$. Indeed, it can be seen by making a change of variables in the last equation

$$\begin{aligned} T(n+1, N, u-N) &= \sum_{k=0}^{N} \#\{\vec{k} \in \mathbb{N}^n | \sum_{i=1}^{n} k_i = N-k, \sum_{i=1}^{n} i \cdot k_i = u-N\} \\ &= \sum_{k=0}^{N} U(n, N-k, u-N) \end{aligned} \tag{4.35}$$

Finally, we obtain the recurrence relation

$$U(n+1, N, u) = \sum_{k=0}^{N} U(n, N-k, u-N), \tag{4.36}$$

with the initial condition

$$U(1, N, u) = \delta_{N,u}. \tag{4.37}$$

This condition implies that if there is one box (with score 1) and N balls, then the total score is necessarily $u = N$. We can examine how this relationship works by computing the first steps:

$$\begin{aligned} U(2, N, u) &= \sum_{k=0}^{N} U(1, N-k, u-N) = \sum_{k=0}^{N} \delta_{N-k,u-N} = \sum_{k=0}^{N} \delta_{k,2N-u} \\ &= \begin{cases} 1 & \text{if } u \in [N, 2N], \\ 0 & \text{else.} \end{cases} \end{aligned} \tag{4.38}$$

Note the small notation abuse, as this function is also 0 if u is not an integer. Finally, for three boxes, we obtain

$$\begin{aligned} U(3, N, u) &= \sum_{k=0}^{N} U(2, N-k, u-N), \\ &= \sum_{k=0}^{N} 1 \text{ if } u-N \in [N-k, 2(N-k)], \\ &= \#\left(\mathbb{N} \cap [0, N] \cap [2N-u, \frac{3N-u}{2}]\right), \\ &= \begin{cases} \left\lfloor \frac{2+u-N}{2} \right\rfloor & \text{if } u \in [N, 2N], \\ \left\lfloor \frac{2+3N-u}{2} \right\rfloor & \text{if } u \in [2N, 3N], \\ 0 & \text{else,} \end{cases} \end{aligned} \tag{4.39}$$

where $\lfloor \cdot \rfloor$ is the floor function. In Appendix D.4, the recurrence relations are implemented in a simple code to compute an array containing all the values of $U(n, N, u)$.
It should be noted that there exist other problems with a combinatorial logic similar to the one we discussed. For instance, $U(n, N, u)$ is the number of combinations of N indistinguishable dices with n faces yielding the total score u. This is different from the classical dice counting problem, as the dice is now indistinguishable in a quantum manner.

4.3.3 Links with the q-binomial coefficients

The recurrence relation we obtained in the previous section is functional but not efficient, as it relies on a sum of N terms at each step (for a depth of one, there are N terms in the summation, $O(N^2)$ in the second step, etc.), which gives a final complexity of the order of $O(N^n)$. Therefore,

we will develop another recurrence relation, which will be easier to compute, and that we will be able to link with already known coefficients, namely, the coefficients of the q-binomial polynomials. The distributions counted by $U(n, N, u)$ can be separated into two subsets: those with at least one ball in the first box, and those with no balls in the first box. The first category was counted by $U(n, N-1, u-1)$. Indeed, if we consider that there is always one ball in the first box, we cannot consider this ball anymore (there are $N-1$ balls left), and we have to subtract the points it gave to the total score $(u-1)$. The second category is represented by $U(n-1, N, u-N)$. Indeed, there are no balls in the first box; therefore, we should only consider $(n-1)$ boxes. There are still N balls, but the score on the boxes are now $2, ..., n$, renumbering the boxes into $1, ..., n-1$ which gives back the expression of U when changing the final score to $u-N$ (the N balls have lost a point). This leads to the following relation

$$U(n, N, u) = U(n, N-1, u-1) + U(n-1, N, u-N). \tag{4.40}$$

The initial condition needed is the following

$$U(n, 0, \alpha) = U(1, N, \alpha + N) = \delta_{\alpha 0}. \tag{4.41}$$

If there are 0 balls, the score should be 0; if there is only 1 box, then the score is the number of balls. Appendix D.4 implements these relations to directly compute the value of U and adapts it to find all the distributions of the balls. The idea is this time to find the $U(n, N-1, u-1)$ combinations with at least one ball in the first box, by considering $U(n, N-1, u-1)$ for $(N-1)$ balls with a score $(u-1)$, and then adding one ball in the first box; similarly, for $U(n-1, N, u-N)$ combinations with no balls in the first box, by taking $U(n-1, N, u-N)$ for N balls in $(n-1)$ boxes, with a score $(u-N)$, and then adding one empty box in front.
Let us now use these relations to show that there is a link with the q-binomial (sometimes called Gaussian binomial) coefficients that are already known and tabulated. All of their properties and some tables are provided in [39].
For reminder, the binomial coefficients are defined through the following recurrence relation

$$\begin{pmatrix} n \\ j \end{pmatrix} = \begin{pmatrix} n-1 \\ j-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ j \end{pmatrix} \quad \forall 1 \leq j \leq n-1, \tag{4.42}$$

with the explicit form being

$$\begin{pmatrix} n \\ j \end{pmatrix} = \frac{n!}{j!(n-j)!}. \tag{4.43}$$

The Gaussian binomial coefficients are polynomials in q defined in a similar way by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \quad \forall 1 \leq k \leq n-1, \tag{4.44}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \quad \forall 1 \leq k \leq n-1. \tag{4.45}$$

Note that these relations are equivalent and that one is sufficient to determine the value of the bracket. The initial values are given by

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1. \tag{4.46}$$

An explicit form of the q-polynomial is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\alpha=0}^{k(n-k)} C(n, k, \alpha) q^\alpha = \frac{[n]_q!}{[k]_q![n-k]_q!}, \tag{4.47}$$

where $C_{n,k,\alpha}$ are the coefficients of the polynomial and $[n]_q!$ is defined as:

$$[n]_q! = \prod_{i=1}^{n} [i]_q, \quad \text{with} \quad [i]_q = \sum_{j=0}^{i-1} q^j. \tag{4.48}$$

In the following, we will show that

$$\begin{cases} U(n,N,u) &= C(n+N-1,N,u-N), \\ C(n,k,\alpha) &= U(n-k+1,k,\alpha+k). \end{cases} \tag{4.49}$$

For this purpose, let us define the polynomial

$$P_{n,k}(q) = \sum_{\alpha=0}^{k(n-k)} U(n-k+1,k,\alpha+k)q^\alpha. \tag{4.50}$$

Then, we can directly match the initial condition for the q-binomial (Equation (4.46)) with the initial condition for the function U (Equation (4.41)):

$$P_{n,0}(q) = \sum_{\alpha=0}^{0} U(n+1,0,\alpha+0)q^\alpha = U(n+1,0,0) = 1 = \begin{bmatrix} n \\ 0 \end{bmatrix}_q, \tag{4.51}$$

$$P_{n,n}(q) = \sum_{\alpha=0}^{0} U(1,n,\alpha+n)q^\alpha = U(1,n,n) = 1 = \begin{bmatrix} n \\ n \end{bmatrix}_q. \tag{4.52}$$

Similarly, Equation (4.44) can be matched with Equation (4.40) as follows:

$$\begin{aligned} P_{n-1,k-1}(q) + q^k P_{n-1,k}(q) &= \sum_{\alpha=0}^{(k-1)(n-k)} U(n-k+1,k-1,\alpha+k-1)q^\alpha \\ &\quad + q^k \sum_{\alpha=0}^{k(n-k-1)} U(n-k,k,\alpha+k)q^\alpha, \\ &= \sum_{\alpha=0}^{(k-1)(n-k)} U(n-k+1,k-1,\alpha+k-1)q^\alpha \\ &\quad + \sum_{\alpha=0}^{k(n-k-1)} U(n-k,k,\alpha+k)q^{\alpha+k}, \\ &= \sum_{\alpha=0}^{(k-1)(n-k)} U(n-k+1,k-1,\alpha+k-1)q^\alpha \\ &\quad + \sum_{\alpha=k}^{k(n-k)} U(n-k,k,\alpha)q^\alpha. \end{aligned}$$

If $\alpha < k$, then $U(n-k,k,\alpha) = 0$ because there is no distribution in which the score is lower than the number of balls, as each ball brings at least one point. Similarly, if $\alpha > (k-1)(n-k)$, $U(n-k+1,k-1,\alpha+k-1) = 0$ because there could not be any combination with a score higher than the accessible score (each ball gives at most as many points as the number of boxes). We can then extend the limits of the summation by adding these null terms.

$$P_{n-1,k-1}(q) + q^k P_{n-1,k}(q) = \sum_{\alpha=0}^{k(n-k)} U(n-k+1,k-1,\alpha+k-1)q^\alpha$$

$$
\begin{aligned}
&+ \sum_{\alpha=0}^{k(n-k)} U(n-k,k,\alpha)q^{\alpha}, \\
&= \sum_{\alpha=0}^{k(n-k)} \left(U(n-k+1,k-1,\alpha+k-1) + U(n-k,k,\alpha)\right) q^{\alpha}.
\end{aligned}
\tag{4.53}
$$

Then, using Equation (4.40), we find

$$
\begin{aligned}
P_{n-1,k-1}(q) + q^k P_{n-1,k}(q) &= \sum_{\alpha=0}^{k(n-k)} U(n-k+1,k,\alpha+k)q^{\alpha} \\
&= P_{n,k}(q).
\end{aligned}
\tag{4.54}
$$

This is the recurrence relation of the q-binomial. As the initial condition is the same and the recurrence relation is also the same, the two polynomials should be equal:

$$
P_{n,k}(q) = \left[\begin{array}{c} n \\ k \end{array} \right]_q . \tag{4.55}
$$

The other recurrence relation for the q-binomial, given by Equation (4.45), can be verified similarly, considering this time either the distribution with at least one ball in the last box or no ball in the last box:

$$
U(n,N,u) = U(n,N-1,u-N) + U(n-1,N,u). \tag{4.56}
$$

4.3.4 Examples and other comments

Finally, to determine the values of $\tilde{g}(n,N,l_z)$ starting from the values of U, Equation (4.31) should be applied. This gives:

$$
\begin{aligned}
\tilde{g}(1,N,l_z) =& \delta_{l_z,0} \text{ and } l_z \in \mathbb{Z}, \\
\tilde{g}(2,N,l_z) =& \begin{cases} 1 & \text{if } l_z \in [-\frac{N}{2},\frac{N}{2}] \cap \mathbb{Z} + \frac{(N+1)\mathrm{mod}(2)}{2}, \\ 0 & \text{else} \end{cases} \\
\tilde{g}(3,N,l_z) =& \begin{cases} \left\lfloor \frac{2+N-|l_z|}{2} \right\rfloor & \text{if } l_z \in \mathbb{Z} \cap [-N,N], \\ 0 & \text{else.} \end{cases}
\end{aligned}
\tag{4.57}
$$

These functions are shown in Figure 4.8 for one to three modes and from left to right 5 particles. As shown in Figure 4.9, $\tilde{g}(n,N,l_z)$ becomes more complex for larger values of n and N. For large values of n and N, the shape of the function appears Gaussian, as shown in Figure 4.10. However, if we make a Gaussian fit conserving the variance, mean, and total sum of the function, we see two divergences. For values of l_z near 0 or $|l_z| \gg 0$ the Gaussian is always larger. In opposition for intermediate values the Gaussian fit is smaller.

One thing interesting to notice is that in some cases l_z has to be an integer and in other cases it should be an half integer. This question can be viewed from two perspectives. First, as $U(n,N,u)$ is non-zero only for integer values of u, one has that $l_z + \frac{(n+1)N}{2}$ must be an integer. Therefore, l_z is an integer, except if neither $(n+1)$ nor N is even. In other words, l_z is half-integer if n is even and N is odd; conversely l_z is integer if n is odd or N is even. Another perspective is that the spin of our N particle state is the sum of the individual spins of each particle. If the number of modes n is odd, then each particle carries an integer spin, and if it is even, then they carry a half-integer spin. The value of the individual spins is related to the eigenvalue of the matrix L_z, which is an

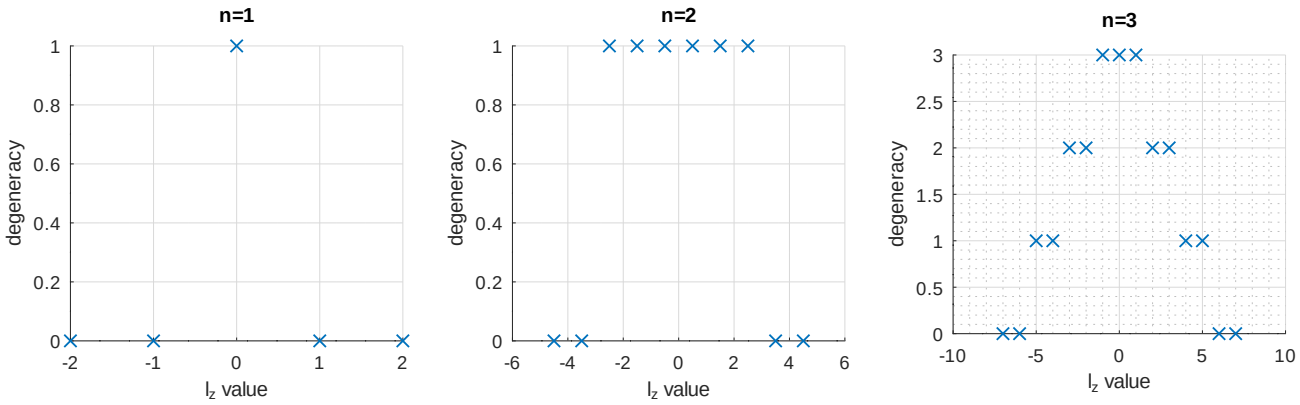

Figure 4.8: Degeneracies of the eigenvalues l_z for one to three modes and 5 particles.

