Supplemental Material: Gaussian work extraction from random Gaussian states is nearly impossible

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In Section I, we use a Gaussian purification argument to prove that the total state can be taken pure with no loss of generality in the argument leading to the definition of the set $\mathcal{L}_{m,E}$. In Sections II and III, we state and prove Lemmas 1 and 2, respectively, which are used in order to prove Theorem 1 in the main text. In Appendix A, we detail the mathematical tools used to prove the results of the main text. Appendix B elaborates on the method to compute average over Haar distributed unitaries following Weingarten calculus.

I. GAUSSIAN PURIFICATION

We have the following proposition on the structure of the set $\mathcal{L}_{m,E}$.

Proposition 1 (Gaussian purification)**.** *For any covariance matrix $\mathbf{\Gamma}_m \in \mathcal{L}_{m,E}$, there exists a $2m$-mode covariance matrix $\mathbf{\Gamma}$ corresponding to pure Gaussian state with $\mathrm{Tr}[\mathbf{\Gamma}] \leq 4E$ such that $\mathbf{\Gamma}_m = \mathbf{\Pi}_{m,m}\mathbf{\Gamma}\mathbf{\Pi}_{m,m}$, where $\mathbf{\Pi}_{m,m} = \mathrm{diag}\{\overbrace{1,\cdots,1}^{m},\overbrace{0,\cdots,0}^{m}\} \oplus \mathrm{diag}\{\overbrace{1,\cdots,1}^{m},\overbrace{0,\cdots,0}^{m}\}$.*

Proof. Given an m-mode covariance matrix $\mathbf{\Gamma}_m$ of system A, using Williamson's theorem, we can write

$$\mathbf{\Gamma}_m = \mathcal{S}\mathbf{\Gamma}_{0m}\mathcal{S}^T, \tag{1}$$

where $\mathcal{S}$ is a symplectic matrix and $\mathbf{\Gamma}_{0m} = \mathrm{diag}\{\nu_1,\cdots,\nu_m\} \oplus \mathrm{diag}\{\nu_1,\cdots,\nu_m\}$ with ν_i being the symplectic eigenvalues. $\mathbf{\Gamma}_{0m}$ is a collection of m single mode thermal states and the mean photon number of the ith thermal state is given by $(2\nu_i - 1)/2$. It is known that a single mode thermal state can be purified using two mode squeezed state. In particular, a two mode squeezer on systems A and R is described as a symplectic transformation $\mathcal{S}_{TMS}$ given by

$$\mathcal{S}_{TMS} = \begin{pmatrix} \cosh r_i \mathbb{I} & \sinh r_i \sigma_z \\ \sinh r_i \sigma_z & \cosh r_i \mathbb{I} \end{pmatrix}, \tag{2}$$

which acts linearly on the two mode quadrature operators $\mathbf{x}_i = (q_i^A, p_i^A, q_i^R, p_i^R)$. The covariance matrix for two mode vacuum state is given by $\frac{1}{2}(\mathbb{I} \oplus \mathbb{I})$, where $\mathbb{I}$ is a 2×2 identity matrix. Then the two mode squeezed vacuum state is given by

$$\begin{aligned} \mathbf{\Gamma}_{TMS} &= \frac{1}{2}\mathcal{S}_{TMS}\mathcal{S}_{TMS}^T \\ &= \frac{1}{2}\begin{pmatrix} \cosh r_i \mathbb{I} & \sinh r_i \sigma_z \\ \sinh r_i \sigma_z & \cosh r_i \mathbb{I} \end{pmatrix}\begin{pmatrix} \cosh r_i \mathbb{I} & \sinh r_i \sigma_z \\ \sinh r_i \sigma_z & \cosh r_i \mathbb{I} \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \cosh 2r_i \mathbb{I} & \sinh 2r_i \sigma_z \\ \sinh 2r_i \sigma_z & \cosh 2r_i \mathbb{I} \end{pmatrix} \\ &= \begin{pmatrix} \nu_i \mathbb{I} & \sqrt{\nu_i^2 - 1/4}\,\sigma_z \\ \sqrt{\nu_i^2 - 1/4}\,\sigma_z & \nu_i \mathbb{I} \end{pmatrix}, \end{aligned} \tag{3}$$

where $\cosh 2r_i = 2\nu_i$. We see that indeed removing one mode (the R mode) gives us a thermal state with mean photon number $(2\nu_i - 1)/2$. Similarly, we can write purification for m modes. In particular, $\widetilde{\mathbf{\Gamma}}_{2m}$ is a purification of

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$\mathbf{\Gamma}_{0m}$, where

$$\widetilde{\mathbf{\Gamma}}_{2m} = \begin{pmatrix} \mathbf{\Gamma}_{0m} & V \\ V & \mathbf{\Gamma}_{0m} \end{pmatrix}, \tag{4}$$

$V = \oplus_{i=1}^{n} \sqrt{\nu_i^2 - 1/4}\sigma_z$, and the order of quadrature operators is given by $(q_1^A, p_1^A \cdots, q_m^A, p_m^A, q_1^R, p_1^R \cdots, q_m^R, p_m^R)$. Thus, we have

$$\begin{aligned} \mathbf{\Gamma}'_{2m} &:= (\mathcal{S} \oplus \mathbb{I}_m)\widetilde{\mathbf{\Gamma}}_{2m}(\mathcal{S} \oplus \mathbb{I}_m)^T \\ &= \begin{pmatrix} \mathcal{S}\mathbf{\Gamma}_{0m}\mathcal{S}^T & \mathcal{S}V \\ V\mathcal{S}^T & \mathbf{\Gamma}_{0m} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{\Gamma}_m & \mathcal{S}V \\ V\mathcal{S}^T & \mathbf{\Gamma}_{0m} \end{pmatrix}, \end{aligned} \tag{5}$$

where $\mathcal{S}$ is the same as in Eq. (1). For consistency of the notation, we further need to perform a permutation P_π on $2m$ indices such that

$$P_\pi(q_1^A, p_1^A \cdots, q_m^A, p_m^A, q_1^R, p_1^R \cdots, q_m^R, p_m^R)^T = (q_1^A, \cdots, q_m^A, q_1^R \cdots, q_m^R, p_1^A, \cdots, p_m^A, p_1^R, \cdots, p_m^R)^T. \tag{6}$$

