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Quantum general covariance as an extended symmetry principle

A study of relative Hamiltonian dynamics

Thesis submitted for the award of the degree of
Master of science in Computer Science and Engineering

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Academic year
2022 - 2023

Abstract

In this Thesis, I will apply to an illustrative toy model a framework which is a group theoretic generalisation that derives reversible transformations between quantum reference frames that hold for general groups. We start by examining the three fundamental characteristics of a quantum reference frame according to this framework. First, a particle that obeys quantum laws can not serve as a quantum reference frame to measure both position and momentum. Second, quantum observers who do not have access to an external reference frame, have access to invariant quantities under a given transformation which live on a so-called invariant subspace. And finally, there exists an "extra-particle" that is part of this invariant subspace and which is necessary to make the quantum reference frame transformations reversible. The main objectives are the following: first, we will investigate the situation when the quantum reference frame is part of the interaction (hence non-inertial). This is the first time this study is being done within this framework. Second, we will study the covariance of the dynamics in this scenario. Finally, we will study the role of this extra particle in the relative Hamiltonian description of the system for the translation, boost, and Galilei groups. In particular, we will study the role of the extra-particle in the relative description of both an inertial and a non-inertial quantum reference frame. To do so, we will start from the perspective of an external massive observer and we will apply appropriate quantum reference frame transformations to get to one frame or another. We will then compare the results between the different groups considered, and we will link our approach with the classical Hamiltonian mechanics one. We will see that this extra-particle can be interpreted as being fundamentally linked to properties of the centre of mass of the system. This link may lead, in future research, to Quantum Gravity implications. Finally, we will also discuss the nature of the concept of quantum reference frame.

Abstract

Dans cette thèse, j'appliquerai à un modèle simple un cadre théorique qui est une généralisation, sur base de la théorie des groupes, des transformations réversibles entre des systèmes de référence quantiques. Ce cadre théorique s'applique à des groupes généraux. Nous commencerons par examiner les trois caractéristiques fondamentales d'un système de référence quantique selon ce cadre. Premièrement, une particule qui obéit aux lois quantiques ne peut pas servir de système de référence quantique pour mesurer simultanément la position et la quantité de mouvement. Deuxièmement, les observateurs quantiques qui n'ont pas accès à un système de référence externe ont accès à des quantités qui sont invariantes pour une transformation donnée. Ces quantités vivent dans ce que l'on appelle un sous-espace invariant. Enfin, il existe une "particule supplémentaire" qui fait partie de ce sous-espace invariant et qui est nécessaire pour rendre les transformations entre systèmes de référence quantiques réversibles. Les principaux objectifs sont les suivants : premièrement, nous étudierons la situation où le système de référence quantique fait partie de l'interaction (donc lorsqu'il est un système de référence non inertiel). C'est la première fois que cette étude est réalisée pour ce cadre théorique. Ensuite, nous étudierons la covariance de la dynamique du système dans ce scénario. Enfin, nous étudierons le rôle de cette particule supplémentaire dans la description hamiltonienne relative du système pour les groupes de translation, d'impulsion et de Galilée. En particulier, nous étudierons le rôle de la particule supplémentaire dans la description relative d'un système de référence quantique inertiel et non inertiel. Pour ce faire, nous partirons du point de vue d'un observateur massif externe et nous appliquerons les transformations appropriées pour se positionner sur un système de référence quantique ou un autre. Nous comparerons ensuite les résultats entre les différents groupes considérés et nous ferons un lien entre notre approche et celle de la Mécanique Hamiltonienne classique. Nous verrons que cette particule supplémentaire peut être interprétée comme étant fondamentalement liée aux propriétés du centre de masse du système. Ce lien pourrait conduire, dans de futures recherches, à des implications en Gravité Quantique. Enfin, nous discuterons également de la nature du concept de système de référence quantique.

Acknowledgements

I want to thank Prof. Ognyan Oreshkov and Dr. Lin-Qing Chen for all the time they took for explanations and discussions about this absolutely fascinating subject. This work would certainly not have been possible without them.
I am extremely grateful and I could never thank them enough for entrusting me with this project for which I have always been passionate about.
Furthermore, I would like to thank all my family and friends for their support and their patience during all my studies of computer science engineering and now for my Thesis in physics.
I would like to address a particular thanks to my parents Olga and Valery and my sister Charlyne for their unconditional love and for never stop believing in me.

