Letter

Gaussian work extraction from random Gaussian states is nearly impossible

Uttam Singh,[1,2,3,*] Jarosław K. Korbicz,[2,†] and Nicolas J. Cerf[1,4,‡]

[1] *Centre for Quantum Information and Communication, École polytechnique de Bruxelles, CP 165, Université libre de Bruxelles, 1050 Brussels, Belgium*

[2] *Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotnikow 32/46, 02-668 Warsaw, Poland*

[3] *Centre of Quantum Science and Technology, International Institute of Information Technology, Hyderabad 500032, India*

[4] *James C. Wyant College of Optical Sciences, University of Arizona, Tucson, Arizona 85721, USA*

(Received 7 December 2022; accepted 7 June 2023; published 17 July 2023)

Quantum thermodynamics can be naturally phrased as a theory of quantum state transformation and energy exchange for small-scale quantum systems undergoing thermodynamical processes, thereby making the resource theoretical approach very well suited. A key resource in thermodynamics is the extractable work, forming the backbone of thermal engines. Therefore it is of interest to characterize quantum states based on their ability to serve as a source of work. From a near-term perspective, quantum optical setups turn out to be ideal test beds for quantum thermodynamics; so it is important to assess work extraction from quantum optical states. Here, we show that Gaussian states are typically useless for Gaussian work extraction. More specifically, by exploiting the "concentration of measure" phenomenon, we prove that the probability that the Gaussian extractable work from a zero-mean energy-bounded multimode random Gaussian state is nonzero is exponentially small. This result can be thought of as an ϵ-no-go theorem for work extraction from Gaussian states under Gaussian unitaries, thereby revealing a fundamental limitation on the quantum thermodynamical usefulness of Gaussian components.

DOI: 10.1103/PhysRevResearch.5.L032010

Introduction. In the wake of the rapid technological advancements making the control and efficient manipulation of single quantum systems experimentally possible, it has become necessary to address the energetics of nanoscale devices [1–9]. Quantum thermodynamics is a burgeoning field of research broadly aimed at systematically addressing this question and, in particular, at challenging the applicability of classical thermodynamics at atomic scales, where quantum effects are inescapable [10–15]. A number of approaches to a theory of quantum thermodynamics have been developed, including, notably, a quantum resource-theory-based formalism [16–21], a purely information theoretic framework [22,23], and open-systems dynamics [24,25] (see also a recent book [10]). To complement the theoretical efforts towards quantum thermodynamics, there are also exciting new experiments [8,9,26,27] that confirm the distinctive features of quantum engines that have been theoretically predicted. Furthermore, quantum effects have been shown to offer advantages in charging quantum batteries [28] and in heat bath algorithmic cooling [29,30].

Despite tremendous experimental progress in designing quantum thermal machines, quantum thermodynamics is still largely a theoretical endeavor, and more experimental models are needed to confirm the theoretical predictions. It is well established that Gaussian quantum optical states can readily be prepared in the laboratory and Gaussian quantum operations can be implemented efficiently; hence quantum optical setups form a uniquely suited test bed for quantum thermodynamics (see, e.g., Ref. [7]). Given that these are central features of quantum thermodynamics, work extraction and battery charging have then been investigated in Refs. [31,32] when restricted to Gaussian operations. More generally, a theory of Gaussian work extraction from multipartite Gaussian states has also been developed in Ref. [20]. Interestingly, the total amount of work that can be extracted using Gaussian unitaries from a (zero-mean) multipartite Gaussian state was proven to be equal to the difference between the trace and symplectic trace of the covariance matrix [20] [see Eq. (3)]. In order to benchmark the experimental usefulness of such a Gaussian framework for quantum thermodynamics, it is therefore essential to resolve the question of what is the amount of work that can be extracted with Gaussian unitaries if we start from a random multimode Gaussian state? Here, we solve this question by exploiting the "concentration of measure" phenomenon [33], which states, broadly speaking, that a sufficiently smooth function on a measurable probability space concentrates around its expected value (see also Refs. [34–44]).

We start by introducing a procedure to sample energy-bounded random covariance matrices corresponding to a uniform measure on the set of multipartite Gaussian states,

*uttam@iiit.ac.in
†jkorbicz@cft.edu.pl
‡ncerf@ulb.ac.be

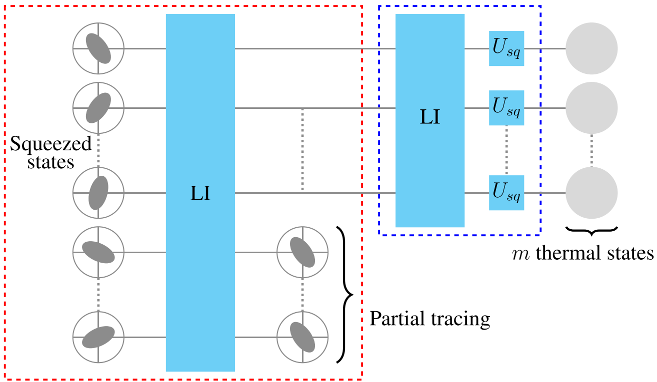

FIG. 1. Schematic of Gaussian work extraction from random Gaussian states. The dashed red rectangle represents the preparation of energy-bounded m-mode random Gaussian mixed states starting from the tensor product of n random squeezed vacuum states, while the dashed blue rectangle represents Gaussian work extraction. In both rectangles, the LI box stands for a linear interferometer (an array of beam splitters and phase shifters). Our main result is the proof that the state emerging from the dashed red rectangle is typically a product of thermal states and hence no work can be extracted by Gaussian means (see Theorem 1).

following Refs. [39,44]. As a first technical result, we then show that such random covariance matrices are, typically, locally thermal. Building on this, we prove that the probability that the Gaussian extractable work from a zero-mean energy-bounded random Gaussian state is nonzero is exponentially small. This can be interpreted as an ϵ-no-go theorem for work extraction from random Gaussian states under Gaussian unitaries (see Fig. 1), where $\epsilon > 0$ denotes the work that could potentially be extracted. We then discuss the impact of this fundamental near impossibility on quantum thermodynamics in the Gaussian regime.