Thus, the desired purification is given by $\mathbf{\Gamma}_{2m} = P_\pi \mathbf{\Gamma}'_{2m} P_\pi^{-1}$ as $\mathbf{\Gamma}_m = \mathbf{\Pi}_{m,m}\mathbf{\Gamma}_{2m}\mathbf{\Pi}_{m,m}$. Now let $\mathbf{\Gamma}_m \in \mathcal{L}_{m,E}$, i.e., $\mathrm{Tr}[\mathbf{\Gamma}_m] \le 2E$. The energy corresponding to covariance matrix $\mathbf{\Gamma}_{2m}$ is given by

$$\begin{aligned} \frac{1}{2}\mathrm{Tr}[\mathbf{\Gamma}_{2m}] &= \frac{1}{2}\mathrm{Tr}[\mathbf{\Gamma}'_{2m}] \\ &= \frac{1}{2}\mathrm{Tr}[\mathbf{\Gamma}_m] + \frac{1}{2}\mathrm{Tr}[\mathbf{\Gamma}_{0m}] \\ &\le \mathrm{Tr}[\mathbf{\Gamma}_m] \\ &\le 2E, \end{aligned} \tag{7}$$

where we used the fact that $\mathrm{Tr}[\mathbf{\Gamma}_m] = \mathrm{Tr}[\mathcal{S}\mathbf{\Gamma}_{0m}\mathcal{S}^T] \ge \min_{\mathcal{S}\in \mathrm{Sp}(2n,\mathbb{R})} \mathrm{Tr}[\mathcal{S}\mathbf{\Gamma}_{0m}\mathcal{S}^T] = \mathrm{Tr}[\mathbf{\Gamma}_{0m}]$ [1]. This completes the proof of the proposition. $\square$

II. PROOF OF LEMMA 1

In this Appendix, we give the precise statement and proof of Lemma 1, which establishes the typicality of the eigenspectra as needed to prove our main result on the Gaussian extractable work, Theorem 1 of the main text. First, let us recall the physical procedure to sample a n-mode energy-constrained random Gaussian pure state. We sample $\mathbf{z}$ satisfying the energy constraint, yielding a squeezed vacuum state $|\Psi(\mathbf{z})\rangle = |\psi_{z_1}\rangle \otimes \cdots \otimes |\psi_{z_n}\rangle$ with covariance matrix $J(\mathbf{z}) \oplus J^{-1}(\mathbf{z})$, and then we apply a random $\mathbf{O} \in \mathrm{K}_n$ sampled from the invariant measure on K_n. This is done via the isomorphism $F : \mathrm{U}(n) \mapsto \mathrm{K}_n$ defined as

$$\mathbf{O} \equiv F(U) := P \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} P^{-1}, \tag{8}$$

where $P = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_n & \mathrm{i}\mathbb{I}_n \\ \mathrm{i}\mathbb{I}_n & \mathbb{I}_n \end{pmatrix}$ and $P^\dagger = P^{-1}$. Using the Gaussian purification argument, we actually replace n with $2n$ in the above. Then, the zero-mean random m-mode pure

$$\mathbf{\Gamma}_m := \frac{1}{2}\mathbf{\Pi}_{m,N+n}\mathbf{O}_{2n}\widetilde{J}_{2n}(\mathbf{z})\mathbf{O}_{2n}^T\mathbf{\Pi}_{m,N+n}, \tag{9}$$

Lemma 1. *Let $\mathbf{\Gamma}_m$ be the covariance matrix as in Eq. (9). Further, assume that $4\beta < 1$. For universal constants $\gamma, \widetilde{\gamma} > 0$ such that $\epsilon > 2\gamma n^{\beta-1}$, the eigenvalues $\{\lambda_i\}_{i=1}^{2m}$ of $\mathbf{\Gamma}_m$ converge in probability to ν_{th}, i.e.,*

$$\Pr\left[\sum_{i=1}^{2m} (\lambda_i - \nu_{\mathrm{th}})^2 > \epsilon\right] \le \exp\left[-\widetilde{\gamma}\epsilon^2 n^{1-4\beta}\right], \tag{10}$$

where ν_{th} is the average energy per mode of the $2n$-mode input pure state, i.e., $\nu_{\mathrm{th}} = \mathrm{Tr}[\widetilde{J}_{2n}(\mathbf{z})]/(8n)$.

Proof. Let us consider a function $T_m : \mathrm{U}(2n) \to \mathbb{R}$ of random unitary matrices defined as

$$T_m(U) := \mathrm{Tr}\left[(\mathbf{\Gamma}_m - \nu_{th}\mathbb{I}_{2m})^2\right], \tag{11}$$

where $\mathbf{\Gamma}_m \equiv \mathbf{\Gamma}_m(U)$ is defined by Eq. (9) and ν_{th} is the average energy of the $2n$-mode input pure state with covariance matrix $\frac{1}{2}\mathbf{O}_{2n}\widetilde{J}_{2n}(\mathbf{z})\mathbf{O}_{2n}^T$. Thus we have $\nu_{\mathrm{th}} = \mathrm{Tr}[\widetilde{J}_{2n}(\mathbf{z})]/(8n)$. Also, note that ν_{th} is uniformly bounded in n by definition of $\widetilde{J}_{2n}(\mathbf{z}) = J_{2n}(\mathbf{z}) \oplus J_{2n}^{-1}(\mathbf{z})$ with $\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty = O\left(n^\beta\right)$. Since $\mathbf{\Gamma}_m$ is a real symmetric matrix, it can be diagonalized by an orthogonal matrix and we have