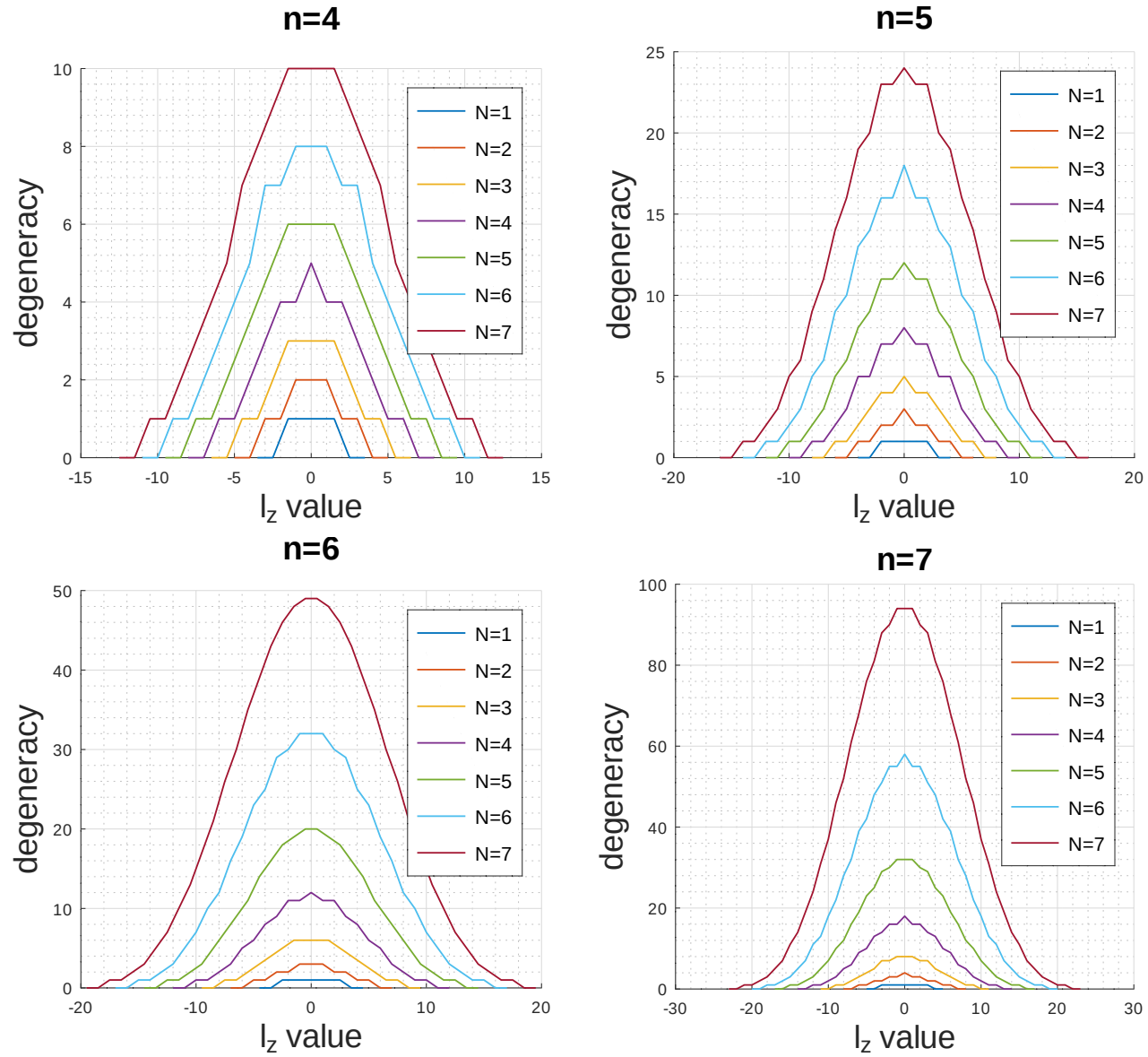

Figure 4.9: Degeneracies of the eigenvalues l_z for four to seven modes and one to seven particles.

integer for odd numbers of modes. The total spin can then only be a half integer if the number of

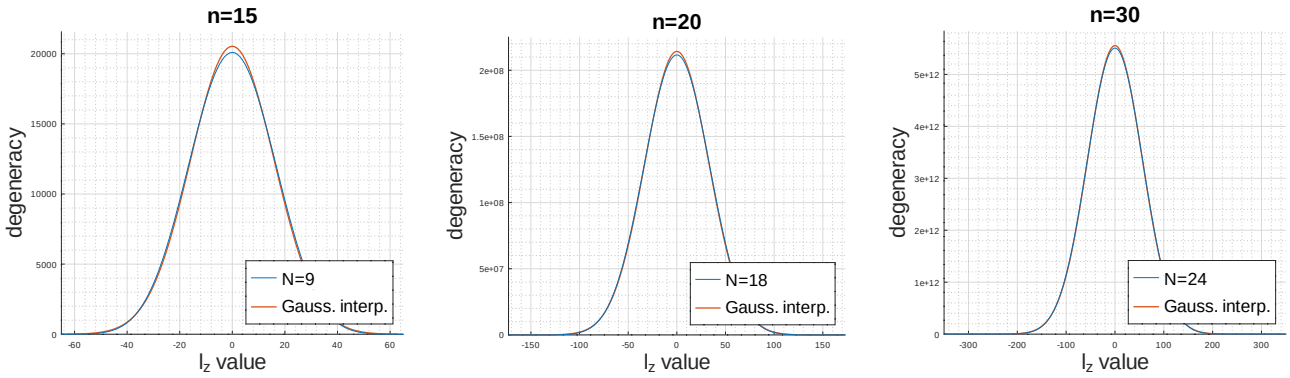

Figure 4.10: Degeneracies of the eigenvalues l_z for 15, 20 and 30 modes and 9, 18, 24 particles respectively, and Gaussian fits preserving the variance and the total sum, showing that the degeneracy distribution is of lower entropy.

	$N \in 2\mathbb{N}+1$	$N \in 2\mathbb{N}$
$n \in 2\mathbb{N}+1$	$l_z \in \mathbb{Z}$	$l_z \in \mathbb{Z}$
$n \in 2\mathbb{N}$	$l_z \in \frac{\mathbb{Z}+1}{2}$	$l_z \in \mathbb{Z}$

Table 4.1: Values of l_z as a function of the parity of n and N

modes is even and the number of particles is odd, as summarized in Table (4.1).
This discussion about the degeneracy of the eigenvalue l_z on W_n can be used to answer several questions:

- How many orthogonal irreducible submodules of $\mathfrak{su}(2)$ can be constructed in W_N for n modes? If the value of l_z is an integer, then all submodules will have one and only one vector giving $l_z = 0$; if there is more than one vector, then the module will not be irreducible. This directly implies that the number of orthogonal irreducible submodules is $\tilde{g}(n, N, 0)$.

- What is the dimension of the largest irreducible module? Because the irreducible modules are eigenspaces of $\hat{L}^2$ their dimensions are given by $(2l+1)$. The largest value of l_z in an irreducible submodule is equal to the value of l for such a submodule, and as the highest value of l_z reachable is $(N(n-1)/2)$, the dimension of the largest submodule is $(N(n-1)+1)$.

The link with the q-binomials is not insignificient, as the q-binomials already have applications in the study of the structure of vector spaces; in the future, it would be interesting to see if this relation could have been found only by geometrical considerations.

4.4 General algorithm

This algorithm constructs the basis of the Hilbert space $\mathcal{H}_n$, where all basis vectors are eigenvectors of $\hat{N}, \hat{L}^2, \hat{L}_z$, and the irreducible submodules are clearly separated, starting from the number basis. The quantum numbers used will be N, l, and l_z with an additional fourth number, called the counting number, used to discriminate all basis vectors that share the same three first quantum numbers. Therefore, each vector of the basis can be written as

$$|N, l, l_z, c\rangle_s \,. \tag{4.58}$$

The subscript s indicates that it is a vector of the new basis, the spin basis, and n indicates the number basis (where the quantum numbers used to describe the state are the occupation numbers

of each mode). In the following, we explicitly develop the first steps of the algorithm and then make a shorter version to present it explicitly. The following notations are used: W_N is the subspace with a fixed total number of particles N, $\mathcal{E}_N^{l_z}$ is the eigenspace of $\hat{L}_z$ of eigenvalue l_z in W_N, and B_{l_z} is the basis of $\mathcal{E}_N^{l_z}$ we construct.

Initial step: Let us consider only W_N, the subspace of $\mathcal{H}_n$ with states with total number of particles N. The maximal value of l_z reachable in this subspace is equal to $l_z = N(n-1)/2$ and as $\tilde{g}(n, N, \frac{N(n-1)}{2}) = 1$, there is only one vector $|\psi\rangle$ realizing it, such that

$$\hat{L}_z |\psi\rangle = \frac{N(n-1)}{2} |\psi\rangle . \tag{4.59}$$

As $\hat{N}$ commutes with all elements of $\mathfrak{su}(2)$, it commutes with $\hat{L}_+$, or equivalently, as W_N is a submodule of the algebra, each operator in the algebra is intern to it, and we have that $\hat{L}_+ |\psi\rangle \in W_N$. The eigenvalue associated with $\hat{L}_+ |\psi\rangle$ is $\left(\frac{N(n-1)}{2} + 1\right)$ and no vectors of W_N have such an eigenvalue (as $\frac{N(n-1)}{2}$ is the largest admissible eigenvalue). This result implies that $\hat{L}_+ |\psi\rangle = \vec{0}$. Therefore, $l = l_z = \frac{N(n-1)}{2}$, because for irreducible modules of $\mathfrak{su}(2)$, l is equal to the largest admissible value of l_z. (This is the classical result presented in Section 2.3.7). This vector is the first vector of the basis. It corresponds to the particular case in which all balls are in the last box, that is, $|0, ..., 0, N\rangle$.We can then make the following identification:

$$|\psi\rangle = |N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2}, 0\rangle_s = |0, ..., 0, N\rangle_n \tag{4.60}$$

We now have the first part of our basis

$$B_{\frac{N(n-1)}{2}} = \left\{ |N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2}, 0\rangle_s \right\}. \tag{4.61}$$

First iteration: We now consider all values of $l_z \geq 0$ from $l_z = N(n-1)/2 - 1$ and at each iteration, its value is decreased by one. Now, considering $l_z = N(n-1)/2 - 1$, once again $\tilde{g}(n, N, \frac{N(n-1)}{2} - 1) = 1$ (if there is more than one mode), this corresponds to the single distribution where all the balls are in the last box, except one that is in the penultimate box. This configuration can be matched with the vector $|0, 0, ...1, N-1\rangle_n$. On the other hand, we can also construct a vector with $l_z = N(n-1)/2 - 1$ by using lowering operators

$$|N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 1, 0\rangle_s \equiv \frac{\hat{L}_- |N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2}, 0\rangle_s}{\left\| \hat{L}_- |N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2}, 0\rangle_s \right\|}. \tag{4.62}$$

As the dimension of the subspace with $l_z = N(n-1)/2 - 1$ is one (the value of $\tilde{g}$), those vector are equal

$$|N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 1, 0\rangle_s = |0, 0, ...1, N-1\rangle_n . \tag{4.63}$$

The second part of the basis is

$$B_{\frac{N(n-1)}{2}-1} = \left\{ |N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 1, 0\rangle_s \right\}. \tag{4.64}$$

For the sake of completeness, one can check using the image of the matrix L_- by the Jordan-Schwinger map $\hat{L}_-$ that the complex phase of the vector is correct.

Second iteration: Subsequently, considering $l_z = N(n-1)/2 - 2$, and if $n \geq 3$, we have $\tilde{g}(n, N, N(n-1)/2 - 2) = 2$. This means that the eigenspace of $\hat{L}_z$ in W_N associated with this eigenvalue $\mathcal{E}_N^{l_z}$ is of dimension 2. There are two distributions of the balls corresponding to this value, and they are the ones with two balls in the penultimate box, and the one with one in the antepenultimate box, which gives two vectors generating that subspace: $|0, 0, ..., 0, 2, N-2\rangle_n$ and $|0, 0, ..., 1, 0, N-1\rangle_n$. In the subspace $\mathcal{E}_N^{l_z}$, there is already a known vector

$$|N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 2, 0\rangle_s \equiv \frac{\hat{L}_- |N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 1, 0\rangle_s}{\left\| \hat{L}_- |N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 1, 0\rangle_s \right\|} \tag{4.65}$$

Knowing this, we need only one other vector to form an orthogonal basis of $\mathcal{E}_N^{l_z}$. One can, for instance, take the Gram-Schmidt algorithm to deduce it, and we name it:

$$|N, \frac{N(n-1)}{2} - 2, \frac{N(n-1)}{2} - 2, 0\rangle_s . \tag{4.66}$$

Let us examine the values of the quantum numbers: the number of particles is directly N, l_z is the number chosen for this step, and $C = 0$ as it is the only one we created here. For l we chose the same value as l_z, the reason is that

$$\hat{L}_+ |N, \frac{N(n-1)}{2} - 2, \frac{N(n-1)}{2} - 2, 0\rangle_s = \vec{0}. \tag{4.67}$$

To prove this, we use the fact that $\hat{L}_+$ commutes with $\hat{N}$ and increases the quantum number l_z by one. Immediately, $\hat{L}_+ |N, \frac{N(n-1)}{2} - 2, \frac{N(n-1)}{2} - 2, 0\rangle_s$ is an element of $\mathcal{E}_N^{\frac{N(n-1)}{2}-1}$. However, this also belongs to $\mathcal{E}_N^{\frac{N(n-1)}{2}-1^{\perp}}$. Indeed the basis of $\mathcal{E}_N^{\frac{N(n-1)}{2}-1}$ only contains $|N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 1, 0\rangle$, and $\hat{L}_+ |N, \frac{N(n-1)}{2} - 2, \frac{N(n-1)}{2} - 2, 0\rangle_s$ is orthogonal to it:

$$\begin{aligned}
&{}_s\langle N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 1, 0| \hat{L}_+ |N, \frac{N(n-1)}{2} - 2, \frac{N(n-1)}{2} - 2, 0\rangle_s , \\
=&{}_s\langle N, \frac{N(n-1)}{2} - 2, \frac{N(n-1)}{2} - 2, 0| \hat{L}_- |N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 1, 0\rangle_s^* , \\
=&{}_s\langle N, \frac{N(n-1)}{2} - 2, \frac{N(n-1)}{2} - 2, 0|N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 2, 0\rangle_s^* , \\
=&{}_s\langle N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 2, 0|N, \frac{N(n-1)}{2} - 2, \frac{N(n-1)}{2} - 2, 0\rangle_s \\
=&0,
\end{aligned} \tag{4.68}$$

where the last equality is true owing to the construction of the basis. As the vector belongs to a vector space and is orthogonal, this vector can only be null. This proves that $\hat{L}_+ |N, \frac{N(n-1)}{2} - 2, \frac{N(n-1)}{2} - 2, 0\rangle_s = 0$. Therefore, the value of l is $\frac{N(n-1)}{2} - 2$. This part of the basis is then

$$B_{\frac{N(n-1)}{2}-2} = \left\{ |N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2} - 2, 0\rangle_s , |N, \frac{N(n-1)}{2} - 2, \frac{N(n-1)}{2} - 2, 0\rangle_s \right\}. \tag{4.69}$$

The general procedure is described in the following.

General method

1. Fix the number of mode n and the total number of particles N a basis should be constructed.

2. Construct the first part of the basis with Equation (4.60):

$$B_{\frac{N(n-1)}{2}} = \left\{ \left| N, \frac{N(n-1)}{2}, \frac{N(n-1)}{2}, 0 \right\rangle_s \right\}. \tag{4.70}$$

3. Loop over the values of l_z from $\frac{N(n-1)}{2}$ to 0 (or $\frac{1}{2}$ for half integer valued spins) by decreasing l_z by one at each step. In this loop, for each value of l_z:

   (a) Find all the vectors $\left|\vec{k}\right\rangle_n$ corresponding to the eigenvalue l_z. These span $\mathcal{E}_N^{l_z}$.

   (b) Construct the first part of the corresponding basis by applying the lowering operator on the basis of $(l_z + 1)$

$$B_{l_z}^0 = \left\{ \frac{\hat{L}_- |\psi\rangle}{\left\| \hat{L}_- |\psi\rangle \right\|}, \forall |\psi\rangle \in B_{l_z+1} \right\}. \tag{4.71}$$

   This set of vectors spans $\hat{L}_- \mathcal{E}_N^{l_z+1}$ and contains $\tilde{g}(n, N, l_z + 1)$ vectors.