Contents

Chapter 1

Introduction

The 20^{th} century was an era of great theoretical physics development where both of the most solid scientific theories have been formalised: General Relativity in 1916 [1] and quantum mechanics for there are still many fundamental questions to be understood. The former studies the universe and how matter interacts at a large scale whereas the latter focuses on subatomic scales of matter and how quantum systems interact with each other. Both theories have made predictions that have been verified over and over and have shown resilience over the ages against refutation attempts.

Moreover scientific research in physics has always been driven by the willingness of unifying concepts to put them in a coherent broader conceptual frame whether it was James Clark Maxwell's unification of light, magnetism, and electricity during the 19^{th} century with the electromagnetism spectrum [2] and one single formalism (the Electromagnetism theory) or Albert Einstein's unification of space and time in 1905 with the Special Relativity theory [3]. Now because General Relativity and Quantum Mechanics seem to be the best candidate to describe reality, numerous attempts to "unify" them (i.e. how to include quantum mechanical effects into gravity on all distances and especially under the Planck length [4]) have been proposed such as Quantum Gravity [5], String theory [4] and many other more.

Reference frame is a central notion in physics and appears almost everywhere. Even in quantum mechanics, the use of it is quite implicit but is necessary if we want for instance to make sense of the position of a wave function. However, in some cases, the typical notion of reference frame as we understand it seems to be not sufficient to describe what we observe experimentally.

For that let's give an example which should motivate the study of such types of more general reference frames. The example relates to an issue regarding the spin in Special Relativity, and it will show that a reference frame with quantum properties may be required to make sense of the situation. The spin S of a particle in Special Relativity is defined as the total angular momentum J in the rest frame of the particle. Indeed, by definition, J = L + S where L is the orbital part, and in the frame where the particle is at rest, L = 0 and by performing the Stern-Gerlach experiment, we could indeed observe and make sense of the spin and treat the latter as a qubit, which could be useful if one would work on Quantum Information Theory. However, a problem arises if this particle is in a state such that its momentum is not properly defined i.e. if it is in a superposition of states where for example the particle is propagating along two different directions. How would we then choose a reference frame in which the particle is at rest? If in that case, a reference frame was in a superposition of momenta, then we could jump into that frame for which the particle would be at rest and for which we could properly define the spin [6] [7].

Now, however, we can underline the fact that in quantum mechanics, some issues seem to appear whenever we start considering an observer at the quantum level. Indeed, let's imagine two classical observers Alice and Bob. Alice wants to measure Bob's characteristics, she can and may measure all kinds of quantities like Bob's position, momentum, etc. because she has her own position and momentum relative to Bob.

Moreover, if we suppose that Alice and Bob are quantum particles and if Alice wants to measure Bob, we will see that several problems will occur. First of all, Alice is the observer and is considered to be quantum, hence she obeys quantum laws and according to the Heisenberg uncertainty principle, a particle can not have both a well-defined position and a well-defined momentum at the same time [8]. So if a measurement is relative, and her characteristics are not well defined, proper measurement can not be done. Hence to remove this problem regarding this quantum observer, it is implicitly suggested in quantum mechanics that the observer or the measurement apparatus is sufficiently heavy [9] such that it can be treated as a classical object for which this uncertainty regarding its position and momentum is small [8].

A second issue called the "paradox of the measuring device" related to this highly massive observer (which is well explained in [9]) stipulates that because of the Heisenberg principle again, it has been known that if a measurement is done on a system from an observer with a certain mass, an uncertain and unpredictable interaction occurs between the object and the observer which acts on the observer and makes its position and momentum unknown again (even though it may have been known initially).

These fundamental issues regarding the observation of a quantum system from a quantum observer motivate the study of quantum reference frames.

