Supplementary material

Here, we provide details on the fermionic formalism and additional remarks for the interested reader.

(a) (Un)physical states

- When allowing $\lambda \neq 0$, the state in Eq. (1) is still Gaussian, *i.e.*, it can be written as $\boldsymbol{\rho} = \exp(-2\langle n\rangle \boldsymbol{a}^\dagger\boldsymbol{a} + \lambda\boldsymbol{a} + \lambda^*\boldsymbol{a}^\dagger + \langle n\rangle - |\lambda|^2)$, but it is neither thermal, nor physical (as we shall see, this actually corresponds to a Gaussian state that is not centered at the origin).

(b) Berezin integrals

- Berezin integrals are integrals over Grassmann-valued variables, which are such that $\int \mathrm{d}\alpha\, 1 = 0$ and $\int \mathrm{d}\alpha\, \alpha = 1$, showing that integration is equivalent to differentiation for Grassmann variables. Integrals of all other powers vanish since $\alpha^n = 0$ for $n \geq 2$.

(c) Fermionic coherent states

- Fermionic coherent states $|\alpha\rangle$ are characterized by Grassmann-valued displacements and thus are unphysical unless $\alpha = 0$, see also [1]. Indeed, using Eq. (3) together with $\boldsymbol{D}(\alpha) = \mathbb{1} + \boldsymbol{a}^\dagger\alpha - \alpha^*\boldsymbol{a} + \left(\frac{1}{2}\mathbb{1} - \boldsymbol{a}^\dagger\boldsymbol{a}\right)\alpha\alpha^*$, we see that coherent states are superpositions $|\alpha\rangle = (1 + \frac{1}{2}\alpha\alpha^*)\,|0\rangle - \alpha\,|1\rangle$. In analogy to the bosonic case, fermionic coherent states are eigenstates of the annihilation operator $\boldsymbol{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$.

- The decomposition of the identity in terms of fermionic coherent states, namely $\mathbb{1} = \int \mathcal{D}\alpha\, |\boldsymbol{\alpha}\rangle\,\langle\boldsymbol{\alpha}|$, can be checked by using $|\alpha\rangle = \beta\,|0\rangle - \alpha\,|1\rangle$ and $\langle\alpha| = \beta^*\langle 0| - \alpha^*\langle 1|$, with $\beta = 1 + \frac{1}{2}\alpha\alpha^*$, together with the fact that $\int \mathcal{D}\alpha\, \alpha\alpha^* = \int \mathcal{D}\alpha\, \beta\beta^* = 1$ as well as $\int \mathcal{D}\alpha\, \alpha\beta^* = \int \mathcal{D}\alpha\, \beta\alpha^* = 0$.

- The expression for the trace in the fermionic coherent basis, namely $\mathbf{Tr}\{\boldsymbol{O}\} = \int \mathcal{D}\alpha\,\langle\alpha|\boldsymbol{O}|-\alpha\rangle$, can also be checked by using $|-\alpha\rangle = \beta\,|0\rangle + \alpha\,|1\rangle$ and $\langle\alpha| = \beta^*\langle 0| - \alpha^*\langle 1|$, with $\beta = 1 + \frac{1}{2}\alpha\alpha^*$, together with the fact that $\int \mathcal{D}\alpha\, \beta^*\beta = \int \mathcal{D}\alpha\,(-\alpha^*)\alpha = 1$ as well as $\int \mathcal{D}\alpha\, \beta^*\alpha = \int \mathcal{D}\alpha\,(-\alpha^*)\beta = 0$. We may easily verify that the minus sign in $|-\alpha\rangle$ is needed by applying the trace formula to the density operator $\boldsymbol{\rho}$. Indeed, we get $\mathbf{Tr}\{\boldsymbol{\rho}\} = 1$ whereas we would get $\mathbf{Tr}\{\boldsymbol{\rho}\} = 1 - 2\langle n\rangle$ by (wrongly) replacing $|-\alpha\rangle$ with $|\alpha\rangle$, so that the excited state would (wrongly) be normalized to -1. This issue is intrinsically linked to the fact that $\langle n|\alpha\rangle$ and $\langle\alpha|n\rangle$ anticommute for $n = 1$ (whereas they commute for $n = 0$), so we have to insert this minus sign when exchanging the two matrix elements (this fixes the problem both for $n = 0$ and $n = 1$).

(d) Fermionic phase-space distributions

- The normalization of the Glauber P-distribution $\int \mathcal{D}\alpha\, P(\alpha) = \mathbf{Tr}\{\boldsymbol{\rho}\} = 1$ is ensured by the normalized projector $\mathbf{Tr}\{|\boldsymbol{\alpha}\rangle\,\langle-\boldsymbol{\alpha}|\} = \langle\alpha|\alpha\rangle = 1$, whereas $\mathbf{Tr}\{|\boldsymbol{\alpha}\rangle\,\langle\boldsymbol{\alpha}|\} = \langle-\alpha|\alpha\rangle = 1 - 2\alpha^*\alpha$. Analogously, $\int \mathcal{D}\alpha\, Q(\alpha) = \mathbf{Tr}\{\boldsymbol{\rho}\} = 1$ is a consequence of the minus sign appearing in the coherent-state trace formula.