   (c) Construct $g(n, N, l_z) = \tilde{g}(n, N, l_z) - \tilde{g}(n, N, l_z + 1)$ normalized vectors orthogonal to each other and to $B_{l_z}^0$ [36]. Those vectors form the set

$$B_{l_z}^1 = \{ |N, l_z, l_z, C\rangle, C \in \{0, 1, ..., g(n, N, l_z) - 1\} \}. \tag{4.72}$$

   They span $\mathcal{E}_N^{l_z} \setminus \hat{L}_- \mathcal{E}_N^{l_z+1}$. The existence of such orthogonal vectors is guaranteed because of the dimensions of $\mathcal{E}_N^{l_z}$. The value of l can be fixed to the same value as l_z for the vectors found (the demonstration is similar to that in Equation (4.68)).

   (d) The basis of $\mathcal{E}_N^{l_z}$ is then simply given by the two bases spanning its two orthogonal subspaces:

$$B_{l_z} = B_{l_z}^0 + B_{l_z}^1. \tag{4.73}$$

4. Loop over the value of l_z from 0 ($-\frac{1}{2}$ for half integer valued spin) to $-\frac{N(n-1)}{2}$, decreasing l_z by one at each step. In this loop, for each value of l_z, the lowering operator is applied on the basis of the previous l_z which do not vanish:

$$B_{l_z} = \left\{ \frac{\hat{L}_- |N, l, l_z + 1, C\rangle}{\left\| \hat{L}_- |N, l, l_z + 1, C\rangle \right\|}, \forall |N, l, l_z + 1, C\rangle \in B_{l_z+1}, \text{ such that } |l_z + 1| \neq l \right\}. \tag{4.74}$$

This algorithm is the main result of this thesis. It is implemented in Octave for positive values of l_z in Appendix D.6. In future research, it will help solve most of the issues we raised in Section 3.6. In the rest of this section we will make some remarks regarding this algorithm, then in Section 4.5 we give an example of the application of the algorithm. In Section 4.6 we will extends this algorithm to other operators than only the spin-like, and in Section 4.7 we show how to consider transformations applied to the matrices used for the Jordan-Schwinger map. In this last section, we will also consider the special case of rotations of the spins which will show the advantages of the decomposition in submodules.

[36] g is larger than 0 for positive values of l_z because of the properties of Gaussian binomial coefficients.

Speeding up the construction of the basis for negative l_z by appropriately choosing the arbitrary bases for positive l_z. In the following, $\vec{\mathsf{K}}$ is an array of n numbers such that $\mathsf{K}_i = k_{n-i}$. Then, in the context of second quantification, we define a mirror symmetry as,

$$\text{mir} : |v\rangle = \sum_{\vec{k}} c_{\vec{k}} |\vec{k}\rangle \to |\text{ʌ}\rangle = \sum_{\vec{k}} c_{\vec{k}} |\vec{\mathsf{K}}\rangle . \tag{4.75}$$

A vector $|v\rangle$ is symmetric in itself with regard to the mirror symmetry if

$$|v\rangle = |\text{ʌ}\rangle . \tag{4.76}$$

In the algorithm in step 3.c, we construct an orthonormal basis, and with an appropriate choice at this step, it may be possible to dramatically speed up step 4. First, by considering only the integer values of l_z, if $\forall |\psi_{l_z}\rangle \in B_{l_z}$, the corresponding vector in B_0, $\hat{L}_-^{l_z} |\psi_{l_z}\rangle$ is symmetric, then we can simply create the basis B_{-l_z} by using the symmetry of the vectors in B_{l_z}. For the half-integer spin values of l_z, we require that

$$\hat{L}_-^{\lceil l_z \rceil} |\psi_{l_z}\rangle = \text{mir}\hat{L}_-^{\lfloor l_z \rfloor} |\psi_{l_z}\rangle . \tag{4.77}$$

Apparition of the irreducible submodules This construction allows us to highlight the irreducible submodules. Indeed, the vector space

$$V(N,l,\vec{\gamma}) = \text{span}\left(\left\{ |\psi_{l_z}\rangle = \sum_{i=0}^{g(n,N,l)-1} \gamma_i |N,l,l_z,i\rangle_s , \forall l_z \text{ such that } |l_z| \leq l \right\}\right), \text{ with } \vec{\gamma} \in \mathbb{C}^{g(n,N,l)} \tag{4.78}$$

is an irreducible $\mathfrak{su}(2)$-module. Intuitively, it is an eigenspace of the Casimir operator $\hat{L}^2$ of eigenvalue $l(l+1)$ spanned from $2(l+1)$ basis vectors orthogonal to each other (they are eigenvectors of the observable $\hat{L_z}$ with different eigenvalues; they are therefore orthogonal) with the following properties:

$$\left\{ \begin{array}{rcl} L_+ |\psi_l\rangle & = & 0, \\ L_- |\psi_{-l}\rangle & = & 0, \\ L + |\psi_{l_z}\rangle & \propto & |\psi_{l_z+1}\rangle . \end{array} \right. \tag{4.79}$$

Not all irreducible submodules can be described in this form because one can imagine an irreducible submodule not contained in a space with a fixed total number of particles. For example, for 3-mode systems:

$$X = \text{Span}\left(\frac{|0,0,0\rangle_s + |2,0,0\rangle_s}{\sqrt{2}} = \frac{1}{\sqrt{2}} |0,0,0\rangle_n + \frac{1}{\sqrt{3}} |1,0,1\rangle_n - \frac{1}{\sqrt{6}} |0,2,0\rangle_n \right), \tag{4.80}$$

is an irreducible submodule (the irreducibility is direct, its dimension being one, we cannot separate it in a direct sum of more than one vector space of non-zero dimension) but cannot be represented by a space of the kind of V. To be complete, we have that

$$X \in V(0,0) \oplus V(3,0). \tag{4.81}$$

The construction of V allows us to easily construct orthogonal submodules. Choosing orthogonal vectors $\vec{\gamma}$ ensures orthogonality between the vector spaces. For $|v\rangle \in V(N,l,\vec{\gamma}_1), |w\rangle \in V(N,l,\vec{\gamma}_2)$ with $\vec{\gamma}_1$ and $\vec{\gamma}_2$ orthogonal we find

$$\langle v|w\rangle \propto \vec{\gamma}_1 \cdot \vec{\gamma}_2 = 0, \quad \text{if } \vec{\gamma}_1 \neq \vec{\gamma}_2.$$

The demonstration is written extensively in Appendix D.5. Finally, based on these considerations, we can construct a full decomposition of $\mathcal{H}_n$ in independent irreducible submodules by considering the following set of submodules:

$$\{V(N,l,\vec{\gamma}_c) | c \in \{0,1,...,g(n,N,l)\}\}, \tag{4.82}$$

where the collection $\{\vec{\gamma}_c | c \in \{0,1,...,g(n,N,l)\}\}$ is an arbitrary orthogonal basis of $\mathbb{C}^{g(n,N,l)}$.

4.5 Application to 3-mode states

We apply the general method described above to create a basis for the three-mode case. In Equation (4.57), it is proven that function $\tilde{g}$ is given by

$$\tilde{g}(3, N, l_z) = \begin{cases} \left\lfloor \frac{2+N-|l_z|}{2} \right\rfloor & \text{if } l_z \in \mathbb{Z} \cap [-N, N], \\ 0 & \text{otherwise}, \end{cases} \tag{4.83}$$

We also find this result graphically. For the 3-mode case, we can represent the vector of the

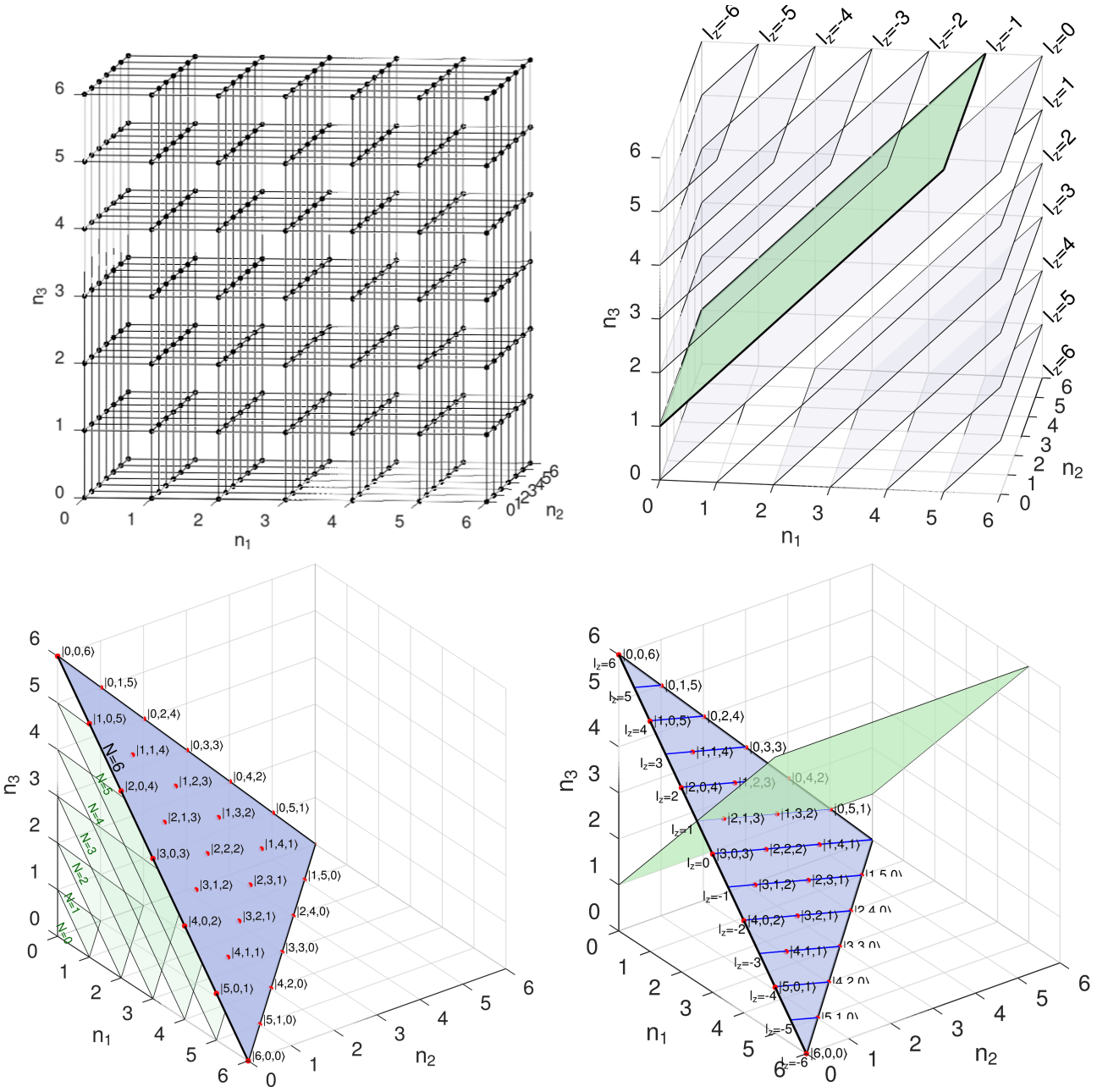

Figure 4.11: Lattice representation of the vectors of the number basis for 3-mode systems in top left figure. In the top right, the eigenspaces of $\hat{N}$ in this representation, and in the bottom left, the eigenspaces of $\hat{L}_z$; their intersection is shown in the bottom right figure.

number basis in a lattice, as shown in the top left of Figure 4.11. Then, the subspaces with a fixed total number of particles N are the surface of the equation $n_1 + n_2 + n_3 = N$, as shown in

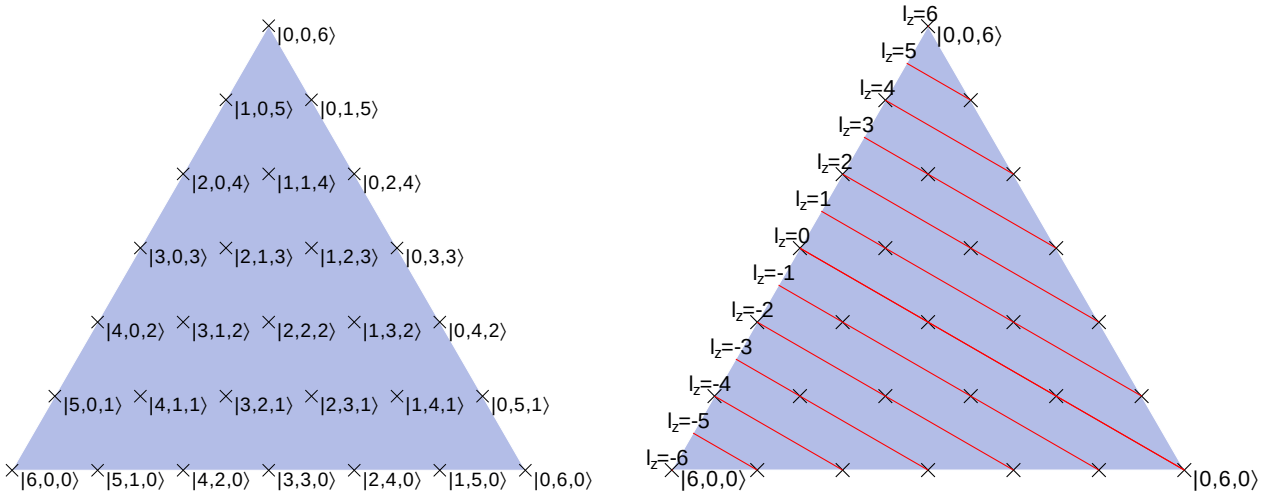

Figure 4.12: Lattice representation of the vectors with total number of particle 6 on the left, and on the right the representation of the eigenspaces of l_z on the same representation.

the bottom left panel of Figure 4.11. In the diagonal representation, the operator $\hat{L}_z$ is given by $\hat{n}_3 - \hat{n}_1$. Therefore, in this representation, the eigenspaces of $\hat{L}_z$ are the planes of the equation $l_z = n_3 - n_1$, which we shox in the top right of Figure 4.11. Then, the value of $g(3, N, l_z)$ is the number of points on the line of intersection between the two planes, see Figure 4.11.
More precisely, when we focus on the case of a fixed total number of particles (as shown on the right side of Figure 4.12), the intersections with the planes of fixed l_z become lines on which the number of states can be counted to obtain $\tilde{g}$. In this case, increasing the value of l_z by one will never increase $\tilde{g}$ by more than one. Therefore, a counting number is not required.
Let us now apply the general method; the initial vector is given by Equation (4.60), namely

$$|N, N, N\rangle_s = |0, 0, N\rangle_n \,. \tag{4.84}$$

Then, Equation (4.63) predicts that

$$|N, N, N-1\rangle_s = |0, 1, N-1\rangle_n \,, \tag{4.85}$$

This can be verified explicitly. In the matrix representation of $\mathfrak{su}(2)$ where L_z is diagonal, the spin matrices for the spin read

$$L_x = 2^{-1/2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = i2^{-1/2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.86}$$

Therefore, L_- is given by

$$L_- = L_x - iL_y = 2^{1/2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.87}$$

Now, using Jordan-Schwinger map, the expression for $\hat{L}_-$ is

$$\hat{L}_- = \sqrt{2} \left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_3 \right). \tag{4.88}$$

Finally ,

$$\hat{L}_- \, |0, 0, N\rangle_n = \sqrt{2} \left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_3 \right) |0, 0, N\rangle_n \,,$$

$$= \sqrt{2}\left(\vec{0} + \sqrt{N}\,|0,1,N-1\rangle_n\right),$$
$$\Rightarrow |N,N,N-1\rangle_s = |0,1,N-1\rangle_n . \tag{4.89}$$

Then, there are two vectors with $l_z = N-2$: $|0,2,N-2\rangle_n$ and $|1,0,N-1\rangle_n$. In the new basis, the first vector corresponding to this eigenvalue is

$$\begin{aligned}\hat{L}_- |N,N,N-1\rangle_s &= \sqrt{2}\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_3\right)|0,1,N-1\rangle_n ,\\ &= \sqrt{2}\left(|1,0,N-1\rangle_n + \sqrt{2}\sqrt{N-1}\,|0,2,N-2\rangle_n\right),\\ |N,N,N-2\rangle_s &= \frac{|1,0,N-1\rangle_n + \sqrt{2(N-1)}\,|0,2,N-2\rangle_n}{\sqrt{2N-1}}.\end{aligned} \tag{4.90}$$

We need to find another vector orthogonal to this one to make the second basis vector with this eigenvalue

$$|N,N-2,N-2\rangle_s = \frac{\sqrt{2(N-1)}\,|1,0,N-1\rangle_n - |0,2,N-2\rangle_n}{\sqrt{2N-1}}. \tag{4.91}$$

We can continue until we obtain $l_z = 0$.
In Appendix D.7, we show the full basis change for 6 particles.