$$T_m(U) = \sum_{k=1}^{2m} (\lambda_k - \nu_{th})^2. \tag{12}$$

Thus, the lemma provides us with an upper bound to the probability $\Pr\left[T_m(U) > \epsilon\right]$. The proof follows from the concentration of measure phenomenon as we show below. First, we compute the average value of the function $T_m(U)$. By definition of $\mathbf{\Gamma}_m$, we have

$$2\mathbf{\Gamma}_m = \frac{1}{2}\begin{pmatrix}\mathbf{\Pi} & \mathrm{i}\mathbf{\Pi}\\ \mathrm{i}\mathbf{\Pi} & \mathbf{\Pi}\end{pmatrix}\mathcal{A}(U)\begin{pmatrix}\mathbf{\Pi} & \mathrm{i}\mathbf{\Pi}\\ \mathrm{i}\mathbf{\Pi} & \mathbf{\Pi}\end{pmatrix}, \tag{13}$$

where

$$\mathcal{A}(U) = \begin{pmatrix} UAU^T & -\mathrm{i}UBU^\dagger \\ -\mathrm{i}U^*BU^T & -U^*AU^\dagger\end{pmatrix} \tag{14}$$

with $A = (J_{2n}(\mathbf{z}) - J_{2n}^{-1}(\mathbf{z}))/2$ and $B = (J_{2n}(\mathbf{z}) + J_{2n}^{-1}(\mathbf{z}))/2$. From Remark 1 in appendix B, we have $\mathbb{E}_{\mathrm{U}}UAU^T = 0$. Further, it is easy to see that

$$\mathbb{E}_{\mathrm{U}}UBU^\dagger = \mathrm{Tr}[B]\frac{\mathbb{I}}{2n}. \tag{15}$$

Now, using $||B||_1 = \mathrm{Tr}[B] = 4n\nu_{\mathrm{th}}$, we have $\mathbb{E}_{\mathrm{U}}\mathbf{\Gamma}_m = \nu_{\mathrm{th}}\begin{pmatrix}\mathbf{\Pi} & 0\\ 0 & \mathbf{\Pi}\end{pmatrix}$. Thus,

$$\mathrm{Tr}\left[\mathbb{E}_{\mathrm{U}}\mathbf{\Gamma}_m\right] = 2m\nu_{\mathrm{th}}. \tag{16}$$

Moreover,

$$4\mathbf{\Gamma}_m^2 = \frac{\mathrm{i}}{2}\begin{pmatrix}\mathbf{\Pi} & \mathrm{i}\mathbf{\Pi}\\ \mathrm{i}\mathbf{\Pi} & \mathbf{\Pi}\end{pmatrix}\begin{pmatrix}\mathcal{Q}(U) & -\mathcal{B}(U)\\ -\mathcal{B}^*(U) & \mathcal{Q}^*(U)\end{pmatrix}\begin{pmatrix}\mathbf{\Pi} & \mathrm{i}\mathbf{\Pi}\\ \mathrm{i}\mathbf{\Pi} & \mathbf{\Pi}\end{pmatrix},$$

where

$$\mathcal{Q}(U) = -\mathrm{i}UBU^\dagger\mathbf{\Pi}UAU^T - \mathrm{i}UAU^T\mathbf{\Pi}U^*BU^T;$$
$$\mathcal{B}(U) = UBU^\dagger\mathbf{\Pi}UBU^\dagger + UAU^T\mathbf{\Pi}U^*AU^\dagger.$$

Using Remark 1 of Appendix B, we have $\mathbb{E}_{\mathrm{U}}\mathcal{Q}(U) = 0$. Thus,

$$4\mathbb{E}_{\mathrm{U}}\mathbf{\Gamma}_m^2 = \begin{pmatrix}\mathbf{\Pi}\mathbb{E}_{\mathrm{U}}\mathcal{B}(U)\mathbf{\Pi} & 0\\ 0 & \mathbf{\Pi}\mathbb{E}_{\mathrm{U}}\mathcal{B}(U)\mathbf{\Pi}\end{pmatrix}. \tag{17}$$

We now compute the $\mathbb{E}_{\mathrm{U}}\mathcal{B}(U)$. We have

$$\mathcal{B}(U) = UBU^\dagger\mathbf{\Pi}UBU^\dagger + UAU^T\mathbf{\Pi}U^*AU^\dagger.$$

We first compute $\mathbb{E}_{\mathrm{U}} UBU^\dagger \mathbf{\Pi} UBU^\dagger$ as follows.