Gaussian states. Consider an n-mode bosonic system described by Hilbert space $\mathcal{H}_n := L^2(\mathbb{R})^{\otimes n}$ and let $\hat{\mathbf{x}} = (\hat{q}_1, \dots, \hat{q}_n, \hat{p}_1, \dots, \hat{p}_n)$ be the canonical position and momentum operators. They satisfy the canonical commutation relations $[\hat{x}_i, \hat{x}_j] = \mathrm{i}\,\Omega_{ij}\,\mathbb{I}_{\mathcal{H}_n} (i, j = 1, \dots, 2n)$, where we set $\hbar = 1$ and $\mathrm{i} = \sqrt{-1}$,

$$\Omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}, \tag{1}$$

and $\mathbb{I}_n$ is an $n \times n$ identity matrix. The Hamiltonian of the ith mode is given by $\hat{H}_i := (\hat{q}_i^2 + \hat{p}_i^2)/2$ assuming that the angular frequencies of all modes are equal to 1. For an arbitrary n-mode state ρ, the $2n$-dimensional mean vector (or coherence vector) $\bar{\mathbf{x}}$ is defined as $\bar{\mathbf{x}} := \langle \hat{\mathbf{x}} \rangle = \mathrm{Tr}(\rho\,\hat{\mathbf{x}})$, where the angular bracket $\langle \bullet \rangle$ denotes the expectation value with respect to ρ. Similarly, the $2n \times 2n$ real positive-definite covariance matrix Γ of state ρ is defined via the second-order moments as $\Gamma_{ij} := \frac{1}{2}\langle\{\hat{x}_i - \langle\hat{x}_i\rangle, \hat{x}_j - \langle\hat{x}_j\rangle\}\rangle$, where $\{\bullet, \bullet\}$ is the anticommutator. Gaussian states are states whose characteristic function is Gaussian; hence they are completely described by their mean vector and covariance matrix [45]. For example, a thermal state is a Gaussian state with $\bar{\mathbf{x}} = 0$ and $\Gamma = (\bar{n} + 1/2)\,\mathbb{I}_{2n}$, where $\bar{n}$ is the average photon number per mode.

Gaussian unitaries. Gaussian unitaries are defined as unitaries that map Gaussian states onto Gaussian states. In particular, a Gaussian unitary U in state space induces an affine map $(\mathcal{S}, \mathbf{d}) : \hat{\mathbf{x}} \to \mathcal{S}\hat{\mathbf{x}} + \mathbf{d}$ in the space of quadrature operators $\hat{\mathbf{x}}$, where $\mathcal{S} \in \mathrm{Sp}(2n, \mathbb{R})$ is a $2n \times 2n$ real symplectic matrix (such that $\mathcal{S}\,\boldsymbol{\Omega}\,\mathcal{S}^T = \boldsymbol{\Omega}$) and $\mathbf{d}$ is a $2n$-dimensional real vector (displacement vector) [45]. Thus a Gaussian unitary can be written as $U_{\mathcal{S},\mathbf{d}} = D_{\mathbf{d}}\,U_{\mathcal{S}}$, where $U_{\mathcal{S}}$ corresponds to the symplectic map $\hat{\mathbf{x}} \to \mathcal{S}\,\hat{\mathbf{x}}$ and the Weyl operator $D_{\mathbf{d}}$ corresponds to the map $\hat{\mathbf{x}} \to \hat{\mathbf{x}} + \mathbf{d}$. Under Gaussian unitaries, the first- and second-order moments transform as $\bar{\mathbf{x}} \to \mathcal{S}\bar{\mathbf{x}} + \mathbf{d}$ and $\Gamma \to \mathcal{S}\Gamma\mathcal{S}^T$. Of special importance to us are the energy-conserving (or passive) Gaussian unitaries, which induce orthogonal symplectic transformations $\mathcal{S} \in \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{O}(2n) \equiv \mathrm{K}_n$ on the quadrature operators, where $\mathrm{Sp}(2n, \mathbb{R})$ is the group of real $2n \times 2n$ symplectic matrices and $\mathrm{O}(2n)$ is the group of real orthogonal $2n \times 2n$ matrices. Physically, passive Gaussian unitaries comprise all linear-optical circuits, also called as linear interferometers (LIs). The other Gaussian unitaries that are relevant here are squeezers. For example, a single-mode squeezer induces the symplectic transformation $\mathcal{S} = \mathrm{diag}\{z, z^{-1}\}$, where $z = e^r$ with $r \in \mathbb{R}$ being the squeezing parameter.