4.6 Further generalization

A generalization of this algorithm considers the eigenbasis of any operator created using the Jordan-Schwinger map (not only representations of $\mathfrak{su}(2)$). This section focuses on Hermitian operators in diagonal form. The non-Hermitian operators are not treated here (the problem is that their eigenbasis is not orthogonal), and the non-diagonal form can be deduced from this section combined with the section on base changes 4.7.
For a diagonal matrix G, the Jordan-Schwinger map transforms this matrix into

$$\hat{G} = \sum_\alpha \hat{n}_\alpha \Lambda_\alpha, \tag{4.92}$$

where $\vec{\Lambda}$ denotes a vector containing the eigenvalues of $\hat{G}$.
As previously suggested, the total number operator $\hat{N}$ commutes with $\hat{G}$ and can therefore be used to establish of a CSCO. Further, any occupation number operator commutes with $\hat{G}$, and therefore, the number basis is always an eigenbasis of $\hat{G}$. Indeed, the number basis is a basis and all its vectors are eigenvectors of $\hat{G}$:

$$\begin{aligned}\hat{G}\,|k_1,...,k_n\rangle &= \sum_\alpha \vec{\Lambda}\hat{n}_\alpha\,|k_1,...,k_n\rangle ,\\ &= \sum_\alpha \Lambda_\alpha k_\alpha\,|k_1,...,k_n\rangle ,\\ &= \vec{\Lambda}\cdot\vec{k}\,|k_1,...,k_n\rangle .\end{aligned} \tag{4.93}$$

If one wants to construct a basis using a CSCO containing this operator, then one must consider constructing the set:

$$V_\lambda = \{\vec{k} \in \mathbb{N}^n | \vec{\Lambda}\cdot\vec{k} = \lambda\} \tag{4.94}$$

The equation $\vec{\Lambda}\cdot\vec{k} = \lambda$ is a linear Diophantine equation[37]. There already exists literature on this problem, as it has applications in crystallography.

[37] A Diophantine equation is a polynomial equation, with two or more variables, integer coefficients, and whose solutions of interest are the integer ones.

Once V_λ is known, the set $\{|\vec{k}\rangle\}$, with $\vec{k} \in V_\lambda$ is the eigenspace of $\hat{G}$ corresponding to the eigenvalue λ. If $\dim(V_\lambda) \leq 1$, $\hat{G}$ is a CSCO. Otherwise, if $\dim\left(V_\lambda \cap \{|\vec{k}|_1 = N\}\right) \leq 1$, then $\{\hat{N}, \hat{G}\}$ is a CSCO because there are no states $|\vec{k}\rangle$ with the same quantum numbers N and λ. Finally, in the other cases, other quantum numbers are required to discriminate the different eigenvectors. In particular, in cases where the eigenvalues are an arithmetic sequence, this operator has to be treated similarly to $\hat{L}_z$.
Here, we provide a particular example. Let us consider G to be a $n \times n$ diagonal matrix, the i^{th} eigenvalue λ_i, is equal to the logarithm of the i^{th} prime number p_i. Then, each basis vector $|\vec{k}\rangle$ is an eigenvector of $\hat{G}$, and its eigenvalue is the logarithm of the number whose decomposition into prime numbers is given by $\vec{k}$:

$$\hat{g}\,|\vec{k}\rangle = \sum_\alpha \Lambda_\alpha \hat{n}_\alpha\,|\vec{k}\rangle = \sum_\alpha \log(p_\alpha) k_\alpha\,|\vec{k}\rangle = \sum_\alpha \log\left(p_\alpha^{k_\alpha}\right)|\vec{k}\rangle = \log\left(\prod_\alpha p_\alpha^{k_\alpha}\right)|\vec{k}\rangle \tag{4.95}$$

This operator is a CSCO because only one quantum number is sufficient to determine each state of a basis; each integer vector corresponds to one state and one prime number. We might speculate that such decomposition could be useful in the context of quantum cryptography.

4.7 Effect of unitaries

4.7.1 General transformations of matrices

Another generalization we consider are non-diagonal matrices, but if we take M to be the unitary matrix diagonalizing a spin matrix L[38], with $\Lambda = \mathrm{diag}(L)$ such that

$$L = M\Lambda M^\dagger, \tag{4.96}$$

as we already know how to construct the basis corresponding to Λ, we can solve the problem for general L by discussing how the unitary acts. In the general case of the action of an unitary U such that

$$G' = UGU^\dagger, \tag{4.97}$$

using the Jordan-Schwinger map on L, we get the operator

$$\hat{G}' = \hat{a}_\alpha^\dagger G'_{\alpha\beta}\hat{a}_\beta = \hat{a}_\alpha^\dagger U_{\alpha t} G_{tu} U^\dagger_{u\beta}\hat{a}_\beta, \tag{4.98}$$

where we used the index notation which implies the summation of repeated indices. We can define two new operators $\hat{b}_u \equiv U^\dagger_{u\beta}\hat{a}_\beta$ and $\hat{b}_t^\dagger \equiv \hat{a}_\alpha^\dagger U_{\alpha t}$, which behave exactly as creation and annihilation operators, but for other modes, that is, the linear combinations of the initial ones. Indeed, by properties of the conjugate transpose, it is straightforward to demonstrate that they are adjoint to each other; however, with properties of unitaries, we can also show that their commuting properties are the ones expected for such operators. Indeed, as $[\hat{b}_u, \hat{b}_t] = [\hat{b}_u^\dagger, \hat{b}_t^\dagger] = 0$ by linearity of the commutators. Further, we have

$$[\hat{b}_u, \hat{b}_t^\dagger] = [U^\dagger_{u\beta}\hat{a}_\beta, \hat{a}_\alpha^\dagger U_{\alpha t}] = U^\dagger_{u\beta}U_{\alpha t}[\hat{a}_\beta, \hat{a}_\alpha^\dagger] = U^\dagger_{u\beta}U_{\alpha t}\delta_{\beta,\alpha} = U^\dagger_{u\alpha}U_{\alpha t} = \delta_{u,t}, \tag{4.99}$$

where the last line is obtained by saying that the product of a unitary matrix and its adjoint is the identity, and therefore, in index notation, a Kronecker delta. With the verified commutation

[38]Note that not all matrices are diagonalizable, and only those with orthogonal eigenspaces are diagonalized by unitaries. Fortunately, all Hermitian matrices match these requirement

relation, we can define the new occupation numbers in the new modes as $\hat{m}_u = \hat{b}_u^\dagger \hat{b}_u \; \forall u$. Fortunately, the total number of particles remains unchanged by the unitaries:

$$\sum_u \hat{m}_u = \sum_u \hat{b}_u^\dagger \hat{b}_u = \hat{a}_\alpha^\dagger U_{\alpha u} U_{u\beta}^\dagger \hat{a}_\beta = \hat{a}_\alpha^\dagger \delta_{\alpha\beta} \hat{a}_\beta = \sum_\alpha \hat{a}_\alpha^\dagger \hat{a}_\alpha = \sum_\alpha m_\alpha = \hat{N}. \tag{4.100}$$

Those new operators $\hat{b}$ and $\hat{b}^\dagger$ create a new 'number' basis $|m_1, m_2, ..., m_n\rangle$, with a similar definition as for Equation (2.86). In this basis, the operator $\hat{G}'$ is obtained as a simple application of the Jordan-Schwinger map (with the new mode operators) on matrix G:

$$\hat{G}' = \hat{b}_t^\dagger G_{tu} \hat{b}_u. \tag{4.101}$$

If G is a diagonal matrix Λ, using the algorithm presented in the previous sections, we can create the eigenbasis of G on the new number basis. However, if we want the eigenbasis to be expressed in the original basis, we must know how to change the basis from $\{|m_1, m_2, ..., m_n\rangle\}$ to $\{|n_1, n_2, ..., n_n\rangle\}$. In order to obtain such transformation, we need to compute the scalar product

$$\langle n_1, n_2, ..., n_n | m_1, m_2, ..., m_n \rangle \,. \tag{4.102}$$

If we write $N = \sum_i n_i$ and $M = \sum_j m_j$ for the total number of particles, we define two vectors $\vec{\beta}$ and $\vec{j}$ of N and M components respectively. The n_1 first components of $\vec{\beta}$ are 1, the following n_2 seconds are 2, *etc.* For instance, the vector β associated with the state $|3, 2, 4\rangle$ is $(1, 1, 1, 2, 2, 3, 3, 3, 3)$. In Appendix D.8, we show that the scalar product is given by

$$\begin{aligned} \langle n_1, n_2, ..., n_n | m_1, m_2, ..., m_n \rangle &= \delta_{N,M} \prod_i^n \sqrt{\frac{n_i!}{m_i!}} \sum_{\vec{\sigma} \in \mathrm{Perm}(\vec{\beta})} \left(\prod_c^N U_{\sigma_c j_c} \right), \\ &= \delta_{N,M} \prod_i^n \sqrt{\frac{n_i!}{m_i!}} \sum_{\vec{\sigma} \in \mathrm{Perm}(\vec{j})} \left(\prod_c^N U_{\beta_c \sigma_c} \right), \end{aligned} \tag{4.103}$$

where $\vec{\sigma}$ are permutations without repetitions[39] (the vector $(1, 1)$ has only one permutation: itself. Thus, it is not counted twice). To understand this expression, we can apply it to the cases where $\{|n'_1, n'_2, ..., n'_n\rangle\}$ and $\{|n_1, n_2, ..., n_n\rangle\}$ belong to the same number basis. We know that the scalar product is given by:

$$\langle n_1, n_2, ..., n_n | n'_1, n'_2, ..., n'_n \rangle = \delta_{n_1, n'_1} \delta_{n_2, n'_2} ... \delta_{n_n, n'_n} \tag{4.104}$$

First, if $N \neq N'$, then the scalar product should be 0. If the basis is the same, the unitary is a Kronecker delta, $U_{\alpha,\beta} = \delta_{\alpha,\beta}$. The product in Equation (4.103) is therefore 1 if $\vec{\sigma} = \vec{\beta}$ and 0 otherwise. The sum is then equal to the number of permutations of $\vec{j}$ which is equal to $\vec{\beta}$. As we consider permutations without repetition, we do not consider two times the same permutations; thus, there cannot be two permutations equal to $\vec{\beta}$: the sum is either 1 if there is one permutation equal to $\vec{\beta}$; otherwise, it is 0. Finally, as the components of $\vec{\beta}$ and $\vec{j}$ are in ascending order, the only permutation $\vec{\sigma}$ that can be equal to $\vec{\beta}$ is $\vec{j}$ itself, if it counts the same number of 1, of 2, *etc.* and therefore if $n_i = n'_i \; \forall i$.

4.7.2 Relation between the spin-eigenstate using Wigner $\mathcal{D}$-matrices

The result in the last subsection was general but it is complex to use, however, we can find a particular expression for rotations in the space of spin operators. For instance, if we want to

[39] More precisely, due to intistinguishability of the particles, all equivalent permutations are counted only ones. This is what we mean by without repetition in this Section, it is not the same meaning as classically.

use the work we have done on $\hat{L}_z$ to deduce the eigenstates of $\hat{L}_y$, using Equation(4.103) is not practical. The basic idea of this subsection is to express the eigenvectors of the operators $\hat{L}_x, \hat{L}_y, \hat{L}_z$ as rotations of each other. Generally, the rotations are symmetries generated by the momentum operators. In the following, we denote $\{g_x, g_y, g_z\}$ a particular basis of $\mathfrak{su}(2)$[40]. One can show that the operator $e^{-i\theta g_j}$ generates a rotation of angle θ around the axis j in $\mathfrak{su}(2)$ [14]. This can be resumed as

$$e^{i\theta g_j} g_i e^{-i\theta g_j} = \begin{cases} \cos\theta g_i + \epsilon_{ijk} \sin\theta g_k & i \neq j, \\ g_i & i = j. \end{cases} \tag{4.105}$$

Without loss of generality we can take $j = z$, to any vector $\vec{g} = (ag_x + bg_y + cg_z) \in \mathfrak{su}(2)$ to obtain a more general expression

$$e^{i\gamma g_z} (ag_x + bg_y + cg_z) e^{-i\gamma g_z} = (a\cos\gamma + b\sin\gamma) g_x + (b\cos\gamma - a\sin\gamma) g_y + cg_z. \tag{4.106}$$

Identifying $\vec{g}$ to the triplet (a, b, c), this can be rewritten as a simple matrix-vector product

$$\mathcal{R}\vec{g} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{4.107}$$

This is the general matrix for the rotation around the z-axis. Now that the rotation around one axis has been derived, one can combine several rotations around different axis to realize arbitrary rotations. In particular, the rotations of any observable can be described using the Euler angles. In the literature [14], we find that

$$\mathcal{R}(\alpha, \beta, \gamma)\, \vec{g} = e^{i\gamma g_z} e^{i\beta g_y} e^{i\alpha g_z} \vec{g} e^{-i\alpha g_z} e^{-i\gamma g_y} e^{-i\gamma g_z}. \tag{4.108}$$

Thus the corresponding operator R can be defined as:

$$R(\alpha, \beta, \gamma) \equiv e^{-i\alpha g_z} e^{-i\gamma g_y} e^{-i\gamma g_z}, \tag{4.109}$$

In the matrix form developed earlier, this can be rewritten as

$$\mathcal{R}(\alpha, \beta, \gamma)\, \vec{g} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{4.110}$$

In particular:

$$\begin{cases} g_x = \mathcal{R}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) g_y = R^\dagger\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) g_y R\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right), \\ g_y = \mathcal{R}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) g_z = R^\dagger\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) g_z R\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right), \\ g_z = \mathcal{R}\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) g_x = R^\dagger\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right) g_x R\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right), \end{cases} \tag{4.111}$$

and

$$\begin{cases} g_x = \mathcal{R}\left(-\frac{\pi}{2}, -\frac{\pi}{2}, 0\right) g_z = R^\dagger\left(-\frac{\pi}{2}, -\frac{\pi}{2}, 0\right) g_z R\left(-\frac{\pi}{2}, -\frac{\pi}{2}, 0\right), \\ g_y = \mathcal{R}\left(-\frac{\pi}{2}, -\frac{\pi}{2}, 0\right) g_x = R^\dagger\left(-\frac{\pi}{2}, -\frac{\pi}{2}, 0\right) g_x R\left(-\frac{\pi}{2}, -\frac{\pi}{2}, 0\right), \\ g_z = \mathcal{R}\left(-\frac{\pi}{2}, -\frac{\pi}{2}, 0\right) g_y = R^\dagger\left(-\frac{\pi}{2}, -\frac{\pi}{2}, 0\right) g_y R\left(-\frac{\pi}{2}, -\frac{\pi}{2}, 0\right). \end{cases} \tag{4.112}$$