$$
\begin{aligned}
&\mathbb{E}_{\mathrm{U}} UBU^\dagger \mathbf{\Pi} UBU^\dagger \\
&= \mathbb{E}_{\mathrm{U}} \sum_{i_1,k_1,i_2,k_2,j_1,l_1,j_2,l_2} U_{i_1k_1} B_{k_1l_1} U^*_{j_1l_1} \mathbf{\Pi}_{j_1i_2} U_{i_2k_2} B_{k_2l_2} U^*_{j_2l_2} |i_1\rangle\langle j_2| \\
&= \sum_{i_1,k_1,i_2,k_2,j_1,l_1,j_2,l_2} B_{k_1l_1} \mathbf{\Pi}_{j_1i_2} B_{k_2l_2} \sum_{\alpha,\beta\in S_2} \prod_{x=1}^{2} \delta_{i_x j_{\alpha(x)}} \prod_{y=1}^{2} \delta_{k_y l_{\beta(y)}} \mathrm{Wg}(2n, \alpha^{-1}\beta) |i_1\rangle\langle j_2| \\
&= \sum_{i_1,k_1,i_2,k_2,j_1,l_1,j_2,l_2} B_{k_1l_1} \mathbf{\Pi}_{j_1i_2} B_{k_2l_2} |i_1\rangle\langle j_2| \left[\delta_{i_1j_1}\delta_{i_2j_2}\delta_{k_1l_1}\delta_{k_2l_2}\mathrm{Wg}(2n,(1)(2)) + \delta_{i_1j_1}\delta_{i_2j_2}\delta_{k_1l_2}\delta_{k_2l_1}\mathrm{Wg}(2n,(12))\right. \\
&\left.+\delta_{i_1j_2}\delta_{i_2j_1}\delta_{k_1l_1}\delta_{k_2l_2}\mathrm{Wg}(2n,(12)) + \delta_{i_1j_2}\delta_{i_2j_1}\delta_{k_1l_2}\delta_{k_2l_1}\mathrm{Wg}(2n,(1)(2))\right] \\
&= \sum_{i_1,k_1,i_2,k_2} \left[B_{k_1k_1} \mathbf{\Pi}_{i_1i_2} B_{k_2k_2} |i_1\rangle\langle i_2| \mathrm{Wg}(2n,(1)(2)) + B_{k_1k_2} \mathbf{\Pi}_{i_1i_2} B_{k_2k_1} |i_1\rangle\langle i_2| \mathrm{Wg}(2n,(12))\right. \\
&\left.+B_{k_1k_1} \mathbf{\Pi}_{i_2i_2} B_{k_2k_2} |i_1\rangle\langle i_1| \mathrm{Wg}(2n,(12)) + B_{k_1k_2} \mathbf{\Pi}_{i_2i_2} B_{k_2k_1} |i_1\rangle\langle i_1| \mathrm{Wg}(2n,(1)(2))\right] \\
&= (\mathrm{Tr}[B])^2 \mathbf{\Pi}\mathrm{Wg}(2n,(1)(2)) + \mathrm{Tr}[B^2]\mathbf{\Pi}\mathrm{Wg}(2n,(12)) + (\mathrm{Tr}[B])^2 \mathrm{Tr}[\mathbf{\Pi}]\mathbb{I}_{2n}\mathrm{Wg}(2n,(12)) + \mathrm{Tr}[B^2]\mathrm{Tr}[\mathbf{\Pi}]\mathbb{I}_{2n}\mathrm{Wg}(2n,(1)(2)) \\
&= (\mathrm{Tr}[B])^2 \left(\mathbf{\Pi}\mathrm{Wg}(2n,(1)(2)) + m\mathbb{I}_{2n}\mathrm{Wg}(2n,(12))\right) + \mathrm{Tr}[B^2]\left(\mathbf{\Pi}\mathrm{Wg}(2n,(12)) + m\mathbb{I}_{2n}\mathrm{Wg}(2n,(1)(2))\right) \\
&= (\mathrm{Tr}[B])^2 \frac{2n\mathbf{\Pi} - m\mathbb{I}_{2n}}{2n(4n^2-1)} + \mathrm{Tr}[B^2]\frac{2mn\mathbb{I}_{2n} - \mathbf{\Pi}}{2n(4n^2-1)}.
\end{aligned}
$$

Now we compute $\mathbb{E}_{\mathrm{U}} UAU^T \mathbf{\Pi} U^* AU^\dagger$ as follows.

$$
\begin{aligned}
&\mathbb{E}_{\mathrm{U}} UAU^T \mathbf{\Pi} U^* AU^\dagger \\
&= \mathbb{E}_{\mathrm{U}} \sum_{i_1,k_1,i_2,k_2,j_1,l_1,j_2,l_2} U_{i_1k_1} A_{k_1k_2} U_{i_2k_2} \mathbf{\Pi}_{i_2j_1} U^*_{j_1l_1} A_{l_1l_2} U^*_{j_2l_2} |i_1\rangle\langle j_2| \\
&= \sum_{i_1,k_1,i_2,k_2,j_1,l_1,j_2,l_2} A_{k_1k_2} \mathbf{\Pi}_{i_2j_1} A_{l_1l_2} \sum_{\alpha,\beta\in S_2} \prod_{x=1}^{2} \delta_{i_x j_{\alpha(x)}} \prod_{y=1}^{2} \delta_{k_y l_{\beta(y)}} \mathrm{Wg}(2n, \alpha^{-1}\beta) |i_1\rangle\langle j_2| \\
&= \sum_{i_1,k_1,i_2,k_2,j_1,l_1,j_2,l_2} A_{k_1k_2} \mathbf{\Pi}_{i_2j_1} A_{l_1l_2} |i_1\rangle\langle j_2| \left[\delta_{i_1j_1}\delta_{i_2j_2}\delta_{k_1l_1}\delta_{k_2l_2}\mathrm{Wg}(2n,(1)(2)) + \delta_{i_1j_1}\delta_{i_2j_2}\delta_{k_1l_2}\delta_{k_2l_1}\mathrm{Wg}(2n,(12))\right. \\
&\left.+\delta_{i_1j_2}\delta_{i_2j_1}\delta_{k_1l_1}\delta_{k_2l_2}\mathrm{Wg}(2n,(12)) + \delta_{i_1j_2}\delta_{i_2j_1}\delta_{k_1l_2}\delta_{k_2l_1}\mathrm{Wg}(2n,(1)(2))\right] \\
&= \sum_{i_1,k_1,i_2,k_2} \left[A_{k_1k_2} \mathbf{\Pi}_{i_2i_1} A_{k_1k_2} |i_1\rangle\langle i_2| \mathrm{Wg}(2n,(1)(2)) + A_{k_1k_2} \mathbf{\Pi}_{i_2i_1} A_{k_2k_1} |i_1\rangle\langle i_2| \mathrm{Wg}(2n,(12))\right. \\
&\left.+A_{k_1k_2} \mathbf{\Pi}_{i_2i_2} A_{k_1k_2} |i_1\rangle\langle i_1| \mathrm{Wg}(2n,(12)) + A_{k_1k_2} \mathbf{\Pi}_{i_2i_2} A_{k_2k_1} |i_1\rangle\langle i_1| \mathrm{Wg}(2n,(1)(2))\right] \\
&= \mathrm{Tr}[A^2] \left(\mathbf{\Pi}\mathrm{Wg}(2n,(1)(2)) + \mathbf{\Pi}\mathrm{Wg}(2n,(12)) + \mathrm{Wg}(2n,(12))m\mathbb{I} + \mathrm{Wg}(2n,(1)(2))m\mathbb{I}\right) \\
&= \mathrm{Tr}[A^2]\frac{1}{2n(2n+1)} \left(\mathbf{\Pi} + m\mathbb{I}\right).
\end{aligned}
$$