Gaussian extractable work. The energy of an arbitrary quantum state only depends on the first two moments, $\bar{\mathbf{x}}$ and $\boldsymbol{\Gamma}$. In particular, the energy of an m-mode state ρ is simply given by $\mathrm{Tr}[\rho\,\hat{H}] = \frac{1}{2}(\mathrm{Tr}[\boldsymbol{\Gamma}] + |\bar{\mathbf{x}}|^2)$, where $\hat{H} = \sum_{i=1}^m \hat{H}_i$ is the total Hamiltonian [45]. Now, for any state ρ (Gaussian or otherwise), we define the Gaussian extractable work as the maximum decrease in energy under Gaussian unitaries. Given the decoupled structure of Gaussian unitaries as $U_{\mathcal{S},\mathbf{d}} = D_{\mathbf{d}}\,U_{\mathcal{S}} = U_{\mathcal{S}}\,D_{\mathbf{d}'}$, we can always separate the Gaussian extractable work into two components associated, respectively, with $\mathcal{S}$ and $\mathbf{d}'$. Starting from a state with $\bar{\mathbf{x}} \neq 0$, it is trivial to extract work first via displacement $D_{\mathbf{d}'}$ up to the point where $\bar{\mathbf{x}} = 0$, thereby making this component uninteresting (it can be viewed as classical). Thus we may restrict ourselves to zero-mean states with no loss of generality. Furthermore, using the Bloch-Messiah decomposition [46,47], $U_{\mathcal{S}}$ can be written as the concatenation of a linear interferometer (LI), a layer of single-mode squeezers, and a second LI. Since the latter leaves the energy unchanged, we may disregard it. Hence the Gaussian extractable work from a zero-mean state ρ is given by (see dashed blue rectangle in Fig. 1)

$$W(\rho) = \max_{\mathrm{LI}, U_{sq}} \mathrm{Tr}[\hat{H}(\rho - U_{sq}\,\mathrm{LI}[\rho]\,U_{sq}^{\dagger})], \tag{2}$$

where U_{sq} denotes the squeezing unitary which corresponds to the tensor product of single-mode squeezers. This maximization yields a particularly simple expression in the phase-space picture, namely [20],

$$W(\rho) \equiv \mathcal{W}(\boldsymbol{\Gamma}) = \tfrac{1}{2}(\mathrm{Tr}[\boldsymbol{\Gamma}] - \mathrm{STr}[\boldsymbol{\Gamma}]), \tag{3}$$

where $\mathrm{STr}[\boldsymbol{\Gamma}]$ denotes the symplectic trace of the covariance matrix $\boldsymbol{\Gamma}$, i.e., the sum of all its symplectic eigenvalues. Note that the Gaussian extractable work $W(\rho)$ is solely a function of $\boldsymbol{\Gamma}$, so we simply note it $\mathcal{W}(\boldsymbol{\Gamma})$.

Random sampling of Gaussian states. In order to establish the typical behavior of the Gaussian extractable work, we need to give a prescription for the random sampling of covariance

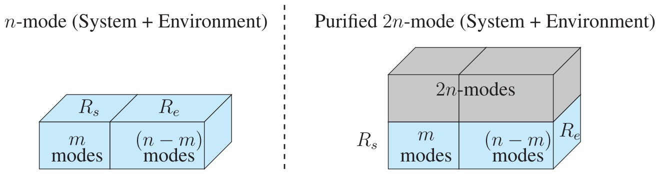

FIG. 2. The schematic on the left depicts the m-mode system of interest (from which work is extracted) denoted as the region R_s, which is part of an n-mode system. The remaining $N = n - m$ modes constitute the environment, denoted as the region R_e. Note that in the most general scenario, the state of the n-mode region $R_s + R_e$ can be mixed. However, from Proposition 1 of the Supplemental Material [50], an energy-bounded mixed Gaussian state can be purified into an energy-bounded pure Gaussian state. The schematic on the right represents such a purification of the n-mode region into a $2n$-mode region. Now, the state of the m-mode system of interest is obtained by performing a partial trace of $(2n - m)$ modes on a pure $2n$-mode state.

matrices. Consider first an n-mode pure Gaussian state. Its covariance matrix $\mathbf{\Gamma}$ can be obtained by applying some Gaussian unitary to the vacuum state; that is, it can be written as $\mathbf{\Gamma} = \mathcal{S}\mathcal{S}^T/2$, where $\mathcal{S} \in \mathrm{Sp}(2n, \mathbb{R})$. From the Bloch-Messiah decomposition, we have $\mathcal{S} = \mathbf{O}\,[Z(\mathbf{z}) \oplus Z^{-1}(\mathbf{z})]\,\mathbf{O}'$, where $\mathbf{O}, \mathbf{O}' \in \mathrm{K}_n$ and $Z(\mathbf{z}) \oplus Z^{-1}(\mathbf{z})$ is a collection of n single-mode squeezers and $Z(\mathbf{z}) = \mathrm{diag}\{z_1, \dots, z_n\}$ with $z_i \geqslant 1$ for all $1 \leqslant i \leqslant n$. Therefore the covariance matrix is written as

$$\mathbf{\Gamma} = \tfrac{1}{2}\mathbf{O}\,[J(\mathbf{z}) \oplus J^{-1}(\mathbf{z})]\,\mathbf{O}^T, \tag{4}$$

where $J(\mathbf{z}) = Z(\mathbf{z})^2$. Now, to define a random covariance matrix, we need to sample $\mathbf{O}$ and $J(\mathbf{z})$ with appropriate probability measures on their respective spaces. While K_n is a compact space and admits an invariant Haar measure, the space of matrices $J(\mathbf{z})$ is not compact and does not admit a natural invariant normalizable measure. To properly define the sampling of a random $\mathbf{\Gamma}$, we need some compactness constraint, which can be provided by imposing an energy bound, $\mathrm{Tr}[\mathbf{\Gamma}] \leqslant 2E$, where E is fixed (remember that $\bar{\mathbf{x}} = 0$). Following Refs. [39,44], we can sample random matrices $J(\mathbf{z})$ by randomly choosing the vector $\mathbf{z}$ via the flat Lebesgue measure on the set

$$\mathcal{G}_E := \left\{(z_1, \dots, z_n)\,|\, z_i \geqslant 1 \text{ and } \sum_{i=1}^{n}\left(z_i^2 + z_i^{-2}\right) \leqslant 4E\right\};$$