- The characteristic function of $\boldsymbol{\rho}$ is calculated using $\mathbf{Tr}\{|\mathbf{0}\rangle\,\langle\mathbf{0}|\,\boldsymbol{D}(\alpha)\} = 1 + \frac{1}{2}\alpha\alpha^*$ as well as $\mathbf{Tr}\{|\mathbf{1}\rangle\,\langle\mathbf{1}|\,\boldsymbol{D}(\alpha)\} = 1 - \frac{1}{2}\alpha\alpha^*$, resulting in $\chi(\alpha) = 1 + (\frac{1}{2} - \langle n\rangle)\alpha\alpha^*$. To get $W(\alpha)$, we perform the Fourier transform of $\chi(\alpha)$ by exploiting the identity $e^{\alpha\beta^* - \beta\alpha^*} = 1 + \alpha\beta^* - \beta\alpha^* + \alpha\alpha^*\beta\beta^*$. The expressions of $P(\alpha)$ and $Q(\alpha)$ are also straightforward to derive by using the expansions of $\boldsymbol{\rho}$ and $|\alpha\rangle$ in the Fock basis.

- We observe all phase-space distributions to be Grassmann-even. More generally, we argue that any physical quantity has to be of definite Grassmann-parity, that is, either fully commute or anti-commute with any Grassmann variable, which we refer to as Grassmann-even or Grassmann-odd, respectively (see also [2]).

- In contrast to the bosonic case, where a complete characterization of the set of Wigner-positive states is an outstanding problem [3–5], the sign of the single-mode fermionic Wigner W-distribution is entirely determined by the particle number $\langle n\rangle$ and thus by the sign of the temperature. Recall that negative temperatures can occur for a Hamiltonian bounded from above when the occupation of the excited states is more likely. In our case, this amounts to $\langle n\rangle > 1/2$. Also, the fermionic Glauber P-distribution is always a real function, while its bosonic counterpart can become distribution-valued involving the Dirac δ-distribution and its derivatives.

(e) Majorization relations

- Without the condition $f(0) = 0$ both sides of the majorization relation would diverge [6].

- Although the proofs of the majorizations relations used the first derivative of f, we do not even have to assume that f is analytic since all higher derivatives of f are multiplied by zero. Further, the existence of its first derivative f' is guaranteed almost everywhere following Rademacher's theorem for Lipschitz-continuous functions, with the exceptions occurring only at the boundary points.

- The equivalence between second-moment and majorization relations can be proven as follows. Any physical distribution $z_1(\alpha_1)$ is related to another distribution $z_2(\alpha_2)$ by $\alpha_2 \to \alpha_1 = \sqrt{M}\,\alpha_2$ such that $z_2(\alpha_2) \to z_1(\alpha_1) = M\,z_2(\alpha_2)$, where $M = z_{2,\mathrm{B}}/z_{1,\mathrm{B}}$ is a real number. Then, the majorization relation Eq. (8) can be rewritten as $\int \mathcal{D}\alpha_1\, f(z_1) = \int \mathcal{D}\alpha_2\, f(M\,z_2)/M \geq \int \mathcal{D}\alpha_2\, f(z_2)$. (Note here that a change of coordinates for Grassmann variables is accompanied by the *inverse* Jacobian.) Since f is concave and fulfills $f(0) = 0$, it is subadditive. Thus, the latter relation is fulfilled if $0 \leq M \leq 1$ which implies $\det\gamma(z_1) \geq (\leq) \det\gamma(z_2)$ when $|z_{1,\mathrm{B}}| \geq (\leq) |z_{2,\mathrm{B}}|$. The converse statement follows from transforming the left-hand side of Eq. (8) instead.

(f) Moments in state space and phase space

- To relate quantities in state space and phase space, we introduce the fermionic variant of the overlap formula for two physical density operators, which reads $\mathbf{Tr}\{\boldsymbol{\rho}_1\boldsymbol{\rho}_2\} = \int \mathcal{D}\alpha\, W_1(\alpha)K_2(\alpha)$, where $W_1(\alpha) = 1/2 - \langle n_1\rangle + \alpha\alpha^*$ and $K_2(\alpha) = 1/2 + (1 - 2\langle n_2\rangle)\alpha\alpha^*$, with $\langle n_1\rangle$ ($\langle n_2\rangle$) denoting the mean particle number in $\boldsymbol{\rho}_1$ ($\boldsymbol{\rho}_2$).