Thus,

$$
\mathbb{E}_{\mathrm{U}}\mathcal{B}(U) = \frac{(2n\mathbf{\Pi} - m\mathbb{I}_{2n})(\mathrm{Tr}[B])^2}{2n(4n^2-1)} + \frac{(2mn\mathbb{I}_{2n} - \mathbf{\Pi})\mathrm{Tr}[B^2]}{2n(4n^2-1)} + \frac{\mathrm{Tr}[A^2]}{2n(2n+1)}\left(\mathbf{\Pi} + m\mathbb{I}\right).
$$

And we then get

$$
\mathbf{\Pi}\mathbb{E}_{\mathrm{U}}\mathcal{B}(U)\mathbf{\Pi} = \left(\frac{(2n-m)}{2n(4n^2-1)}(\mathrm{Tr}[B])^2 + \frac{(2mn-1)}{2n(4n^2-1)}\mathrm{Tr}\left[B^2\right] + \frac{(m+1)\,\mathrm{Tr}\left[A^2\right]}{2n(2n+1)}\right)\mathbf{\Pi}.
$$

For large n, we have

$$
\mathrm{Tr}\left[\mathbf{\Pi}\mathbb{E}_{\mathrm{U}}\mathcal{B}(U)\mathbf{\Pi}\right] = \frac{m}{4n^2}\left(1 + O\left(n^{-1}\right)\right)(\mathrm{Tr}[B])^2 + O\left(n^{-2}\right)\mathrm{Tr}\left[B^2\right] + O\left(n^{-2}\right)\mathrm{Tr}\left[A^2\right]. \tag{18}
$$

Using $||B||_1 = \mathrm{Tr}[B] = 4n\nu_{\rm th}$, $||B||_\infty = O\left(n^\beta\right)$, and $\mathrm{Tr}\left[A^2\right] \le \mathrm{Tr}\left[B^2\right] \le ||B||_1\,||B||_\infty = \nu_{\rm th} O\left(n^{\beta+1}\right)$, we have

$$\begin{aligned}\mathrm{Tr}\left[\mathbf{\Pi}\mathbb{E}_{\rm U}\mathcal{B}(U)\mathbf{\Pi}\right] &= 4m\nu_{\rm th}^2\left(1+O\left(n^{-1}\right)\right)+\nu_{\rm th}O\left(n^{\beta-1}\right)\\ &= 4m\nu_{\rm th}^2+O\left(n^{\beta-1}\right). \end{aligned}\tag{19}$$

Thus, we have

$$4\mathrm{Tr}\left[\mathbb{E}_{\rm U}\mathbf{\Gamma}_m^2\right] = 8m\nu_{\rm th}^2+O\left(n^{\beta-1}\right). \tag{20}$$

Combining Eqs. (16) and (20), we have

$$\begin{aligned}\mathbb{E}_{\rm U}T(U) &= \mathbb{E}_{\rm U}\mathrm{Tr}\left[\mathbf{\Gamma}_m^2\right]-2\nu_{\rm th}\mathbb{E}_{\rm U}\mathrm{Tr}\left[\mathbf{\Gamma}_m\right]+2m\nu_{\rm th}^2\\ &= 2m\nu_{\rm th}^2+O\left(n^{\beta-1}\right)-4m\nu_{\rm th}^2+2m\nu_{\rm th}^2\\ &= O\left(n^{\beta-1}\right).\end{aligned}$$

Thus, there exists a universal constant $\gamma > 0$ such that $\mathbb{E}_{\rm U}T(U) \le \gamma n^{\beta-1}$.

Next, we bound the Lipschitz constant for the function $T(U)$. Let $\mathbf{\Gamma}_m(U)$ and $\mathbf{\Gamma}_m(V)$ be two covariance matrices generated via unitaries U and V, respectively. Also, let us denote $\mathbf{\Gamma}_m(U)$ by $\mathbf{\Gamma}_m$ and $\mathbf{\Gamma}_m(V)$ by $\widetilde{\mathbf{\Gamma}}_m$. Then we have