that is, we define the measure $d\mu_{\mathbf{z}} = dz_1 \cdots dz_n/\mathrm{Vol}(\mathcal{G}_E)$, where $\mathrm{Vol}(\mathcal{G}_E)$ is the volume of $\mathcal{G}_E$ [48]. Then, using the fact that K_n is isomorphic to the complex unitary group $\mathrm{U}(n) := \{U \in \mathbb{C}^{n\times n} : U^\dagger U = \mathbb{I}_n\}$, we generate random $\mathbf{O} \in \mathrm{K}_n$ via the invariant measure on K_n induced by the Haar measure on $\mathrm{U}(n)$. Second, a natural probability measure on the set of energy-bounded random Gaussian mixed states can be induced by performing a partial trace on the random pure state where $\mathbf{z}$ and $\mathbf{O}$ are sampled according to the above measures [49]. Note that a Gaussian purification argument can be used as illustrated in Fig. 2 in order to show that the full state can be taken to be pure with no loss of generality (see Proposition 1 of the Supplemental Material [50]). The partial trace of $N = n - m$ out of n modes (see Fig. 2) corresponds to the map $\mathbf{\Pi}_{m,N}$ on the covariance matrix defined as

$$\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_m := \Pi_{m,N}\,\mathbf{\Gamma}\,\mathbf{\Pi}_{m,N}, \tag{5}$$

where $\mathbf{\Pi}_{m,N} = \widetilde{\mathbf{\Pi}} \oplus \widetilde{\mathbf{\Pi}}$ and $\widetilde{\mathbf{\Pi}} = \mathrm{diag}\{\overbrace{1, \dots, 1}^{m}, \overbrace{0, \dots, 0}^{N=n-m}\}$. Let $\mathcal{L}_{m,E}$ be the resulting set of covariance matrices $\mathbf{\Gamma}_m$ for m-mode energy-constrained Gaussian mixed states, i.e., $\mathrm{Tr}[\mathbf{\Gamma}_m] \leqslant 2E$ for $\mathbf{\Gamma}_m \in \mathcal{L}_{m,E}$. We shall now establish the typicality of the extractable work in $\mathcal{L}_{m,E}$.

Typicality of Gaussian extractable work. We consider a physical scenario where the system of interest is interacting with an inaccessible large environment. This is a common setting in the theory of decoherence responsible for loss of coherence in quantum systems as a consequence of the partial trace of the environment [51]. In particular, in our case the system of interest is a zero-mean energy-bounded m-mode Gaussian system embedded in a large Gaussian environment comprising $N \gg m$ modes, so we must characterize the scaling in $n = m + N$ of the energy constraint on the full n-mode system as follows.

Definition 1. An m-mode random Gaussian state is said to be polynomially energy bounded of degree $\beta \geqslant 0$ if it results from partial tracing an n-mode random Gaussian pure state over $N = n - m$ modes and its covariance matrix can be written as [cf. Eq. (5)]

$$\mathbf{\Gamma}_m := \tfrac{1}{2}\,\Pi_{m,N}\,\mathbf{O}_n\,\widetilde{J}_n(\mathbf{z})\,\mathbf{O}_n^T\,\Pi_{m,N}, \tag{6}$$

where $\mathbf{O}_n \in \mathrm{K}_n$ and the sequence $\widetilde{J}_n(\mathbf{z}) := J_n(\mathbf{z}) \oplus J_n^{-1}(\mathbf{z})$ is such that $||\widetilde{J}_n(\mathbf{z})||_\infty = O(n^\beta)$ [52].

Since $\mathbf{O}_n$ is passive, the n-mode random Gaussian pure state of covariance matrix $\mathbf{\Gamma}_n := \frac{1}{2}\mathbf{O}_n\widetilde{J}_n(\mathbf{z})\mathbf{O}_n^T$ has energy $E_\beta = O(n^{\beta+1})$. Then, it can be used to generate a polynomially energy-bounded m-mode random Gaussian mixed state according to Definition 1 (see Fig. 2).

Before proving our main result, we first need to establish the asymptotic behavior of the eigenspectrum and symplectic eigenspectrum of the energy-bounded random covariance matrices $\mathbf{\Gamma}_m$ from Definition 1, which is the content of the following two lemmas (their proofs are provided in the Supplemental Material [50]).

Lemma 1. Let $\mathbf{\Gamma}_m$ be the covariance matrix of an m-mode polynomially energy-bounded random Gaussian state of degree $\beta < 1/4$ (see Definition 1). For universal constants $\gamma, \widetilde{\gamma} > 0$ such that $\epsilon > 2\gamma n^{\beta-1}$, the eigenvalues $\{\lambda_i\}_{i=1}^{2m}$ of $\mathbf{\Gamma}_m$ converge in probability to ν_{th}, i.e.,

$$\mathrm{Pr}\left[\sum_{i=1}^{2m}(\lambda_i - \nu_{\mathrm{th}})^2 > \epsilon\right] \leqslant \exp[-\widetilde{\gamma}\epsilon^2 n^{1-4\beta}], \tag{7}$$

where $\nu_{\mathrm{th}} = \mathrm{Tr}[\widetilde{J}_{2n}(\mathbf{z})]/(8n)$.

Lemma 2 [44]. Let $\mathbf{\Gamma}_m$ be the covariance matrix of an m-mode polynomially energy-bounded random Gaussian state of degree $\beta < 1/8$ (see Definition 1). For universal constants $C, c > 0$ such that $\epsilon > Cn^{\beta-1}$, the symplectic eigenvalues $\{\nu_i\}_{i=1}^{m}$ of $\mathbf{\Gamma}_m$ converge in probability to ν_{th}, i.e.,

$$\mathrm{Pr}\left[\sum_{i=1}^{m}\left(\nu_i^2 - \nu_{\mathrm{th}}^2\right)^2 > \epsilon\right] \leqslant \exp[-c\epsilon^2 n^{1-8\beta}], \tag{8}$$

where $\nu_{\mathrm{th}} = \mathrm{Tr}[\widetilde{J}_{2n}(\mathbf{z})]/(8n)$.