All this discussion is valid for any representation of $\mathfrak{su}(2)$. In particular, it is true for the matrix representation, where the basis of $\mathfrak{su}(2)$ we are using are the matrices L_x, L_y and L_z. If we want to

[40] In Section 2.3.7, we explained that the expected commutation relations for this basis was given by Equation 2.109

$$[g_i, g_j] = i\epsilon_{ijk} g_k.$$

make the base change using Equation (4.103), we need to know the unitary U corresponding to the rotation, and we can find it thanks to Equation (4.108) as:

$$\begin{aligned} G' =& R^\dagger(\alpha,\beta,\gamma)\, G R(\alpha,\beta,\gamma) \\ =& e^{i\gamma L_z} e^{i\beta L_y} e^{i\alpha L_z} G e^{-i\alpha L_z} e^{-i\beta L_y} e^{-i\gamma L_z} \\ \Rightarrow U =& e^{i\gamma L_z} e^{i\beta L_y} e^{i\alpha L_z}. \end{aligned} \tag{4.113}$$

However, we can proceed in a more efficient way. If we do not consider the matrix representation, but directly the operator representation, we directly obtain that the the rotation operator $\hat{R}$ is given by:

$$\hat{R}(\alpha,\beta,\gamma) \equiv e^{-i\alpha \hat{L}_z} e^{-i\gamma \hat{L}_y} e^{-i\gamma \hat{L}_z}, \tag{4.114}$$

Applying this operator on a state gives the state after rotation[41]. We denote the eigenvector of $\hat{L}_z$ on a particular submodule with N, l and C fixed by an index z

$$|N,l,l_z,C\rangle_z, \tag{4.115}$$

and similarly y for $\hat{L}_y$ etc. We then introduce the Wigner $\mathcal{D}$-matrices as

$$\mathcal{D}^l_{l'_z,l_z}(\alpha,\beta,\gamma) \equiv_z \langle N,l,l'_z,C|R(\alpha,\beta,\gamma)|N,l,l_z,C\rangle_z.$$

Due to Wigner-Eckart's theorem, we know that these matrices are not depending on the quantum numbers N and C. Stronger, they do not depend on the representation, therefore those matrices are the same for our particular case of rotations of spin-like operators, and for classical rotations. With those matrices we can simply compute the rotations as

$$R(\alpha,\beta,\gamma)|N,l,m,C\rangle_z = \sum_{l'_z} \mathcal{D}^l_{l'_z,l_z}(\alpha,\beta,\gamma)|N,l,l'_z,C\rangle_z. \tag{4.116}$$

These matrices can be factorized in a simpler form using the Wigner small d matrices

$$\mathcal{D}^l_{m',m}(\alpha,\beta,\gamma) \equiv e^{-i\alpha m'} d^l_{m',m}(\beta)\, e^{-i\gamma m}, \tag{4.117}$$

Allowing us to have a very concise formula for rotations:

$$R(\alpha,\beta,\gamma)|N,l,l_z,C\rangle_z = \sum_{l'_z} e^{-i\alpha l'_z} d^l_{l'_z,l_z}(\beta)\, e^{-i\gamma l_z}(\alpha,\beta,\gamma)|N,l,l'_z,C\rangle_z. \tag{4.118}$$

This expression justifies our decompositions into submodules. We have found a basis in which rotations are expressed as a sum of a small number of vectors only. Moreover only one quantum number is not left unchanged, and this is the case because the subspace with all the others fixed is an irreducible submodule. To give a small practical example, we can compute the eigenstates of $\hat{L}_y$. They are obtained by rotating the eigenstates of $\hat{L}_z$ with angles $\alpha = -\pi/2, \beta = -\pi/2, \gamma = 0$:

$$|N,l,l_y,C\rangle_y = \sum_{l_z} e^{-i\frac{\pi}{2}l_z} d^l_{l_z,l_y}\left(-\frac{\pi}{2}\right)|N,l,l_z,C\rangle_z. \tag{4.119}$$

[41] Let us give a brief reminder about the Heisenberg picture and the Schrödinger picture. A symmetry operation $\mathcal{S}$ is applied in the Heisenberg picture as $\hat{A} \to S^\dagger \hat{A} S = \mathcal{S}\hat{A}$, which is equivalent to the Schrödinger picture where the operation is applied on the state $|\psi\rangle \to S|\psi\rangle = |\mathcal{S}\psi\rangle$, since $\langle\phi|\mathcal{S}A|\psi\rangle = \langle\phi|S^\dagger A S|\psi\rangle = \langle\mathcal{S}\phi|A|\mathcal{S}\psi\rangle$.

5. Conclusion and outlook

This thesis was structured in two parts: first, in Chapter 3, we have been able to find a general way of producing multimode uncertainty relations using the Jordan-Schwinger map. Many further calculations could still be performed and future works could consider other operators (representations of $\mathfrak{su}(2)$ with more modes, for instance) or other inequalities (for example, the Schrödinger-Robertson inequality applied to the operators of the Jordan-Schwinger map or relations on bounded observables). It would be interesting to determine whether the equations generated in this way can (or cannot) be deduced from the classical Schrödinger-Robertson inequality or the symplectic uncertainty relations. One may further investigate the inequality for general matrices and check whether using a basis of Hermitian matrices could lead to a simpler result. We also obtained different expressions of these uncertainties for several kinds of states, and it could be interesting to check if the violation of a stronger inequality obtained for a single-party state could be translated into an entanglement criterion for bipartite states. Finally, we left open the question of the entropic uncertainty relations as this requires the knowledge of the eigenbasis of the constructed operators.

We then naturally dedicated the second part of this thesis to the construction, in Chapter 4, of the aforementioned basis using the tools given by Lie algebra theory. We first rederived the already known result for 2-mode states, and found a generalization for any number of modes. This result is new and, beyond the computation of the entropy of the observables, could help settle several questions that arose in different works about entanglement and non-classicality. We also gave a new mathematical background which may lead to a general way to design optical circuits. As an outlook, future work could look deeper into the generalization of the Jordan-Schwinger map by replacing the creation and annihilation operators by other operators sharing the same commutation properties, but which will give new representations of $\mathfrak{su}(2)$ that are easier to use to make basis changes and to simplify many calculations in quantum information research.

In this thesis, we tried to take a general approach by focusing on mathematical rigor, keeping it general enough so it should encapsulate various physical situations. Future works should focus on specific examples, or making a stronger link between the mathematical developments and real physical systems, giving more illustrations of the theoretical concepts discussed in the thesis. For instance, even if we suggested a link between unitaries in the matrix space and quantum circuits, the correspondence should be made explicit so to yield a general method to design the quantum circuit associated with any unitary.

Future studies could further investigate the fundamental link between our spin-like observables and the real spin of a bosonic system. Indeed, if the matrix l_z describes the spin of a single boson, then the total spin of the system is the single-body observable, obtained via the Jordan-Schwinger map. In this way, some of our results allow for a new physical interpretation. It becomes obvious that the commutation relations should be preserved, that the rotation in the spin phase space is similar to a real rotation in space, and, moreover, our algorithm can be interpreted as a general procedure to find the eigenbasis of the total spin of a system with many bosons. This could help solve problems such as the behaviour of several bosons immersed in a magnetic field. Our results can also be generalized to fermionic systems, as there exists a generalization of the Jordan-Schwinger map to fermionic creation operators [33]. The only difference is that the statistics of the particles will change; in each box there cannot be more than one fermion by Pauli's exclusion principle.

This thesis highlighted the fundamental relations between bosonic systems and spin operators. As the properties of the latter are well-known, we hope that our results will help to answer fundamental questions arising in the context of continuous-variables quantum information.

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A. Hilbert spaces in quantum theories

A.1 Decomposition of a linear form in function of the components of the dual vector

The components of the dual of a vector $|\psi\rangle$ on the dual basis can be deduced from the components of the vector itself. We write $\langle\psi| = \sum_i d_i \langle v_i|$ the decomposition of $\langle\psi|$ on the dual basis. Let us compute the application of the linear form $\langle\psi|$ on any vector $|\phi\rangle$:

$$\begin{aligned}
\langle\psi|\phi\rangle &= \langle\phi|\psi\rangle^*, \quad \text{Hermitian symmetry} \\
\left(\sum_i d_i \langle v_i|\right)\phi &= \left(\langle\phi|\left(\sum_i c_i |v_i\rangle\right)\right)^*, \quad \text{Decomposition in bases} \\
\sum_i d_i \langle v_i|\phi\rangle &= \sum_i c_i^* \langle\phi|v_i\rangle^*, \quad \text{Linearity} \\
\sum_i d_i \langle v_i|\phi\rangle &= \sum_i c_i^* \langle v_i|\phi\rangle \quad \forall\phi, \quad \text{Hermitian symmetry} \\
\Rightarrow d_i &= c_i^*.
\end{aligned} \tag{A.1}$$

We find that the components of the vector and its dual are the complex conjugate of each other.

A.2 Decomposition of a vector in an orthonormal basis

Let's be any vector $|\psi\rangle$, and an orthonormal basis $\{n_i\}$. As $\{n_i\}$ is a basis it is possible to define ψ_i, the components of $|\psi\rangle$, as

$$|\psi\rangle = \sum_j \psi_j |n_j\rangle . \tag{A.2}$$

Taking the scalar product of this vector with any vector of the basis will directly allow us to express the components in the basis:

$$\begin{aligned}
\langle n_i|\psi\rangle &= \langle n_i| \sum_j \psi_j |n_j\rangle , \\
&= \sum_j \psi_j \langle n_i|n_j\rangle , \quad \text{by linearity at right for the scalar product} \\
&= \sum_j \psi_j \delta_{ij}, \quad \text{by definition of orthonormality} \\
&= \psi_i .
\end{aligned} \tag{A.3}$$

This imposes that the components of a vector in an orthonormal basis is given by its scalar products with the basis vectors:

$$\psi_i = \langle n_i|\psi\rangle . \tag{A.4}$$

A.3 Independence of the trace of operators with respect to the basis

If $\{|n_i\rangle\}$ and $\{|m_i\rangle\}$ are two orthonormal bases, we can use the closure relation to switch from one to the other:

$$\begin{aligned}
\sum_i \langle n_i|\hat{A}|n_i\rangle &= \sum_i \langle n_i| \left(\sum_j |m_j\rangle \langle m_j|\right) \hat{A} \left(\sum_k |m_k\rangle \langle m_k|\right) |n_i\rangle\,, \quad \text{Closure Relation} \\
&= \sum_{ijk} \langle n_i|m_j\rangle \langle m_j|\hat{A}|m_k\rangle \langle m_k|n_i\rangle\,,, \quad \text{Distributivity} \\
&= \sum_{ijk} \langle m_j|\hat{A}|m_k\rangle \langle m_k|n_i\rangle \langle n_i|m_j\rangle\,, \quad \text{Commutation of the scalars} \\
&= \sum_{jk} \langle m_j|\hat{A}|m_k\rangle \langle m_k| \left(\sum_i |n_i\rangle \langle n_i|\right) |m_j\rangle\,, \quad \text{Distributivity} \\
&= \sum_{jk} \langle m_j|\hat{A}|m_k\rangle \langle m_k|m_j\rangle\,, \quad \text{Closure relation} \\
&= \sum_{jk} \langle m_j|\hat{A}|m_k\rangle \delta_{kj}, \quad \text{Orthonormality of the basis} \\
&= \sum_j \langle m_j|\hat{A}|m_j\rangle\,. \qquad (A.5)
\end{aligned}$$

The definition is consistent independently of the basis.

B. Lie algebras

B.1 Determination of the action of the Lie bracket, based on its action on some basis

Let's be $\mathfrak{g}$ some Lie algebra defined on some field F doted of the Lie bracket $[\cdot,\cdot]$. Let's be $\{g_i\}$ a basis of such an algebra. Given that

$$\forall g_i, g_j \quad \text{basis vectors} : [g_i, g_j] = f_{ij}^k g_k \quad \text{with} \quad f_{ij}^k \in F, \tag{B.1}$$

it is possible to deduce the effect of the Lie bracket on any vector $v \in \mathfrak{g}$. Indeed as $\{g_i\}$ is a basis of a vector space, it is possible to write:

$$v = v_i g_i \quad \forall v \in \mathfrak{g} \quad \text{with} \quad v_i \in F, \tag{B.2}$$

where we used the index notation (with automatical sum over the repeated indices). Therefore:

$$\begin{aligned} x &= [v, w], \\ &= [v_i g_i, w_j g_j], \\ &= v_i \cdot w_j \cdot [g_i, g_j], \quad \text{by bilinearity} \\ &= v_i w_j f_{ij}^k g_k. \end{aligned} \tag{B.3}$$

B.2 Conservation of the defining relations of an algebra by its representations

Let us show that for any representation $\phi_0 : \mathfrak{V} \to \mathfrak{W}$, mapping a basis $\{g_i\}$ to a set $\{G_i\}$, if the Lie brackets were defined on this basis, then the defining relations are preserved.

$$\begin{aligned} [g_i, g_j] &= f_{ij}^k g_k, \\ \phi_0([g_i, g_j]) &= \phi_0(f_{ij}^k g_k), \\ [\phi_0(g_i), \phi_0(g_j)] &= f_{ij}^k \phi_0(g_k), \\ [G_i, G_j] &= f_{ij}^k G_k, \end{aligned} \tag{B.4}$$

where we used the convention that repeated indices are automatically summed.

B.3 An element c_ϕ commutes with all elements of L

Let's be β any non-degenerate symmetric associative bilinear form on L; $\{x_i\}$ a basis of L and $\{y_i\}$ the dual basis relative to β (*i.e.* $\beta(x_i, y_j) = \delta_{ij}$). Finally, let's be any vector v. We can define the coefficients a_{ij} and b_{ij} such that

$$\begin{cases} [v, x_i] &= \sum_j a_{ij} x_j, \\ [v, y_i] &= \sum_j b_{ij} y_j, \end{cases} \tag{B.5}$$

then, one can show that $a_{ik} = -b_{ki}$,

$$\begin{aligned} a_{ik} &= \sum_j a_{ij}\delta_{jk}, \\ &= \sum_j a_{ij}\beta(x_j, y_k), \quad x \text{ and } y \text{ are dual basis} \\ &= \beta([v, x_i], y_k), \quad \text{Linearity and definition of } a_{ij} \\ &= -\beta([x_i, v], y_k), \quad \text{Antisymmetry of the Lie bracket and linearity of } \beta \\ &= -\beta(x_i, [v, y_k]) \quad \text{Thanks to associativity and symmetry} \\ &= -\sum_j b_{kj}\beta(x_i, y_j), \quad \text{Linearity and definition of } b_{ij} \\ &= -\sum_j b_{kj}\delta_{ij}, \quad x \text{ and } y \text{ are dual basis} \\ &= -b_{ki} \end{aligned} \tag{B.6}$$

This being shown, let's compute the commutator between $c_\phi(\beta)$ and $\phi(v)$ for some $v \in L$.

$$\begin{aligned} [\phi(v), c_\phi(\beta)] &= \sum_i \phi(v)\phi(x_i)\phi(y_i) - \phi(x_i)\phi(y_i)\phi(v), \quad \text{Definition of the Casimir element} \\ &= \sum_i (\phi(v)\phi(x_i)\phi(y_i) - \phi(x_i)\phi(v)\phi(y_i)) + (\phi(x_i)\phi(v)\phi(y_i) - \phi(x_i)\phi(y_i)\phi(v)) \\ &= \sum_i [(\phi(v)\phi(x_i)]\phi(y_i) + \phi(x_i)[\phi(v)\phi(y_i)], \quad \text{Definition of the commutator} \end{aligned} \tag{B.7}$$

$$= \sum_{ij} a_{ij}\phi(x_j)\phi(y_i) + \phi(x_i)b_{ij}\phi(y_j), \quad \text{Definition of } a_{ij} \text{ and } b_{ij} \tag{B.8}$$

$$\begin{aligned} &= \sum_{ij} a_{ij}\phi(x_j)\phi(y_i) - a_{ji}\phi(x_i)b_{ij}\phi(y_j) \\ &= 0. \end{aligned} \tag{B.9}$$

This shows that for any β, c_ϕ commutes with $\phi(L)$.