- The mean fields of any physical state vanish $\mathbf{Tr}\{\boldsymbol{\rho}\boldsymbol{a}\} = \int \mathcal{D}\alpha\, W(\alpha)\,\alpha^* = 0$ and $\mathbf{Tr}\{\boldsymbol{\rho}\boldsymbol{a}^\dagger\} = \int \mathcal{D}\alpha\, W(\alpha)\,\alpha = 0$. Hence, physical states are Gaussian states that are centered at the origin, see also Eq. (2), and thus all second-order moments $\gamma_{jj'}(z)$ are centered moments.

- By setting $\langle n_2\rangle$ to either 0 or 1, the overlap formula shows that the state's covariance matrix defined as $\gamma_{jj'}(\boldsymbol{\rho}) \equiv \mathbf{Tr}\{\boldsymbol{\rho}[\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j'}]\}/(2i)$, where $\boldsymbol{\xi}_1 = \boldsymbol{a}$ and $\boldsymbol{\xi}_2 = \boldsymbol{a}^\dagger$, agrees with the one of the Wigner W-distribution, *i.e.*, $\gamma(\boldsymbol{\rho}) = \gamma(W)$. More precisely, we find for the off-diagonal elements $\mathbf{Tr}\{\boldsymbol{\rho}[\boldsymbol{a}, \boldsymbol{a}^\dagger]\} = \int \mathcal{D}\alpha\, W(\alpha)[\alpha, \alpha^*]$, although similar relations do *not* hold for the individual terms, consider, *e.g.*, $\mathbf{Tr}\{\boldsymbol{\rho}\boldsymbol{a}\boldsymbol{a}^\dagger\} = \int \mathcal{D}\alpha\, W(\alpha)(1/2 + \alpha\alpha^*) = 1 - \langle n\rangle$.

- The covariance matrix is also often defined in terms of the field quadratures. In contrast to bosons, fermionic quadratures $\boldsymbol{x} = (\boldsymbol{a} + \boldsymbol{a}^\dagger)/\sqrt{2}$ and $\boldsymbol{p} = (\boldsymbol{a} - \boldsymbol{a}^\dagger)/(\sqrt{2}i)$, which are commonly referred to as Majorana operators, anti-commute $\{\boldsymbol{x}, \boldsymbol{p}\} = 0$ and square to a constant $\boldsymbol{x}^2 = \boldsymbol{p}^2 = \mathbb{1}/2$. Noting that $[\boldsymbol{x}, \boldsymbol{p}] = i[\boldsymbol{a}, \boldsymbol{a}^\dagger]$ shows that the covariance matrix takes the form $\gamma_{jj'}(\boldsymbol{\rho}) = \mathbf{Tr}\{\boldsymbol{\rho}[\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j'}]\}/2$, where now $\boldsymbol{\xi}_1 = \boldsymbol{x}$ and $\boldsymbol{\xi}_2 = \boldsymbol{p}$. We note that the diagonal elements are still zero since every operator commutes with itself. Hence, the variances defined via $\sigma^2(\boldsymbol{x}) \equiv \langle \boldsymbol{x}^2\rangle - \langle \boldsymbol{x}\rangle^2$ (which are constant $\sigma^2(\boldsymbol{x}) = \sigma^2(\boldsymbol{p}) = 1/2$) do not appear in the covariance matrix. Instead, $\gamma(\boldsymbol{\rho}) = i\sigma(\boldsymbol{x}, \boldsymbol{p})\sigma_y$, where $\sigma(\boldsymbol{x}, \boldsymbol{p}) \equiv \langle[\boldsymbol{x}, \boldsymbol{p}]\rangle/2$ denotes the anti-symmetrized covariance and σ_y is the second Pauli matrix. Hence, we have $\det\gamma(\boldsymbol{\rho}) = \sigma^2(\boldsymbol{x}, \boldsymbol{p}) = \langle[\boldsymbol{x}, \boldsymbol{p}]\rangle^2/4$.

(g) Entropies and entropic uncertainty relations

- Continuous-variable entropies of classical probability distributions are commonly denoted by $h(\cdot)$. Still, we choose the convention $S(\cdot)$ that is preferred in the mathematical literature, see, *e.g.*, [6–10].

- Defining entropies with an additional minus sign to render them concave would make them lose their meaning as uncertainty measures, since, *e.g.*, $S(W)$ would then tend to $-\infty$ when approaching the maximally mixed state $\langle n\rangle \to 1/2$.

- For a comparison of the entropic uncertainty relations to the bosonic case, we note that the corresponding inequalities for a bosonic mode read $S_r(W) \geq (r-1)^{-1}\ln r + \ln\pi$ and $S_r(Q) \geq (r-1)^{-1}\ln r$. Since the fermionic Wigner W-distribution comes without the normalization factor of 2π that prevails in the bosonic case, the corrected bosonic lower bound on $S_r(W)$ would read $(r-1)^{-1}\ln r - \ln 2$, which is precisely the fermionic lower bound with an overall minus sign.

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