$$\begin{aligned}|T(U)-T(V)| &\le \left|\mathrm{Tr}\left[\mathbf{\Gamma}_m^2-\widetilde{\mathbf{\Gamma}}_m^2\right]\right|+2\nu_{th}\left|\mathrm{Tr}\left[\mathbf{\Gamma}_m-\widetilde{\mathbf{\Gamma}}_m\right]\right|\\ &\le \left|\left|\mathbf{\Gamma}_m^2-\widetilde{\mathbf{\Gamma}}_m^2\right|\right|_1+2\nu_{th}\left|\left|\mathbf{\Gamma}_m-\widetilde{\mathbf{\Gamma}}_m\right|\right|_1\\ &\le \left(||\mathbf{\Gamma}_m||_\infty+\left|\left|\widetilde{\mathbf{\Gamma}}_m\right|\right|_\infty+2\nu_{th}\right)\left|\left|\mathbf{\Gamma}_m-\widetilde{\mathbf{\Gamma}}_m\right|\right|_1\\ &\le 2\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty\left|\left|\mathbf{\Gamma}_m-\widetilde{\mathbf{\Gamma}}_m\right|\right|_1\\ &\le 2\sqrt{2m}\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty\left|\left|\mathbf{\Gamma}_m-\widetilde{\mathbf{\Gamma}}_m\right|\right|_2,\end{aligned}\tag{21}$$

where we have used $\max\left\{||\mathbf{\Gamma}_m||_\infty, \left|\left|\widetilde{\mathbf{\Gamma}}_m\right|\right|_\infty\right\} \le \left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty/2$ and $2\nu_{th} \le \left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty$. Further, we have

$$\begin{aligned}\left|\left|\mathbf{\Gamma}_m-\widetilde{\mathbf{\Gamma}}_m\right|\right|_2 &\le \frac{1}{2}\left|\left|F(U)\widetilde{J}_{2n}(\mathbf{z})F(U)^T-F(V)\widetilde{J}_{2n}(\mathbf{z})F(V)^T\right|\right|_2\\ &\le \frac{1}{2}\left|\left|F(U)\widetilde{J}_{2n}(\mathbf{z})\left(F(U)-F(V)\right)^T\right|\right|_2+\frac{1}{2}\left|\left|\left(F(U)-F(V)\right)\widetilde{J}_{2n}(\mathbf{z})F(V)^T\right|\right|_2\\ &\le \left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty||F(U)-F(V)||_2\\ &= \left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty||U\oplus U^*-V\oplus V^*||_2\\ &\le 2\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty||U-V||_2.\end{aligned}\tag{22}$$

Thus,

$$\begin{aligned}|T(U)-T(V)| &\le 4\sqrt{2m}\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty^2||U-V||_2\\ &= O\left(n^{2\beta}\right)||U-V||_2,\end{aligned}$$

where we have used the fact that $\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty = O\left(n^\beta\right)$. Thus, the Lipschitz constant L for the function $T(U)$ is equal to $O\left(n^{2\beta}\right)$. Now, we use concentration of measure phenomenon to the function $T(U)$ of random unitaries U. Let us take

$$\epsilon > 2\gamma n^{\beta-1},$$

where γ is a universal constant. Then we have

$$\begin{aligned}\Pr\left[T(U)>\epsilon\right] &\le \Pr\left[T(U)>\frac{\epsilon}{2}+\mathbb{E}_{\rm U}T(U)\right]\\ &\le \exp\left[-\frac{n\epsilon^2}{48L^2}\right]\\ &\le \exp\left[-\widetilde{\gamma}\epsilon^2 n^{1-4\beta}\right],\end{aligned}$$

where the second inequality follows from the concentration of measure phenomenon (see Appendix A) and $\widetilde{\gamma}$ is a suitable universal constant. This concludes the proof of the Lemma 1. □

III. PROOF OF LEMMA 2

In this Appendix, we give the precise statement and proof of Lemma 2. In a similar way as Lemma 1, Lemma 2 establishes the typicality of the symplectic eigenspectra (see also Ref. [2]). Let us consider a function $\mathfrak{T}_m : \mathrm{U}(2n) \to \mathbb{R}$ of random unitary matrices defined as

$$\mathfrak{T}_m(U) := \mathrm{Tr}\left[\left((\mathbf{\Omega\Gamma}_m)^2 + \nu_{th}^2 \mathbb{I}_{2m}\right)^2\right] = 2\sum_{k=1}^{m}\left(-\nu_k^2 + \nu_{th}^2\right)^2, \tag{23}$$

where $\{\nu_k\}_{k=1}^m$ are the symplectic eigenvalues of $\mathbf{\Gamma}_m$ and $\{\pm \mathrm{i}\nu_k\}_{k=1}^m$ comprises the spectra of matrix $\mathbf{\Omega\Gamma}_m$. Also, $\mathbf{\Gamma}_m \equiv \mathbf{\Gamma}_m(U)$ is defined by Eq. (9) and $\nu_{\mathrm{th}} = \mathrm{Tr}[\widetilde{J}_{2n}(\mathbf{z})]/(8n)$ as before.

Lemma 2 (Ref. [2])**.** *Let $\mathbf{\Gamma}_m$ be the covariance matrix as in Eq. (9). Further, assume that $8\beta < 1$. For universal constants $C, c > 0$ such that $\epsilon > Cn^{\beta-1}$, the symplectic eigenvalues $\{\nu_i\}_{i=1}^m$ of $\mathbf{\Gamma}_m$ converge in probability to ν_{th}, i.e.,*

$$\Pr\left[\sum_{i=1}^{m}\left(\nu_i^2 - \nu_{\mathrm{th}}^2\right)^2 > \epsilon\right] \le \exp\left[-c\epsilon^2 n^{1-8\beta}\right]. \tag{24}$$

The proof of the above lemma follows from the concentration of measure phenomenon applied to $\mathfrak{T}_m(U)$. The key steps include the calculation of the Lipschitz constant for $\mathfrak{T}_m(U)$ and its average with respect to the unitaries. For completeness, we show that $\mathbb{E}_U \mathfrak{T}_m(U) = O\left(n^{\beta-1}\right)$ and the Lipschitz constant for $\mathfrak{T}_m(U)$ is given by $O\left(n^{4\beta}\right)$. Note that these results easily follow from Ref. [2].