C. Inequalities

C.1 Conservation of the commutator by the Jordan-Schwinger map

The demonstration that the Jordan-Schwinger map preserves the commutation relations, and is therefore an acceptable representation of the algebra, is obtained only using linearity and the expulsion properties of the commutator, the definition of the Jordan-Schwinger map, and the commutation relation of annihilation and creation operators. Therefore any generalization of the Jordan-Schwinger using operators with the same commutation relation will also preserve the commutation relations.
Letbe an algebra such that

$$[G_i, G_j] = f_{ij}^k G_k, \tag{C.1}$$

using the index notation (implicit summation over repeated indices). The Jordan-Schwinger map gives:

$$\hat{G}_i = \hat{a}_\alpha^\dagger (G_i)_{\alpha\beta} \hat{a}_\beta. \tag{C.2}$$

Then we can develop the commutation relation of the operators by replacing their definition with the Jordan-schwinger map.

$$\begin{aligned}
[\hat{G}_i, \hat{G}_j] &= [\hat{a}_\alpha^\dagger \left(G_i\right)_{\alpha,\beta} \hat{a}_\beta, \hat{a}_\gamma^\dagger \left(G_j\right)_{\gamma,\delta} \hat{a}_\delta] \\
&= \left(G_i\right)_{\alpha,\beta} \left(G_j\right)_{\gamma,\delta} [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger \hat{a}_\delta] \\
&= \left(G_i\right)_{\alpha,\beta} \left(G_j\right)_{\gamma,\delta} \left(\hat{a}_\alpha^\dagger \hat{a}_\gamma^\dagger [\hat{a}_\beta, \hat{a}_\delta] + \hat{a}_\alpha^\dagger [\hat{a}_\beta, \hat{a}_\gamma^\dagger] \hat{a}_\delta + \hat{a}_\gamma^\dagger [\hat{a}_\alpha^\dagger, \hat{a}_\delta] \hat{a}_\beta + [\hat{a}_\alpha^\dagger, \hat{a}_\gamma^\dagger] \hat{a}_\beta \hat{a}_\delta\right) \\
&= \left(G_i\right)_{\alpha,\beta} \left(G_j\right)_{\gamma,\delta} \left(0 + \hat{a}_\alpha^\dagger \delta_{\beta\gamma} \hat{a}_\delta + \hat{a}_\gamma^\dagger (-\delta_{\alpha\delta}) \hat{a}_\beta + 0\right) \\
&= \left(G_i\right)_{\alpha,\epsilon} \left(G_j\right)_{\epsilon,\delta} \hat{a}_\alpha^\dagger \hat{a}_\delta - \left(G_i\right)_{\epsilon,\beta} \left(G_j\right)_{\gamma,\epsilon} \hat{a}_\gamma^\dagger \hat{a}_\beta \\
&= \left(G_i G_j\right)_{\mu,\nu} \hat{a}_\mu^\dagger \hat{a}_\nu - \left(G_j G_i\right)_{\mu,\nu} \hat{a}_\mu^\dagger \hat{a}_\nu \\
&= \left([G_i, G_j]\right)_{\mu,\nu} \hat{a}_\mu^\dagger \hat{a}_\nu \\
&= f_{ij}^k \hat{a}_\mu^\dagger (G_k)_{\mu,\nu} \hat{a}_\nu \\
&= f_{ij}^k \hat{G}_k
\end{aligned} \tag{C.3}$$

This proves that the commutation relation of operators created thanks to the Jordan-Schwinger map are similar to those of the initial algebra.

C.2 Simplification of the general uncertainty relation for symmetric matrices

In this appendix we simplify the cross terms appearing in the term with two matrices R in Equation 3.45, using the symmetry properties of R.

$$
\begin{aligned}
& R_{\alpha\beta}R_{\gamma\delta}\left(\langle\hat{p}_\alpha\hat{p}_\beta\hat{x}_\gamma\hat{x}_\delta\rangle + \langle\hat{x}_\alpha\hat{x}_\beta\hat{p}_\gamma\hat{p}_\delta\rangle\right) \\
=& R_{\alpha\beta}R_{\gamma\delta}\left\langle\{\hat{x}_\alpha\hat{x}_\beta, \hat{p}_\gamma\hat{p}_\delta\}\right\rangle \\
=& R_{\alpha\beta}R_{\gamma\delta}\left\langle\hat{x}_\alpha[\hat{x}_\beta, \hat{p}_\gamma\hat{p}_\delta] + \{\hat{x}_\alpha, \hat{p}_\gamma\hat{p}_\delta\}\hat{x}_\beta\right\rangle \\
=& R_{\alpha\beta}R_{\gamma\delta}\left\langle\hat{x}_\alpha\hat{p}_\gamma[\hat{x}_\beta, \hat{p}_\delta] + \hat{x}_\alpha[\hat{x}_\beta, \hat{p}_\gamma]\hat{p}_\delta - \hat{p}_\gamma[\hat{x}_\alpha, \hat{p}_\delta]\hat{x}_\beta + \{\hat{x}_\alpha, \hat{p}_\gamma\}\hat{p}_\delta\hat{x}_\beta\right\rangle \\
=& R_{\alpha\beta}R_{\gamma\delta}\left\langle i\hat{x}_\alpha\hat{p}_\gamma\delta_{\beta,\delta} + i\hat{x}_\alpha\delta_{\beta,\gamma}\hat{p}_\delta - i\hat{p}_\gamma\delta_{\alpha,\delta}\hat{x}_\beta + \{\hat{x}_\alpha, \hat{p}_\gamma\}\hat{p}_\delta\hat{x}_\beta\right\rangle \\
=& R_{\alpha\beta}R_{\gamma\delta}\left\langle i\delta_{\beta,\delta}(2\hat{x}_\alpha\hat{p}_\gamma - \hat{p}_\gamma\hat{x}_\alpha) + \{\hat{x}_\alpha, \hat{p}_\gamma\}\hat{p}_\delta\hat{x}_\beta\right\rangle \\
=& R_{\alpha\beta}R_{\gamma\delta}\left\langle i\delta_{\beta,\delta}([\hat{x}_\alpha, \hat{p}_\gamma] + \hat{x}_\alpha\hat{p}_\gamma) + \{\hat{x}_\alpha, \hat{p}_\gamma\}\hat{p}_\delta\hat{x}_\beta\right\rangle \\
=& R_{\alpha\beta}R_{\gamma\delta}\left\langle -\delta_{\beta,\delta}\delta_{\alpha,\gamma} + i\delta_{\beta,\delta}\hat{x}_\alpha\hat{p}_\gamma + \{\hat{x}_\alpha, \hat{p}_\gamma\}\hat{p}_\delta\hat{x}_\beta\right\rangle \\
=& R_{\alpha\beta}R_{\gamma\delta}\left\langle -\delta_{\beta,\delta}\delta_{\alpha,\gamma} + i\frac{\delta_{\beta,\delta}}{2}([\hat{x}_\alpha, \hat{p}_\gamma] + \{\hat{x}_\alpha, \hat{p}_\gamma\}) + \frac{1}{2}\{\hat{x}_\alpha, \hat{p}_\gamma\}([\hat{p}_\delta, \hat{x}_\beta] + \{\hat{p}_\delta, \hat{x}_\beta\})\right\rangle \\
=& R_{\alpha\beta}R_{\gamma\delta}\left\langle -\frac{3}{2}\delta_{\beta,\delta}\delta_{\alpha,\gamma} + i\frac{\delta_{\beta,\delta}}{2}\{\hat{x}_\alpha, \hat{p}_\gamma\}\frac{\delta_{\beta,\delta}}{2} - i\frac{\delta_{\beta,\delta}}{2}\{\hat{x}_\alpha, \hat{p}_\gamma\} + \frac{1}{2}\{\hat{x}_\alpha, \hat{p}_\gamma\}\{\hat{p}_\delta, \hat{x}_\beta\}\right\rangle \\
=& R_{\alpha\beta}R_{\gamma\delta}\frac{1}{2}\left\langle\{\hat{x}_\alpha, \hat{p}_\gamma\}\{\hat{x}_\beta, \hat{p}_\delta\}\right\rangle - \frac{3}{2}\mathrm{tr}\left(R^2\right)
\end{aligned}
\tag{C.4}
$$

We see a constant term appearing that will be used as a lower bound.

D. Relating number and spin eigenstates

D.1 Relations between $\hat{L}^2$ in the 2-mode system Jordan-Schwinger representation of $\mathfrak{su}(2)$

To show the relation between the operator $\hat{L}^2$ and the operator $\hat{N}$, we can develop the operator $\hat{L}^2$ as the sum over the $\hat{L}_i^2$, and then develop those operators through their definition by the Jordan-Schwinger map, for a particular representation using Pauli matrices. As the Casimir operator does not depend on the particular basis of the algebra that we use, this result will be true for all representations of 2×2 matrices.

$$
\begin{aligned}
\hat{L}^2 =& \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \\
=& \frac{1}{4}(\hat{a}_1^\dagger \hat{a}_2 \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_1 \\
& - \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 \hat{a}_1^\dagger \hat{a}_2 \\
& + \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_2), \\
=& \frac{1}{4}(2\hat{a}_1^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_1 + 2\hat{a}_2^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_2 \\
& + \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 - 2\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_2), \\
=& \frac{1}{4}(2(\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_1) + 2(\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2) \\
& + \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 - 2(\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2) + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_2), \\
=& \frac{1}{4}((\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1^\dagger \hat{a}_1) + 2\hat{a}_1^\dagger \hat{a}_1 + 2\hat{a}_2^\dagger \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2^\dagger \hat{a}_2), \\
=& \frac{\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2}{2} + \left(\frac{\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2}{2} \right)^2, \\
=& \frac{\hat{n}_1 + \hat{n}_2}{2} \left(\frac{\hat{n}_1 + \hat{n}_2}{2} + \hat{\mathbb{1}} \right), \\
=& \frac{\hat{N}}{2} \left(\frac{\hat{N}}{2} + \hat{\mathbb{1}} \right).
\end{aligned}
\tag{D.1}
$$

We therefore have a clear link between the Casimir operator $\hat{L}^2$ and the total number operator for the particular case of 2-mode systems.

D.2 Commutation relations between $\hat{N}$ and Jordan-Schwinger single body operators

In this appendix, we want to compute the commutation relation between $\hat{N}$ and $\hat{G} = \hat{a}^\dagger_\alpha G_{\alpha\beta}\hat{a}_\beta$ (using the implicit summation notation). We can develop the commutator using the classical properties of it:

$$\begin{aligned}
[\hat{N}, \hat{G}] &= [\hat{a}^\dagger_\alpha \hat{a}_\alpha, \hat{a}^\dagger_\beta G_{\beta\gamma}\hat{a}_\gamma], \\
&= G_{\beta\gamma}[\hat{a}^\dagger_\alpha \hat{a}_\alpha, \hat{a}^\dagger_\beta \hat{a}_\gamma], \\
&= G_{\beta\gamma}\left(\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta [\hat{a}_\alpha, \hat{a}_\gamma] + \hat{a}^\dagger_\alpha [\hat{a}_\alpha, \hat{a}^\dagger_\beta]\hat{a}_\gamma + \hat{a}^\dagger_\beta [\hat{a}^\dagger_\alpha, \hat{a}_\gamma]\hat{a}_\alpha + [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta]\hat{a}_\alpha \hat{a}_\gamma\right), \\
&= G_{\beta\gamma}\left(\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \cdot 0 + \hat{a}^\dagger_\alpha \delta_{\alpha\beta}\hat{a}_\gamma - \hat{a}^\dagger_\beta \delta_{\alpha\gamma}\hat{a}_\alpha + 0 \cdot \hat{a}_\alpha \hat{a}_\gamma\right), \\
&= G_{\beta\gamma}\left(\hat{a}^\dagger_\beta \hat{a}_\gamma - \hat{a}^\dagger_\beta \hat{a}_\gamma\right), \\
&= 0.
\end{aligned} \tag{D.2}$$

This is an explicit proof that the total number operator commutes with all operators created thanks to the Jordan-Schwinger map.

D.3 Generalization of the Jordan-Schwinger map

In this section, we investigate generalizations of the Jordan-Schwinger map using more mode operators to obtain higher-order functions of $\hat{N}$. The only requirement for such a map is to conserve the commutation relations to preserve the algebra $\mathfrak{su}(2)$. The first idea to generalize involves replacing $n \times n$ matrices with $2f$ entry tensors[42] with n values for each index; that is, by using operators of the form

$$\hat{G} = \sum_{\substack{\alpha_1, ..., \alpha_f \\ \beta_1, ..., \beta_f}} \left(\prod_{i=1}^{f} \hat{a}^\dagger_{\alpha_i} \right) G^{\alpha_1 ... \alpha_f}{}_{\beta_1 ... \beta_f} \left(\prod_{j=1}^{f} \hat{a}_{\beta_j} \right). \tag{D.3}$$

The convention for the product of two tensors used in this master thesis is the one conserving the dimensions of the tensor

$$(GH)^{\alpha_1 ... \alpha_f}{}_{\gamma_1 ... \gamma_f} = \sum_{\beta_1, ..., \beta_f} G^{\alpha_1 ... \alpha_f}{}_{\beta_1 ... \beta_f} H^{\beta_1 ... \beta_f}{}_{\gamma_1 ... \gamma_f}, \tag{D.4}$$

and the definition of the commutator of two tensors derives naturally from it as

$$[G, H] = GH - HG. \tag{D.5}$$

We consider a particular subclass of tensors vanishing if all covariant and contravariant indices are not the same. We refer to this family of tensors as the plane diagonal tensor (because the nonzero entries describe particular planes in the tensors):

$$G^{\alpha_1 ... \alpha_f}{}_{\beta_1 ... \beta_f} = \sum_{\alpha, \beta} G^\alpha{}_\beta \left(\prod_{i=1}^{f} \delta_{\alpha \alpha_i} \right) \left(\prod_{j=1}^{f} \delta_{\beta \beta_j} \right). \tag{D.6}$$

The product of two tensors of this form is given by

$$(GH)^{\alpha_1 ... \alpha_f}{}_{\gamma_1 ... \gamma_f} = \sum_{\alpha, \gamma} \sum_{\beta} G^\alpha{}_\beta H^\beta{}_\gamma \left(\prod_{i=1}^{f} \delta_{\alpha \alpha_i} \right) \left(\prod_{j=1}^{f} \delta_{\gamma \gamma_j} \right). \tag{D.7}$$

Therefore, the product of two plane diagonal tensors is plane diagonal, and trivially, this is also the case for their commutator. Using Equation (D.3), this family of tensors generates a specific family of operators, referred to as plane diagonal operators, which are given by

$$\hat{G} = \sum_{\alpha, \beta} \left(\hat{a}^\dagger_\alpha \right)^f G^\alpha{}_\beta \left(\hat{a}_\beta \right)^f. \tag{D.8}$$

If we compute the commutator of this family of tensors, we find that the commutator of such operators is no longer a plane diagonal operator, and is not even a tensor in the form of Equation (D.3). Without details, if $\hat{G}$ and $\hat{H}$ are two plane diagonal operators, their commutator is given by

$$\begin{aligned} [\hat{G}, \hat{H}] &= \sum_{\alpha, \gamma} \sum_{\beta} \left(\hat{a}^\dagger_\alpha \right)^f \left(\prod_{i=1}^{f} (\hat{n}_\beta + i) - \prod_{i=1}^{f} (\hat{n}_\beta + 1 - i) \right) \left(G^\alpha{}_\beta H^\beta{}_\gamma - H^\alpha{}_\beta G^\beta{}_\gamma \right) \left(\hat{a}_\gamma \right)^f, \\ &= \sum_{\alpha, \gamma} \sum_{\beta} \left(\prod_{i=1}^{f} (\hat{n}_\beta + i - f - k) - \prod_{i=1}^{f} (\hat{n}_\beta + 1 - i - f) \right) \left(\hat{a}^\dagger_\alpha \right)^f \left(G^\alpha{}_\beta H^\beta{}_\gamma - H^\alpha{}_\beta G^\beta{}_\gamma \right) \left(\hat{a}_\gamma \right)^f. \end{aligned} \tag{D.9}$$

[42] These are tensors with n^{2f} elements.