Our main result lies in the following ϵ-no-go theorem for the Gaussian extractable work from a polynomially energy-bounded random Gaussian state.

Theorem 1 (typicality of the Gaussian extractable work). Let $\mathbf{\Gamma}_m$ be the covariance matrix of an m-mode polynomially energy-bounded random Gaussian state of degree $\beta < 1/8$ resulting from performing a partial trace on a random n-mode state as in Definition 1. Then, for universal constants $\tilde{c}, C > 0$ such that $\epsilon > \sqrt{2Cmn^{\beta-1}}$, the Gaussian extractable work $\mathcal{W}(\mathbf{\Gamma}_m)$ satisfies

$$\Pr[\mathcal{W}(\mathbf{\Gamma}_m) > \epsilon] \leqslant \exp[-\tilde{c}\epsilon^4 n^{1-8\beta}]. \tag{9}$$

As noted above, the energy of $\mathbf{\Gamma}_n$ scales as $O(n^{\beta+1})$, so the energy of $\mathbf{\Gamma}_m$ scales as $O(n^{\beta})$ after tracing over the $N = \Omega(n)$ environmental modes [49]. The extractable work tolerance ϵ in Theorem 1 thus scales as $\sqrt{\frac{m}{n^{1-\beta}}}$, which goes to zero in the limit $n \to \infty$ as $\beta < 1/8$. Furthermore, the right-hand side of Eq. (9) in Theorem 1 exponentially goes to zero in the limit $n \to \infty$ as $\beta < 1/8$. These two elements make Theorem 1 meaningful.

Proof of the theorem. Using the Gaussian purification argument (see Proposition 1 of the Supplemental Material [50]), i.e., $n \mapsto 2n$, Eq. (6) becomes

$$\mathbf{\Gamma}_m := \tfrac{1}{2}\mathbf{\Pi}_{m,N+n}\mathbf{O}_{2n}\widetilde{J}_{2n}(\mathbf{z})\mathbf{O}_{2n}^T\mathbf{\Pi}_{m,N+n}, \tag{10}$$

where $\mathbf{O}_{2n} \in \mathrm{K}_{2n}$ is a $4n \times 4n$ orthogonal symplectic matrix. Also, we have (see Refs. [44,50])

$$\mathbf{O}_{2n} = P\begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}P^{-1}, \tag{11}$$

where $P = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{I}_{2n} & i\mathbb{I}_{2n} \\ i\mathbb{I}_{2n} & \mathbb{I}_{2n} \end{pmatrix}$ and U is a $2n \times 2n$ random unitary matrix. We define the function $T_m : \mathrm{U}(2n) \to \mathbb{R}$ of random unitary matrices as

$$T_m(U) := \sum_{k=1}^{2m} (\lambda_k - \nu_{\mathrm{th}})^2, \tag{12}$$

where $\{\lambda_k\}_{k=1}^{2m}$ are the eigenvalues of $\mathbf{\Gamma}_m \equiv \mathbf{\Gamma}_m(U)$ as defined from Eqs. (10) and (11) and ν_{th} is the average energy per mode of the $2n$-mode input pure state with covariance matrix $\frac{1}{2}\mathbf{O}_{2n}\widetilde{J}_{2n}(\mathbf{z})\mathbf{O}_{2n}^T$, that is, $\nu_{\mathrm{th}} = \mathrm{Tr}[\widetilde{J}_{2n}(\mathbf{z})]/(8n)$. By exploiting the concentration of measure, Lemma 1 then gives us an exponentially small upper bound on $\Pr[T_m(U) > \epsilon]$. Similarly, we define the function $\mathfrak{T}_m : \mathrm{U}(2n) \to \mathbb{R}$ as

$$\mathfrak{T}_m(U) := 2\sum_{k=1}^{m} \left(\nu_k^2 - \nu_{\mathrm{th}}^2\right)^2, \tag{13}$$

where $\{\nu_k\}_{k=1}^m$ are the symplectic eigenvalues of $\mathbf{\Gamma}_m(U)$, and use Lemma 2 to obtain an exponentially small upper bound on $\Pr[\mathfrak{T}_m(U) > \epsilon]$ (see also Ref. [44]). Thus, together, Lemmas 1 and 2 imply that energy-bounded random Gaussian states are typically locally thermal with the same average energy in each mode (since both the symplectic and regular eigenspectra concentrate around a thermal spectrum).

This is the key to our proof of the near impossibility of Gaussian work extraction from energy-bounded random Gaussian states. Consider the function $\Delta_m : \mathrm{U}(2n) \to \mathbb{R}$ as $\Delta_m(U) := T_m(U) + \mathfrak{T}_m(U)$. From Lemmas 1 and 2 we know that both $T(U)$ and $\mathfrak{T}(U)$ are Lipschitz continuous functions on $\mathrm{U}(2n)$ [50]. Then, for any two unitaries $U, V \in \mathrm{U}(2n)$, we have

$$\begin{aligned} |\Delta(U) - \Delta(V)| &\leqslant |T(U) - T(V)| + |\mathfrak{T}(U) - \mathfrak{T}(V)| \\ &\leqslant O(n^{4\beta})||U - V||_2. \end{aligned}$$