Proof. We first compute the average of the function $\mathfrak{T}_m(U)$ over random unitaries. Following Ref. [2], we have

$$\begin{aligned}4\mathbb{E}_U \mathrm{Tr}\left[(\mathbf{\Omega\Gamma}_m)^2\right] &= -2m\left[\frac{2n-m}{2n(4n^2-1)}\left(\mathrm{Tr}[B]\right)^2 - \frac{m+1}{2n(2n+1)}\mathrm{Tr}[A^2] + \frac{mn-1}{2n(4n^2-1)}\mathrm{Tr}[B^2]\right] \\ &= -2m\left[\frac{1}{4n^2}\left(1 + O\left(n^{-1}\right)\right)\left(\mathrm{Tr}[B]\right)^2 + O\left(n^{-2}\right)\mathrm{Tr}[A^2] + O\left(n^{-2}\right)\mathrm{Tr}[B^2]\right].\end{aligned}$$

Using $||B||_1 = \mathrm{Tr}[B] = 4n\nu_{\mathrm{th}}$, $||B||_\infty = O\left(n^\beta\right)$, and $\mathrm{Tr}\left[A^2\right] \le \mathrm{Tr}\left[B^2\right] \le ||B||_1\,||B||_\infty = \nu_{\mathrm{th}} O\left(n^{\beta+1}\right)$, we have

$$\mathbb{E}_U \mathrm{Tr}\left[(\mathbf{\Omega\Gamma}_m)^2\right] = -2m\nu_{\mathrm{th}}^2 + O\left(n^{\beta-1}\right).$$

Similarly, following Ref. [2], we have

$$\mathbb{E}_U \mathrm{Tr}\left[(\mathbf{\Omega\Gamma}_m)^4\right] = 2m\nu_{\mathrm{th}}^4 + O\left(n^{\beta-1}\right).$$

Thus,

$$\mathbb{E}_U \mathfrak{T}_m(U) = O\left(n^{\beta-1}\right). \tag{25}$$

Now, we compute the Lipschitz constant for the function $\mathfrak{T}_m(U)$. Let $\mathbf{\Gamma}_m(U) \equiv \mathbf{\Gamma}_m$ and $\mathbf{\Gamma}_m(V) \equiv \widetilde{\mathbf{\Gamma}}_m$. Then, again from Ref. [2], we have

$$\begin{aligned}|\mathfrak{T}(U) - \mathfrak{T}(V)| &\le \left(4\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty^3 + 4\nu_{th}^2\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty\right)\left|\left|\mathbf{\Gamma}_m - \widetilde{\mathbf{\Gamma}}_m\right|\right|_1 \\ &\le 5\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty^3\left|\left|\mathbf{\Gamma}_m - \widetilde{\mathbf{\Gamma}}_m\right|\right|_1 \\ &\le 5\sqrt{2m}\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty^3\left|\left|\mathbf{\Gamma}_m - \widetilde{\mathbf{\Gamma}}_m\right|\right|_2,\end{aligned}$$

where we have used $\max\left\{||\mathbf{\Gamma}_m||_\infty , \left|\left|\widetilde{\mathbf{\Gamma}}_m\right|\right|_\infty\right\} \leq \left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty /2$ and $2\nu_{th} \leq \left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty$. In Lemma 1, we have proved $\left|\left|\mathbf{\Gamma}_m - \widetilde{\mathbf{\Gamma}}_m\right|\right|_2 \leq 2\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty ||U-V||_2$, therefore

$$\begin{aligned} |\mathfrak{T}(U) - \mathfrak{T}(V)| &\leq 10\sqrt{2m}\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty^4 ||U-V||_2 \\ &= O\left(n^{4\beta}\right)||U-V||_2 , \end{aligned}$$

where we have used the fact that $\left|\left|\widetilde{J}_{2n}(\mathbf{z})\right|\right|_\infty = O\left(n^{\beta}\right)$. Thus, the Lipschitz constant L for the function $\mathfrak{T}(U)$ is equal to $O\left(n^{4\beta}\right)$.

Now, we use concentration of measure phenomenon to the function $\mathfrak{T}(U)$ of random unitaries U. Let C be a universal constant such that $\mathbb{E}_U\mathfrak{T}_m(U) \leq Cn^{\beta-1}$ and let $\epsilon > Cn^{\beta-1}$. Then we have

$$\begin{aligned} \Pr\left[\mathfrak{T}(U) > 2\epsilon\right] &\leq \Pr\left[T(U) > \epsilon + \mathbb{E}_{\mathrm{U}}T(U)\right] \\ &\leq \exp\left[-\frac{n\epsilon^2}{12L^2}\right] \\ &\leq \exp\left[-c\epsilon^2 n^{1-8\beta}\right], \end{aligned}$$

where c is a suitable universal constant. This concludes the proof of the Lemma 2.

□

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Appendix A: Norms, Lipschitz continuity and concentration of measure phenomenon

Matrix norms:– Let us consider a vector space V_n of complex $n \times n$ matrices. Let $X, Y \in V_n$, then a matrix norm on V_n is a real-valued non-negative function $||\cdot|| : V_n \to \mathbb{R}$ satisfying the following properties:

1. $||X|| \geq 0$ while the equality holds if and only if $X = 0$.
2. $||\alpha X|| = |\alpha|\,||X||$ for all $\alpha \in \mathbb{C}$.
3. $||X+Y|| \leq ||X|| + ||Y||$.
4. $||XY|| \leq ||X||\,||Y||$.