The polynomial of degree $(f-1)$ in $\hat{n}_\beta$ that appears in this expression changes the order of the operators in terms of $\hat{a}$; therefore, the commutator is not intern in the class of operators described by (D.3). Therefore, the commutation relation is not preserved for this generalization of the Jordan-Schwinger map. This would be the case if the polynomial was 1 and if $f=1$, that is, for the classical Jordan-Schwinger map.

Even if this generalization does not work, there may be another that will. To find this, let us try another generalization to increase the order of the Jordan-Schwinger map in terms of the operators $\hat{a}$ and $\hat{a}^\dagger$ to the second order. In the demonstration that the Jordan-Schwinger map preserves the commutation relation in Appendix C.1, we showed that the only property of $\hat{a}_i$ and $\hat{a}_j^\dagger$ that mattered was their commutation relation:

$$[\hat{a}_i, \hat{a}_j] = 0, \quad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \quad \text{and} \quad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}. \tag{D.10}$$

For the first order, we consider the operator $\hat{b} = \chi(\hat{a}) = u\hat{a} + v\hat{a}^\dagger + c$. The commutation relations are preserved if $|u|^2 - |v|^2 = 1$, and such transformations are called Bogoliubov transformations. Now, if we consider second-order operators of the shape $\hat{O} = A\hat{a}^2 + B\hat{n} + C\hat{a}^{\dagger^2} + D\hat{a} + E\hat{a}^\dagger + F$, the conditions for the coefficients are more complex.

$$AB^* = BC^*, \quad |A|^2 = |C|^2, \quad 2AD^* + DB^* = 2EC^* + BE^*, \quad |D|^2 = |E|^2 + 1. \tag{D.11}$$

These equations have no general solutions; however, we can obtain two particular solutions:

$$\begin{aligned} \hat{O}_1 &= e^{i\phi}\left(K\left(\hat{a} - \hat{a}^\dagger\right)^2 + \hat{a} + L\right) = e^{i\phi}\left(K'\hat{p}^2 + \hat{a} + L\right), \\ \hat{O}_2 &= e^{i\phi}\left(iK\left(\hat{a} + \hat{a}^\dagger\right)^2 + \hat{a} + L\right) = e^{i\phi}\left(iK'\hat{x}^2 + \hat{a} + L\right), \end{aligned} \tag{D.12}$$

where $K, k', L, \phi \in \mathbb{R}$. In both solutions, we add a Hermitian or anti-Hermitian operator and a constant to the annihilation operator (and a complex phase). These solutions allow us to postulate a new form of operator $\hat{O} = \hat{F} + \hat{a}$ for higher-order generalization. $\hat{F}$ is an operator that must be built as a function of $\hat{a}$ and $\hat{a}^\dagger$. The condition $[\hat{O}, \hat{O}^\dagger] = 1$ implies

$$[\hat{F}, \hat{F}^\dagger] + [\hat{F}, \hat{a}^\dagger] - [\hat{F}^\dagger, \hat{a}] = 0. \tag{D.13}$$

If we postulate that $\hat{F}^\dagger = e^{i\phi}\hat{F}$ with $|\lambda| = 1$, this expression can be further simplified. This family of operators $\hat{F}$ includes Hermitian operators if $\phi = 0$ and anti-Hermitian operators if $\phi = \pi$. The condition becomes

$$[\hat{F}, a^\dagger - e^{i\phi}a] = [\hat{F}, (1 - e^{i\phi})\hat{x} + i(1 + e^{i\phi})\hat{p}] = 0. \tag{D.14}$$

This condition is verified automatically if $\hat{F} = f(\hat{a}^\dagger - e^{i\phi}\hat{a}) = g((1 - e^{i\phi})\hat{x} + i(1 + e^{i\phi})\hat{p})$. If $f(x) = \sum_{k=0}^{\infty} c_k e^{i\psi_k} x^k$ with $c_k, \psi_k \in \mathbb{R}$, then the condition $\hat{F}^\dagger = e^{i\phi}\hat{F}$ becomes

$$\psi_k = \frac{-1}{2}((k+1)\phi + k\pi). \tag{D.15}$$

Therefore, a general family of operators that can generalize the Jordan-Schwinger map are operators of the following form:

$$\hat{F} = e^{-i\frac{\phi}{2}} \sum_{k=0}^{\infty} c_k (ie^{i\frac{\phi}{2}}\hat{a} - ie^{-i\frac{\phi}{2}}\hat{a}^\dagger)^k = e^{-i\frac{\phi}{2}} \sum_{k=0}^{\infty} c'_k \left(\sin\frac{\phi}{2}\hat{x} + \cos\frac{\phi}{2}\hat{p}\right)^k. \tag{D.16}$$

In particular, if we want a Hermitian operator ($\phi = 0$), we have $\hat{F} = \sum_{k=0}^{\infty} c'_k \hat{p}^k = f(\hat{p})$, where f is a real function, and if we want an anti-Hermitian operator ($\phi = \pm\pi$), we have $\hat{F} = i\sum_{k=0}^{\infty} c'_k \hat{x}^k = if(\hat{x})$. This describes a family of operators suitable for the Jordan-Schwinger map, as follows:

$$\hat{O}(\hat{a}) = e^{-i\frac{\phi}{2}} \sum_{k=0}^{\infty} c_k (ie^{i\frac{\phi}{2}}\hat{a} - ie^{-i\frac{\phi}{2}}\hat{a}^\dagger)^k + \hat{a}, \quad \text{with } c_k, \phi \in \mathbb{R}. \tag{D.17}$$

More generally, we can also consider multiplication by a complex phase and the addition of a constant to get the transformation

$$\hat{A} = \hat{O}(\hat{a}) = e^{i(-\frac{\phi}{2}+\psi)} \sum_{k=0}^{\infty} c_k (ie^{i\frac{\phi}{2}}\hat{a} - ie^{-i\frac{\phi}{2}}\hat{a}^\dagger)^k + e^{i\psi}\hat{a} + L, \quad \text{with } c_k, \phi, \psi \in \mathbb{R} \text{ and } L \in \mathbb{C}. \tag{D.18}$$

If $c_k = 0\ \forall k \geq 2$, we find the Bogoliubov transformation; hence, the transformation $\hat{O}_{\phi,\psi,\{c_k\},L}$ is a generalization of these transformations. Equations (D.12) are particular cases of this transformation, with $c_k = 0$ if $k \neq 2$. It has not been proven whether the composition of several transformations $\hat{O}_{\phi^i,\psi^i,\{c_k^i\},L^i}$ is another transformation $\hat{O}_{\tilde{\phi},\tilde{\psi},\{\tilde{c}_k\},\tilde{L}}$ or if it describes a larger family of transformations. Using transformations $\hat{O}$, we can generalize the Jordan-Schwinger map with a higher order in terms of mode numbers as follows:

$$\hat{G} = \sum_{\alpha,\beta} \hat{A}_\alpha^\dagger G_{\alpha\beta} \hat{A}_\beta, \tag{D.19}$$

with

$$\hat{A}_\alpha = \hat{O}_{\phi^\alpha,\psi^\alpha,\{c_k^\alpha\},L^\alpha}\left(\hat{a}_\alpha\right). \tag{D.20}$$

Note that the coefficients $\phi^\alpha, \psi^\alpha, \{c_k^\alpha\}, L^\alpha$ do not need to be equal for each mode.
We will not use this generalization in this thesis, but further research should investigate if this map has applications in several problems; for instance, one should consider the submodules of the representation of $\mathfrak{su}(2)$ generated with this map.

D.4 Computations of the values of $U(n, N, u)$

This codes implements the Equation (4.36) with the initial condition given by Equation (4.37) to create a 3-dimensional array containing all the values of U for 1 to n boxes, 1 to N balls and scores from 1 to $n \times N$.

```
function U=ballInBinStocked(n,N);
  U=zeros(n,N,n*N);
  for j=1:N
    U(1,j,j)=1;
  endfor
  for i=2:n
    for j=1:N
      for k=j+1:i*j
        U(i,j,k)+=sum(U(i-1,(j:-1:1),k-j));
      endfor
      U(i,j,j)+=1;
    endfor
  endfor

```

The Equation (4.40) is implemented in the following code, with the conditions given by Equation (4.41). A stronger initial condition is added: if $u < N$, then $U = 0$. This is not necessary, but will help speed up the algorithm.

```
function U=ballInBinNotStocked(n,N,u);
  if ((u<=N)||(N==0)|| (n==1))
    U=(N==u);
    return;
  endif
  U=ballInBinNotStocked(n,N-1,u-1)+ballInBinNotStocked(n-1,N,u-N);
endfunction
```

This same relation can be a bit modified to find all the possible combinations explicitly.

```
#BallsCombi
function C=BallsCombi(n,N,u)
  if N==0
    if u==0
      C=zeros(1,n);
    else
      C=zeros(0,n);
    endif
  elseif n==1
    if N==u
      C=[N];
    else
      C=zeros(0,n);
    endif
  else
    C1=BallsCombi(n,N-1,u-1);
    C1(:,1)+=1;
```

```
18      C2=BallsCombi(n-1,N,u-N);
19      C2=[zeros(size(C2)(1),1) C2];
20      C=[C2;C1];
21    endif
22  endfunction
23
```

D.5 Orthogonality of submodules defined by orthogonal vectors $\vec{\gamma}$

We note $V(N,l,\vec{\gamma})$ the submodules in the form of

$$V(N,l,\vec{\gamma}) = \text{span}\left(\left\{|\psi_{l_z}\rangle = \sum_{i=0}^{g(n,N,l)-1} \gamma_i \left|N,l,l_z,i\right\rangle_s, \forall l_z \text{ such that } |l_z| \leq l\right\}\right), \text{ with } \vec{\gamma} \in \mathbb{C}^{g(n,N,l)}. \tag{D.21}$$

If you consider two of those modules with different values of N or l, they are automatically orthogonal; indeed, they are included in two orthogonal spaces. Specifically, $V(N,l,\vec{\gamma})$ is included in the subspace of $\hat{N}$ with eigenvalue N, and because $\hat{N}$ is Hermitian, its eigenspaces are orthogonal; thus, $V(N)$ and $V(N')$ lie in two orthogonal spaces if $N \neq N'$, they are orthogonal.
Now, we compare $V(N,l,\vec{\gamma}_1)$ and $V(N,l,\vec{\gamma}_2)$, where $|v\rangle \in V(N,l,\vec{\gamma}_1), |w\rangle \in V(N,l,\vec{\gamma}_2)$, and $\vec{v}$ and $\vec{w}$ are the vectors containing their components. We can compute their scalar product

$$\begin{aligned}
\langle v|w\rangle &= \left(\sum_{l_z=-l}^{l} v_{l_z}^* \langle\psi_{l_z}(\vec{\gamma}_1)|\right)\left(\sum_{l'_z=-l}^{l} w_{l'_z} |\psi_{l'_z}(\vec{\gamma}_2)\rangle\right), \\
&= \sum_{l_z=-l}^{l}\sum_{l'_z=-l}^{l} v_{l_z}^* w_{l'_z} \langle\psi_{l_z}(c_1)|\psi_{l'_z}(c_2)\rangle, \\
&= \sum_{l_z=-l}^{l}\sum_{l'_z=-l}^{l} v_{l_z}^* w_{l'_z} \left(\sum_{i=0}^{g(n,N,l)-1} \gamma_{1\,i}^* \langle N,l,l_z,i|_s\right)\left(\sum_{j=0}^{g(n,N,l)-1} \gamma_{2j} \left|N,l,l'_z,j\right\rangle_s\right), \\
&= \sum_{l_z=-l}^{l}\sum_{l'_z=-l}^{l} v_{l_z}^* w_{l'_z} \left(\sum_{i=0}^{g(n,N,l)-1}\sum_{j=0}^{g(n,N,l)-1} \gamma_{1\,i}^*\gamma_{2j} \langle N,l,l_z,i|N,l,l'_z,j\rangle\right), \\
&= \sum_{l_z=-l}^{l}\sum_{l'_z=-l}^{l} v_{l_z}^* w_{l'_z} \left(\sum_{i=0}^{g(n,N,l)-1}\sum_{j=0}^{g(n,N,l)-1} \gamma_{1\,i}^*\gamma_{2j}\delta_{ij}\delta_{l_z l'_z}\right), \\
&= \sum_{l_z=-l}^{l} v_{l_z}^* w_{l_z} \left(\sum_{i=0}^{g(n,N,l)-1} \gamma_{1\,i}^*\gamma_{2i}\right), \\
&= (\vec{v}\cdot\vec{w})\,(\vec{\gamma}_1\cdot\vec{\gamma}_2).
\end{aligned}$$

The expression that we obtain is convenient. The scalar product of two vectors in two different spaces $V(N_1,l_1,\vec{\gamma}_1)$ and $V(N_2,l_2,\vec{\gamma}_2)$ is zero if $N_1 \neq N_2$ or $l_1 \neq L_2$, else it is the product of the scalar product of the defining vectors $\vec{\gamma}_i$ and the scalar product of the vector of their components.