Thus $\Delta_m(U)$ is a Lipschitz continuous function on $\mathrm{U}(2n)$ with a Lipschitz constant given by $\theta\, n^{4\beta}$, where θ is a universal constant. Furthermore, we have

$$\begin{aligned} \mathbb{E}_U \Delta(U) &= \mathbb{E}_U T(U) + \mathbb{E}_U \mathfrak{T}(U) \\ &= O(n^{\beta-1}) \leqslant Cn^{\beta-1}, \end{aligned}$$

where C is a universal constant [50]. Next, we can exploit the concentration of measure for $\Delta_m(U)$ in a manner similar to that followed for $T_m(U)$ and $\mathfrak{T}_m(U)$. For universal constants $C, c' > 0$ such that $\delta > 2Cn^{\beta-1}$, we have

$$\begin{aligned} \Pr[\Delta(U) \geqslant \delta] &\leqslant \Pr\left[\Delta(U) > \frac{\delta}{2} + \mathbb{E}_U \Delta(U)\right] \\ &\leqslant \exp\left[-\frac{\delta^2 n}{48\theta^2 n^{8\beta}}\right] \\ &\leqslant \exp[-c'\delta^2 n^{1-8\beta}], \end{aligned} \tag{14}$$

where c' is a universal constant. Then, we express the Gaussian extractable work $\mathcal{W}(\mathbf{\Gamma}_m)$ as a function of $\Delta_m(U)$. Noting that $\mathcal{W}(\mathbf{\Gamma}_m) = |\mathcal{W}(\mathbf{\Gamma}_m)|$, we have

$$\begin{aligned} \mathcal{W}(\mathbf{\Gamma}_m) &= \left|\frac{1}{2}\sum_{k=1}^{2m}(\lambda_k - \nu_{\mathrm{th}}) + \sum_{k=1}^{m}(\nu_{\mathrm{th}} - \nu_k)\right| \\ &\leqslant \frac{1}{2}\left(\sum_{k=1}^{2m}|\lambda_k - \nu_{\mathrm{th}}| + 2\sum_{k=1}^{m}\left|\nu_k^2 - \nu_{\mathrm{th}}^2\right|\right) \\ &\leqslant \sqrt{m}\sqrt{\sum_{k=1}^{2m}|\lambda_k - \nu_{\mathrm{th}}|^2 + 2\sum_{k=1}^{m}\left|\nu_k^2 - \nu_{\mathrm{th}}^2\right|^2} \\ &= \sqrt{m\,\Delta_m(U)}, \end{aligned}$$

where the first inequality follows from the triangle inequality and the fact that $\nu_{\mathrm{th}} + \nu_k \geqslant 1$. The second inequality follows from the inequality $(\sum_{i=1}^n x_i)^2 \leqslant n\sum_{i=1}^n x_i^2$. Letting $\delta > 2Cn^{\beta-1}$ and using Eq. (14), we have

$$\begin{aligned} \Pr[\mathcal{W}(\mathbf{\Gamma}_m) \leqslant \sqrt{m\delta}] &\geqslant \Pr[\Delta_m(U) \leqslant \delta] \\ &\geqslant 1 - \exp[-c'\delta^2 n^{1-8\beta}]. \end{aligned}$$

Furthermore, letting $\epsilon = \sqrt{m\delta} > \sqrt{2Cmn^{\beta-1}}$, we have

$$\Pr[\mathcal{W}(\mathbf{\Gamma}_m) > \epsilon] \leqslant \exp[-\widetilde{c}\epsilon^4 n^{1-8\beta}],$$

where $\widetilde{c}$ is a universal constant, which concludes our proof. ■

Conclusion. We have established the near impossibility—in a strong sense—of extracting work from (zero-mean) polynomially energy-bounded random multimode Gaussian states using Gaussian unitaries. Qualitatively, this follows from the fact that these states are typically locally thermal as a consequence of the "concentration of measure" phenomenon, so one cannot extract work from such states. This is a probabilistic statement in the sense that there remains an exponentially small probability that it does not hold. In this regard, our ϵ-no-go theorem slightly contrasts with the well-known Gaussian no-go theorems, e.g., for Gaussian universal quantum computation [53], for the distillation of entanglement from Gaussian states using Gaussian local operations and classical communication [54–56], or for Gaussian quantum error correction [57].

Our findings reveal a fundamental limitation on the processing of Gaussian states using Gaussian operations and show that harnessing quantum thermodynamical processes is typically impossible in the Gaussian regime. This limitation even goes beyond the extractable work as a similar ϵ-no-go theorem can be proven for the single-mode relative entropy of activity (an alternative measure of the distance from a Gaussian thermal state defined in Ref. [20]) of random Gaussian states; see Ref. [58].

Keeping in mind that quantum optical setups are among the leading platforms for experimental quantum thermodynamics, our results point to an essential requirement to consider non-Gaussian (or even perhaps nontypical Gaussian) components in quantum thermodynamics. A natural question that arises in this regard is the following: Which nontypical or non-Gaussian states can get around the ϵ-no-go theorems presented here and hence be useful for work extraction? It would be very interesting to address this question, which, in turn, could open up exciting developments in quantum thermodynamics with non-Gaussian resources, a topic that has hardly been explored to date.

On a final note, our results are reminiscent to the de Finetti theorem for quantum states that are invariant under orthogonal symplectic transformations [59]. Such a de Finetti theorem states that if we perform a partial trace on a state that obeys this invariance, the resulting state approaches a mixture of products of (independent and identically distributed) thermal states. However, we note that the random states considered here are not, in general, invariant under orthogonal symplectic transformations; yet we are able to show that they concentrate around a product of thermal states with the same mean photon number. In fact, we have a stronger bound here as the error probability is exponentially small, while it is polynomially small in the de Finetti theorem of Ref. [59]. On the other hand, our result is only concerned with Gaussian states, while the de Finetti theorem of Ref. [59] holds for any state with the right invariance. This suggests that it would be interesting to explore the relationship between de Finetti theorems and the "concentration of measure" phenomenon.