The last property is called the submultiplicativity [3]. An important family of matrix norms, called Schatten p-norms with $p \geq 1$, is defined as

$$||X||_p := \left(\sum_{i=1}^{n} s_i^p(X)\right)^{1/p}, \tag{A1}$$

where $\{s_i\}$ are the singular values of $X \in V_n$. These norms are unitarily invariant, i.e., for unitaries $U, V \in V_n$, $||UXV||_p = ||X||_p$. Of particular importance to us are the cases with $p = 1, 2, \infty$, which correspond to trace, Hilbert-Schmidt, and operator norms, respectively. In particular,

$$||X||_1 := \mathrm{Tr}\left[\sqrt{X^\dagger X}\right]; \tag{A2a}$$

$$||X||_2 := \sqrt{\mathrm{Tr}\,[X^\dagger X]}; \tag{A2b}$$

$$||X||_\infty := \max_{\vec{x}\neq 0} \frac{||X\vec{x}||}{||\vec{x}||}, \tag{A2c}$$

where $\vec{x}$ is an n dimensional vector and $||\cdot||$ is usual Euclidean norm for vectors. We list some of the relations between these norms that we will be using. Let $X \in V_n$, then

$$||X||_1 \leq \sqrt{n}\, ||X||_2 \leq n\, ||X||_\infty\,. \tag{A3}$$

Moreover, for $X, Y, Z \in V_n$, we have

$$||XYZ||_p \leq ||X||_\infty\, ||Y||_p\, ||Z||_\infty\,. \tag{A4}$$

Lipschitz continuity:– Let us consider two metric spaces (X, d_X) and (Y, d_Y), where d_X (or d_Y) denotes the metric on X (or Y). A function $F : X \to Y$ is said to be a Lipschitz continuous function if for any $x, x' \in X$

$$d_Y(F(x) - F(x')) \leq L\; d_X(x, x'), \tag{A5}$$

where the positive constant L is called the Lipschitz constant [4]. Note that any other constant $L' \geq L$ is also a valid Lipschitz constant. For this work, we are interested in functions $F : \mathrm{U}(n) \to \mathbb{R}$, where $\mathrm{U}(n)$ is the set of $n \times n$ unitary matrices and $\mathbb{R}$ is the set of real numbers. Such a function F is a Lipschitz continuous function with Lipschitz constant L if for any $U, V \in \mathrm{U}(n)$ we have

$$|F(U) - F(V)| \leq L\; ||U - V||_2\,. \tag{A6}$$

Concentration of the measure phenomenon:– The concentration of the measure phenomenon refers to the collective phenomenon of certain smooth functions defined over measurable vector spaces taking values close to their average values almost surely [5]. There are various versions of concentration inequalities depending on the input measurable space and there are various ways to prove them. A very general technique to prove such inequalities is via logarithmic Sobolev inequalities together with the Herbst argument (this is also called the "entropy method", see e.g. [5–7]). Since we are interested in functions on the unitary group $\mathrm{U}(n)$, a particularly suitable concentration inequality is given as follows [8] (see also [2]):

Theorem 1 ([8])**.** *Let* $\mathrm{U}(n)$ *be the group of* $n \times n$ *unitary matrices which is equipped with the Hilbert-Schmidt norm. Let* $F : \mathrm{U}(n) \to \mathbb{R}$ *be a Lipschitz continuous function with Lipschitz constant* L*. Then for any* $\epsilon > 0$

$$\Pr\left[F(U) > \mathbb{E}_U F + \epsilon\right] < \exp\left[-\frac{n\epsilon^2}{12L^2}\right], \tag{A7}$$

where $\mathbb{E}_U$ *denotes the average with respect to Haar measure on* $\mathrm{U}(n)$*.*

Appendix B: Average over unitaries and Weingarten calculus

Computing averages over the Haar measure on the unitary group is an essential part for establishing concentration inequalities for functions on the unitary group. In this section, we present briefly a method of Ref. [9] to compute averages (see also Ref. [10]). Let $\mathrm{U}(n)$ be the group of $n \times n$ unitary matrices equipped with the normalized Haar measure and S_d be the symmetric group of d objects. Let $U_{ij} = \langle i|\, U\, |j\rangle$ be the matrix elements of $U \in \mathrm{U}(n)$ in the computational basis. Then we have the following formula for the averages:

$$\mathbb{E}_U\left[\prod_{a=1}^{d} U_{i_a j_a} \prod_{b=1}^{d} U^*_{i'_b j'_b}\right] = \sum_{\pi,\sigma\in S_d} \prod_{a=1}^{d} \delta_{i_a i'_{\pi(a)}} \prod_{b=1}^{d} \delta_{j_b j'_{\sigma(b)}} \mathrm{Wg}\left(n, \pi^{-1}\sigma\right), \tag{B1}$$

where π and σ are permutations and the function $\mathrm{Wg}\left(n,\pi^{-1}\sigma\right)$ is called the Weingarten function, defined as

$$\mathrm{Wg}\left(n,\pi\right)=\frac{1}{d!^2}\sum_{\substack{\lambda\vdash d\\ l(\lambda)\leq n}}\frac{\chi^\lambda(\pi)(\chi^\lambda(1))^2}{s_{\lambda,n}(1)}. \tag{B2}$$

In the above expression λ is a Young tableaux and the sum is over all the Young tableaux with d boxes and rows $l(\lambda)\leq n$. For a given λ, χ^λ is the character corresponding to the irreducible representation labeled as λ of S_d. $s_{\lambda,n}(1)$ is the dimension of the representation of $\mathrm{U}(n)$ corresponding to a tableaux λ. In this work, we will need to compute averages for $d=2$ case. In this case,

$$\mathrm{Wg}\left(n,(1)(2)\right)=\frac{1}{n^2-1}; \tag{B3}$$

$$\mathrm{Wg}\left(n,(12)\right)=-\frac{1}{n(n^2-1)}. \tag{B4}$$

Remark 1. *From Eq.* (B1) *if the number of U terms is different than that of U^*, then the expectation in Eq.* (B1) *is zero.*