D.6 Implementation of the general algorithm in Octave

We present here the implementation of the general algorithm of Section 4.4 on Octave. The coefficients for the basis change are stocked in the matrix C (and must be related to the matrix V to find the corresponding vector in the number basis). It is not optimized, but is implemented in order to be as close as possible to the algorithm we presented.

```
clear, close all, clc

% step 1
n = 4; % number of modes
N = 7; % number of particles

hs = (mod(n + 1, 2) && mod(N, 2)) / 2; % value to correct the indices for half
↪  spin valued

l_z_max = (n - 1) / 2 * N; % maximum value reachable for the spin
Lm = sqrt((n - (1:n-1)).* (1:n-1)); % non zero coefficients of the operator L_-
V = zeros(l_z_max + 1 - hs, 1, n); % V(lz + 1, num, ind) This matrix contains
↪  all the vectors of the number basis corresponding to a value Lz. They are
↪  all referenced by a number num. Therefore V(lz + 1, num, ind) gives the
↪  number of particles in the mode ind for the num_th vector of a given lz
C = zeros(l_z_max + 1 - hs, 1, l_z_max + 1 - hs, 2); % C(l + 1, C, lz + 1, num)
↪  contains the coefficients for the new basis lz.C(l+1, C, lz+1, num) gives
↪  the coefficients for a vector | N, l, lz, C > _s for the vector num (cf
↪  array V)
g = zeros(1, l_z_max + 1 - hs); % function g(lz) in the notes

% step 2
V(l_z_max + 1 - hs, 1, end) = N; % only vector with lz=l_z_max:| 0, 0, ..., N
↪  >_n
C(l_z_max + 1 - hs, 1, l_z_max + 1 - hs, 1) = 1; % identification | N, l_z_max,
↪  l_z_max, 1 > _s = | 0, 0, ..., N > _n
g(end) = 1;
gt = g; % dimension of the eigenspaces of lz

% step 3
for lz=l_z_max:-1: 1 % lz est en fait lz + 1
  count = 0;
  [n1, n2, n3] = size(V);
  % step 3a and 3b
  for num=1:n2
    for ind=2:n
      if V(lz + 1 - hs, num, ind) > 0
        VP = V(lz + 1 - hs, num,:);
        VP(ind) -= 1;
        VP(ind - 1) += 1;
        test = 0;
        for i=1:count
          if sum(VP == V(lz - hs, i,: )) == n
            for l=lz+1:l_z_max + 1
```

```
36                for j=1:g(l - hs)
37                  C(l - hs, j, lz - hs, i) += C(l - hs, j, lz + 1 - hs, num) *
                     ↪  sqrt( V(lz + 1 - hs, num, ind) * (V(lz + 1 - hs, num, ind -
                     ↪  1) + 1)) * Lm(ind - 1); % creation of the first part of the
                     ↪  new basis by applying L -
38                  test = 1;
39                endfor
40              endfor
41            break;
42            endif
43          endfor
44          if !test % In this case we find a new vector of the number basis we had
             ↪  not considered
45            count += 1;
46            V(lz - hs, count,:)=VP;
47            for l=lz+1:l_z_max + 1
48              for j=1:g(l - hs)
49                C(l - hs, j, lz - hs, count) = C(l - hs, j, lz + 1 - hs, num) *
                   ↪  sqrt( V(lz + 1 - hs, num, ind) * (V(lz + 1 - hs, num, ind - 1)
                   ↪  + 1)) * Lm(ind - 1);
50              endfor
51            endfor
52          endif
53        endif
54      endfor
55    endfor
56    % normalization
57    for l=lz+1:l_z_max + 1
58      for j=1:g(l - hs)
59        C(l - hs, j, lz - hs,:)=C(l - hs, j, lz - hs,:) / sqrt(sum(C(l - hs, j, lz
           ↪  - hs,:).^ 2));
60      endfor
61    endfor
62    % Step 3C
63    gt(lz - hs) = count;
64    g(lz - hs) = gt(lz - hs) - gt(lz + 1 - hs);
65    for j=1:g(lz - hs) % Gram-Schmidt to construct the new vectors
66      C(lz - hs, j, lz - hs, 1: n2)=rand(1, n2);
67      C(lz - hs, j, lz - hs,:)=C(lz - hs, j, lz - hs,:) / sqrt(sum(C(lz - hs, j,
         ↪  lz - hs,:).^ 2));
68      for l=lz+1:l_z_max + 1
69        for k=1:g(l - hs)
70          C(lz - hs, j, lz - hs,:) -= sum(C(lz - hs, j, lz - hs,:).*C(l - hs, k,
             ↪  lz - hs,:))*C(l - hs, k, lz - hs,:);
71          C(lz - hs, j, lz - hs,:)=C(lz - hs, j, lz - hs,:) / sqrt(sum(C(lz - hs,
             ↪  j, lz - hs,:).^ 2));
72        endfor
73      endfor
74      for k=1:j - 1
```

```
75        C(lz - hs, j, lz - hs,:) -= sum(C(lz - hs, j, lz - hs,:).*C(lz - hs, k, lz
          ↪  - hs,:))*C(lz - hs, k, lz - hs,:);
76        C(lz - hs, j, lz - hs,:)=C(lz - hs, j, lz - hs,:) / sqrt(sum(C(lz - hs, j,
          ↪  lz - hs,:).^ 2));
77      endfor
78    endfor
79  endfor
80  % printing of the results
81  for lz=l_z_max+1:-1: 1
82    for l=l_z_max+1:-1: lz
83      for i=1:g(l - hs)
84        printf(sprintf('|%d,%d,%d,%d>_s = ', N, l - 1, lz - 1, i))
85        for num=1:gt(lz - hs)
86          printf(sprintf('%d|', C(l - hs, i, lz - hs, num)))
87          for j=1:n
88            printf(sprintf('%d', V(lz - hs, num, j)))
89            if j != n
90              printf(',')
91            endif
92          endfor
93          printf('>_n ')
94          if num != gt(lz - hs) && (C(l - hs, i, lz - hs, num + 1) >= 0)
95            printf('+')
96          endif
97        endfor
98        printf('\n')
99      endfor
100   endfor
101 endfor
```

D.7 Spin Basis for 3 modes and 6 particles

We computed explicitly the vectors of the spin basis for 3-mode and 6 particles using the general algorithm described in Section 4.4. If you use algorithm of Appendix D.6, the results are the same.

$$
\begin{aligned}
|6,6,6\rangle_s &= |0,0,6\rangle_n \,, \\
|6,6,5\rangle_s &= |0,1,5\rangle_n \,, \\
|6,6,4\rangle_s &= \frac{1}{\sqrt{11}} \left(|1,0,5\rangle_n + \sqrt{10}\, |0,2,4\rangle_n \right) , \\
|6,4,4\rangle_s &= \frac{1}{\sqrt{11}} \left(\sqrt{10}\, |1,0,5\rangle_n - |0,2,4\rangle_n \right) , \\
|6,6,3\rangle_s &= \frac{1}{\sqrt{11}} \left(\sqrt{3}\, |1,1,4\rangle_n + \sqrt{8}\, |0,3,3\rangle_n \right) , \\
|6,4,3\rangle_s &= \frac{1}{\sqrt{11}} \left(\sqrt{8}\, |1,1,4\rangle_n - \sqrt{3}\, |0,3,3\rangle_n \right) , \\
|6,6,2\rangle_s &= \frac{1}{\sqrt{33}} \left(|2,0,4\rangle_n + 4\, |1,2,3\rangle + 4\, |0,4,2\rangle_n \right) , \\
|6,4,2\rangle_s &= \frac{1}{\sqrt{77}} \left(4\, |2,0,4\rangle_n + 5\, |1,2,3\rangle - 6\, |0,4,2\rangle_n \right) , \\
|6,2,2\rangle_s &= \frac{1}{\sqrt{21}} \left(4\, |2,0,4\rangle_n - 2\, |1,2,3\rangle + |0,4,2\rangle_n \right) , \\
|6,6,1\rangle_s &= \frac{1}{\sqrt{33}} \left(\sqrt{5}\, |2,1,3\rangle_n + \sqrt{20}\, |1,3,2\rangle_n + \sqrt{8}\, |0,5,1\rangle_n \right) , \\
|6,4,1\rangle_s &= \frac{1}{\sqrt{77}} \left(\sqrt{35}\, |2,1,3\rangle_n + \sqrt{1}\, |1,3,2\rangle_n - \sqrt{40}\, |0,5,1\rangle_n \right) , \\
|6,2,1\rangle_s &= \frac{1}{\sqrt{21}} \left(\sqrt{8}\, |2,1,3\rangle_n - \sqrt{8}\, |1,3,2\rangle_n + \sqrt{5}\, |0,5,1\rangle_n \right) , \\
|6,6,0\rangle_s &= \frac{1}{\sqrt{231}} \left(\sqrt{5}\, |3,0,3\rangle_n + \sqrt{90}\, |2,2,2\rangle_n + \sqrt{120}\, |1,4,1\rangle_n + \sqrt{16}\, |0,6,0\rangle_n \right) , \\
|6,4,0\rangle_s &= \frac{1}{\sqrt{385}} \left(\sqrt{54}\, |3,0,3\rangle_n + \sqrt{147}\, |2,2,2\rangle_n - \sqrt{64}\, |1,4,1\rangle_n - \sqrt{120}\, |0,6,0\rangle_n \right) , \\
|6,2,0\rangle_s &= \frac{1}{\sqrt{21}} \left(\sqrt{16}\, |3,0,3\rangle_n + \sqrt{0}\, |2,2,2\rangle_n - \sqrt{3}\, |1,4,1\rangle_n + \sqrt{10}\, |0,6,0\rangle_n \right) , \\
|6,0,0\rangle_s &= \frac{1}{\sqrt{35}} \left(\sqrt{16}\, |3,0,3\rangle_n - \sqrt{8}\, |2,2,2\rangle_n + \sqrt{6}\, |1,4,1\rangle_n - \sqrt{5}\, |0,6,0\rangle_n \right) .
\end{aligned}
$$

The negative l_z vectors can be obtained by symmetry, inverting the number of particles in each mode. For instance:

$$
|6,4,-1\rangle_s = \frac{1}{\sqrt{77}} \left(\sqrt{35}\, |3,1,2\rangle_n + \sqrt{1}\, |2,3,1\rangle_n - \sqrt{40}\, |1,5,0\rangle_n \right) .
$$

D.8 Demonstration for the scalar products of numbers between different number basis

In this section we compute

$$\langle n_1, n_2, ..., n_n | m_1, m_2, ..., m_n \rangle \,, \tag{D.22}$$

where $|n_1, n_2, ..., n_n\rangle$ and $|m_1, m_2, ..., m_n\rangle$ are two Fock vectors corresponding to two different Fock bases, for whose creation and annihilation operators are related by the relation

$$\hat{b}_u \equiv U^\dagger_{u\beta} \hat{a}_\beta \quad \text{and} \quad \hat{b}^\dagger_t \equiv \hat{a}^\dagger_\alpha U_{\alpha t} \tag{D.23}$$

To do this, we need to develop the states with their definitions in Equation (2.86):

$$\begin{aligned}
\langle n_1, n_2, ..., n_n | m_1, m_2, ..., m_n \rangle &= \langle 0 | \left(\prod_\alpha^n \sqrt{n_j!}^{-1} (\hat{a}_\alpha)^{n_\alpha} \right) \left(\prod_i^n \sqrt{m_i!}^{-1} \left(\hat{b}^\dagger_i \right)^{m_i} \right) |0\rangle \\
&= \left(\prod_\alpha^n \sqrt{n_\alpha!} \right)^{-1} \left(\prod_i^n \sqrt{m_i!} \right)^{-1} \langle 0 | \left(\prod_\beta^n \hat{a}^{n_\beta}_\beta \right) \left(\prod_j^n \hat{b}^{\dagger^{m_j}}_j \right) |0\rangle
\end{aligned} \tag{D.24}$$

Let us now focus on the braket. We consider $N = \sum_n \beta n_\beta$ and $M = \sum_j m_j$ and $\vec{\beta}$ and $\vec{j}$ the vectors explained in Section 4.7.

$$\begin{aligned}
\langle 0 | \left(\prod_\beta^n \hat{a}^{n_\beta}_\beta \right) \left(\prod_j^n \hat{b}^{\dagger^{m_j}}_j \right) |0\rangle &= \langle 0 | \left(\prod_a^N \hat{a}_{\beta_a} \right) \left(\prod_b^M \hat{b}^\dagger_{j_b} \right) |0\rangle \\
&= \langle 0 | \left(\prod_a^N \hat{a}_{\beta_a} \right) \left(\prod_b^M \sum_\gamma^n \hat{a}^\dagger_\gamma U_{\gamma j_b} \right) |0\rangle \\
&= \langle 0 | \left(\prod_a^N \hat{a}_{\beta_a} \right) \left(\sum_{\vec{k} \in \{1,...,n\}^M} \prod_b^M \hat{a}^\dagger_{k_b} U_{k_b j_b} \right) |0\rangle \\
&= \sum_{\vec{k} \in \{1,...,n\}^M} \langle 0 | \left(\prod_a^N \hat{a}_{\beta_a} \right) \left(\prod_b^M \hat{a}^\dagger_{k_b} U_{k_b j_b} \right) |0\rangle \\
&= \sum_{\vec{k} \in \{1,...,n\}^M} \left(\prod_c^M U_{k_c j_c} \right) \langle 0 | \left(\prod_a^N \hat{a}_{\beta_a} \right) \left(\prod_b^M \hat{a}^\dagger_{k_b} \right) |0\rangle \\
&= \sum_{\vec{k} \in \{1,...,n\}^M} \left(\prod_c^M U_{k_c j_c} \right) \langle 0 | \left(\prod_\beta^n \hat{a}^{n_\beta}_\beta \right) \left(\prod_\gamma^n \hat{a}^{\dagger^{n_\gamma(\vec{k})}}_\gamma \right) |0\rangle \\
&= \sum_{\vec{k} \in \{1,...,n\}^M} \left(\prod_c^M U_{k_c j_c} \right) \langle 0 | \left(\prod_\beta^n \hat{a}^{n_\beta}_\beta \right) \left(\prod_\gamma^n \hat{a}^{\dagger^{n_\gamma(\vec{k})}}_\gamma \right) |0\rangle \\
&= \sum_{\vec{k} \in \{1,...,n\}^M} \left(\prod_c^M U_{k_c j_c} \right) \prod_\beta^n \delta_{n_\beta, n_\gamma(\vec{k})} n_\beta! \\
&= \delta_{N,M} \sum_{\vec{\sigma} \in \mathrm{Perm}(\vec{\beta})} \left(\prod_c^N U_{\sigma_c j_c} \right) \prod_\beta^n n_\beta!
\end{aligned}$$

$$= \delta_{N,M} \prod_{\beta}^{n} n_\beta! \sum_{\vec{\sigma} \in \mathrm{Perm}(\vec{\beta})} \left(\prod_{c}^{N} U_{\sigma_c j_c} \right) \tag{D.25}$$

Finally, the matrix element is given by

$$\begin{aligned} \langle n_1, n_2, ..., n_n | m_1, m_2, ..., m_n \rangle &= \delta_{N,M} \prod_{i}^{n} \sqrt{\frac{n_i!}{m_i!}} \sum_{\vec{\sigma} \in \mathrm{Perm}(\vec{\beta})} \left(\prod_{c}^{N} U_{\sigma_c j_c} \right) \\ &= \delta_{N,M} \prod_{i}^{n} \sqrt{\frac{n_i!}{m_i!}} \sum_{\vec{\sigma} \in \mathrm{Perm}(\vec{j})} \left(\prod_{c}^{N} U_{\beta_c \sigma_c} \right) \end{aligned} \tag{D.26}$$