Acknowledgments. U.S. and J.K.K. acknowledge support from the Polish National Science Center (NCN; Grant No. 2019/35/B/ST2/01896). N.J.C. acknowledges support from Fonds de la Recherche Scientifique - FNRS under Grant No. T.0224.18 and from the European Union under project ShoQC within ERA-NET Cofund in Quantum Technologies (QuantERA) program.

[1] H. E. D. Scovil and E. O. Schulz-DuBois, Three-Level Masers as Heat Engines, Phys. Rev. Lett. **2**, 262 (1959).
[2] J. E. Geusic, E. O. Schulz-DuBios, and H. E. D. Scovil, Quantum equivalent of the carnot cycle, Phys. Rev. **156**, 343 (1967).
[3] J. Howard, Molecular motors: Structural adaptations to cellular functions, Nature (London) **389**, 561 (1997).
[4] M. O. Scully, Quantum Afterburner: Improving the Efficiency of an Ideal Heat Engine, Phys. Rev. Lett. **88**, 050602 (2002).
[5] M. O. Scully, M. S. Zubairy, G. S. Agarwal, and H. Walther, Extracting work from a single heat bath via vanishing quantum coherence, Science **299**, 862 (2003).
[6] P. Hänggi and F. Marchesoni, Artificial Brownian motors: Controlling transport on the nanoscale, Rev. Mod. Phys. **81**, 387 (2009).
[7] A. Dechant, N. Kiesel, and E. Lutz, All-Optical Nanomechanical Heat Engine, Phys. Rev. Lett. **114**, 183602 (2015).
[8] J. Roßnagel, S. T. Dawkins, K. N. Tolazzi, O. Abah, E. Lutz, F. Schmidt-Kaler, and K. Singer, A single-atom heat engine, Science **352**, 325 (2016).
[9] J. Klatzow, J. N. Becker, P. M. Ledingham, C. Weinzetl, K. T. Kaczmarek, D. J. Saunders, J. Nunn, I. A. Walmsley, R. Uzdin, and E. Poem, Experimental Demonstration of Quantum Effects in the Operation of Microscopic Heat Engines, Phys. Rev. Lett. **122**, 110601 (2019).
[10] *Thermodynamics in the Quantum Regime*, edited by F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso, Fundamental Theories of Physics (Springer, Cham, Switzerland, 2018).
[11] S. Deffner and S. Campbell, *Quantum Thermodynamics* (Morgan & Claypool, Kentfield, CA, 2019).
[12] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, The role of quantum information in thermodynamics—a topical review, J. Phys. A: Math. Theor. **49**, 143001 (2016).
[13] S. Vinjanampathy and J. Anders, Quantum thermodynamics, Contemp. Phys. **57**, 545 (2016).
[14] M. Lostaglio, An introductory review of the resource theory approach to thermodynamics, Rep. Prog. Phys. **82**, 114001 (2019).
[15] P. Talkner and P. Hänggi, Colloquium: Statistical mechanics and thermodynamics at strong coupling: Quantum and classical, Rev. Mod. Phys. **92**, 041002 (2020).
[16] M. Horodecki and J. Oppenheim, Fundamental limitations for quantum and nanoscale thermodynamics, Nat. Commun. **4**, 2059 (2013).
[17] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, Resource Theory of Quantum States Out of Thermal Equilibrium, Phys. Rev. Lett. **111**, 250404 (2013).
[18] P. Skrzypczyk, A. J. Short, and S. Popescu, Work extraction and thermodynamics for individual quantum systems, Nat. Commun. **5**, 4185 (2014).
[19] F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, The second laws of quantum thermodynamics, Proc. Natl. Acad. Sci. USA **112**, 3275 (2015).
[20] U. Singh, M. G. Jabbour, Z. Van Herstraeten, and N. J. Cerf, Quantum thermodynamics in a multipartite setting: A resource theory of local Gaussian work extraction for multimode bosonic systems, Phys. Rev. A **100**, 042104 (2019).

[21] U. Singh, S. Das, and N. J. Cerf, Partial order on passive states and Hoffman majorization in quantum thermodynamics, Phys. Rev. Res. **3**, 033091 (2021).

[22] M. N. Bera, A. Riera, M. Lewenstein, and A. Winter, Generalized laws of thermodynamics in the presence of correlations, Nat. Commun. **8**, 2180 (2017).

[23] M. N. Bera, A. Riera, M. Lewenstein, Z. B. Khanian, and A. Winter, Thermodynamics as a consequence of information conservation, Quantum **3**, 121 (2019).

[24] R. Alicki, The quantum open system as a model of the heat engine, J. Phys. A: Math. Gen. **12**, L103 (1979).

[25] R. Uzdin, A. Levy, and R. Kosloff, Equivalence of Quantum Heat Machines, and Quantum-Thermodynamic Signatures, Phys. Rev. X **5**, 031044 (2015).

[26] G. Clos, D. Porras, U. Warring, and T. Schaetz, Time-Resolved Observation of Thermalization in an Isolated Quantum System, Phys. Rev. Lett. **117**, 170401 (2016).

[27] G. Maslennikov, S. Ding, R. Hablützel, J. Gan, A. Roulet, S. Nimmrichter, J. Dai, V. Scarani, and D. Matsukevich, Quantum absorption refrigerator with trapped ions, Nat. Commun. **10**, 202 (2019).

[28] D. Ferraro, M. Campisi, G. M. Andolina, V. Pellegrini, and M. Polini, High-Power Collective Charging of a Solid-State Quantum Battery, Phys. Rev. Lett. **120**, 117702 (2018).

[29] L. J. Schulman, T. Mor, and Y. Weinstein, Physical Limits of Heat-Bath Algorithmic Cooling, Phys. Rev. Lett. **94**, 120501 (2005).

[30] N. A. Rodríguez-Briones, J. Li, X. Peng, T. Mor, Y. Weinstein, and R. Laflamme, Heat-bath algorithmic cooling with correlated qubit-environment interactions, New J. Phys. **19**, 113047 (2017).

[31] E. G. Brown, N. Friis, and M. Huber, Passivity and practical work extraction using Gaussian operations, New J. Phys. **18**, 113028 (2016).

[32] N. Friis and M. Huber, Precision and work fluctuations in Gaussian battery charging, Quantum **2**, 61 (2018).

[33] M. Ledoux, *The Concentration of Measure Phenomenon*, Mathematical Surveys and Monographs (American Mathematical Society, Providence, RI, 2005), Vol. 89.

[34] G. W. Anderson, A. Guionnet, and O. Zeitouni, *An Introduction to Random Matrices*, Cambridge Studies in Advanced Mathematics (Cambridge University Press, Cambridge, 2009).

[35] P. Hayden, D. W. Leung, and A. Winter, Aspects of generic entanglement, Commun. Math. Phys. **265**, 95 (2006).

[36] S. Popescu, A. J. Short, and A. Winter, Entanglement and the foundations of statistical mechanics, Nat. Phys. **2**, 754 (2006).

[37] G. Adesso, Generic Entanglement and Standard Form for N-Mode Pure Gaussian States, Phys. Rev. Lett. **97**, 130502 (2006).

[38] A. Serafini and G. Adesso, Standard forms and entanglement engineering of multimode Gaussian states under local operations, J. Phys. A: Math. Theor. **40**, 8041 (2007).

[39] A. Serafini, O. C. O. Dahlsten, and M. B. Plenio, Teleportation Fidelities of Squeezed States from Thermodynamical State Space Measures, Phys. Rev. Lett. **98**, 170501 (2007).

[40] B. Collins and P. Śniady, Integration with respect to the Haar measure on unitary, orthogonal and symplectic group, Commun. Math. Phys. **264**, 773 (2006).

[41] B. Collins and I. Nechita, Random quantum channels I: Graphical calculus and the Bell state phenomenon, Commun. Math. Phys. **297**, 345 (2010).

[42] B. Collins, C. E. González-Guillén, and D. Pérez-García, Matrix product states, random matrix theory and the principle of maximum entropy, Commun. Math. Phys. **320**, 663 (2013).

[43] M. Raginsky and I. Sason, Concentration of measure inequalities in information theory, communications, and coding, Found. Trends Commun. Inf. Theory **10**, 1 (2013).

[44] M. Fukuda and R. König, Typical entanglement for Gaussian states, J. Math. Phys. **60**, 112203 (2019).

[45] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. **84**, 621 (2012).

[46] Arvind, B. Dutta, N. Mukunda, and R. Simon, The real symplectic groups in quantum mechanics and optics, Pramana **45**, 471 (1995).

[47] S. L. Braunstein, Squeezing as an irreducible resource, Phys. Rev. A **71**, 055801 (2005).

[48] We will not be needing this probability measure in this Research Letter as we will prove our main result for any choice of vector **z** satisfying the energy constraint (see also Ref. [44]).

[49] We note that an alternative way to define a probability measure on the set of nonzero-mean energy-bounded Gaussian mixed states has been considered in Ref. [60].

[50] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevResearch.5.L032010 for additional results, detailed proofs, and extended discussions.

[51] M. Schlosshauer, *Decoherence* (Springer, Berlin, 2007).

[52] We say that a function $f : \mathbb{R} \to \mathbb{R}^+$ scales as $O(n)$ when for all sufficiently large $x \in \mathbb{R}$, there exists a constant $c > 0$ such that $f(x) \leqslant cn$. Similarly, we say that a function $f : \mathbb{R} \to \mathbb{R}^+$ scales as $\Omega(n)$ when for all sufficiently large $x \in \mathbb{R}$, there exists a constant $c > 0$ such that $f(x) \geqslant cn$.

[53] S. Lloyd and S. L. Braunstein, Quantum Computation over Continuous Variables, Phys. Rev. Lett. **82**, 1784 (1999).

[54] J. Eisert, S. Scheel, and M. B. Plenio, Distilling Gaussian States with Gaussian Operations is Impossible, Phys. Rev. Lett. **89**, 137903 (2002).

[55] J. Fiurášek, Gaussian Transformations and Distillation of Entangled Gaussian States, Phys. Rev. Lett. **89**, 137904 (2002).

[56] G. Giedke and J. I. Cirac, Characterization of Gaussian operations and distillation of Gaussian states, Phys. Rev. A **66**, 032316 (2002).

[57] J. Niset, J. Fiurášek, and N. J. Cerf, No-Go Theorem for Gaussian Quantum Error Correction, Phys. Rev. Lett. **102**, 120501 (2009).

[58] U. Singh, J. K. Korbicz, and N. J. Cerf, Gaussian work extraction from random Gaussian states is nearly impossible, arXiv:2212.03492.

[59] A. Leverrier and N. J. Cerf, Quantum de Finetti theorem in phase-space representation, Phys. Rev. A **80**, 010102(R) (2009).

[60] C. Lupo, S. Mancini, A. De Pasquale, P. Facchi, G. Florio, and S. Pascazio, Invariant measures on multimode quantum Gaussian states, J. Math. Phys. **53**, 122209 (2012).