In addition to this fundamental interest, the study of quantum reference frames has actually a wide range of potential applications in various domains [10]. Some are related to Thermodynamics [11], quantum interferometry [12] or quantum communication [13], while others are more Computer Science related (and in particular in Quantum Information Processing [14]) where quantum reference frames are used for quantum computations and quantum algorithms [15] or even cryptography [16].

There is a way that has been adopted by [17] which solves the issues raised by [8] [9] but at the same time raises fundamental questions regarding the foundations of quantum mechanics. This approach consists in considering that one reference frame can not be used to measure both position and momentum since there seems to be some evidence that those are fundamentally independent variables. This approach will be used in Section 4 and it will in particular be discussed in Section 6. Moreover, this Thesis will explore the consequences of the model introduced by [17] by analyzing the relative Hamiltonian dynamics of a simple system with non-accelerating and accelerating particles.

Chapter 2

Basic theoretical background

In this section, we will briefly introduce some fundamental concepts in quantum mechanics and Group theory (relevant to this work) to help the reader get better insights into this work. This reminder does not aim and has no purpose in being exhaustive.

2.1 Quantum Mechanics

From the first postulate we know that "at each instant the state of a physical system is represented by a ket $|\psi\rangle$ in the space of states" [18] i.e. a complex Hilbert space $\mathcal{H}$ on which is defined inner product. The latter associates a complex number to two states $|\psi\rangle$, $|\phi\rangle \in \mathcal{H}$ [19] as such : $\langle\phi|\psi\rangle = \int \phi^*(x)\psi(x)dx$ where $\phi^*(x)$ is the complex conjugate of $\phi(x)$. From this first postulate, we can also introduce the superposition principle which states that if $|\psi_1\rangle$, $|\psi_2\rangle \in \mathcal{H}$ then $\lambda_1|\psi_1\rangle + \lambda_2|\psi_2\rangle \in \mathcal{H}$ [19].

The second postulate states that "every observable attribute of a physical system is described by an operator that acts on the kets that describe the system" [18]. An operator acting on a state $|\psi\rangle$ is conventionally defined as: $\hat{A} : |\psi\rangle \to |\psi\rangle' = \hat{A}|\psi\rangle$.
We also know that "acting with an operator on a state in general changes the state" [18].

For every operator $\hat{A}$, there are states $|\psi\rangle$ for which the action of $\hat{A}$ is such that $\hat{A}|\psi\rangle = a|\psi\rangle$. These states are called the eigenstates and the a are the corresponding eigenvalues of the operator $\hat{A}$ [18].

The operators for which eigenvalues are real are hermitian. The eigenstates of hermitian operators are orthogonal (the inner product of two states gives the Kronecker or Dirac delta whether it is a discrete or continuous basis) and because they span the associated Hilbert space in which they are defined they form a basis [18]. An arbitrary state $|\psi\rangle$ can be expressed in the $|x\rangle$ basis as $|\psi\rangle = \int dx\psi(x)|x\rangle$.

As we already said, the states of one given system A are defined on a Hilbert space $\mathcal{H}_A$. Consider now a second system B, the associated states corresponding to the latter are defined on a second Hilbert space $\mathcal{H}_B$. The associated Hilbert space $\mathcal{H}_{AB}$ of the combined system is defined as: $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ (where $\otimes$ is called a tensor product) such that for states $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$ the combined state $|\psi_{AB}\rangle (\in \mathcal{H}_{AB})$ can be written as $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ or even simpler as $|\psi_{AB}\rangle = |\psi_A\rangle|\psi_B\rangle$ [19].

One of quantum mechanics postulates also defines how a state $|\psi\rangle$ evolves in time according to Schrödinger's equation which can be written as such: $i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$ with $\hat{H}$ being a hermitian operator (and a Hamiltonian). This operator is the total energy observable. The solution to this equation gives the evolution of the state as: $|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}|\psi(0)\rangle$ [20] where $e^{-i\frac{\hat{H}t}{\hbar}}$ is the time evolution operator which can be seen as a translation in time [21] (we will come back to transformations as such). And for an isolated system i.e. a time-independent Hamiltonian, we can write this in the eigenbasis of $\hat{H}$ as: $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$ with E_n being the associated eigenvalues of $\hat{H}$ and which correspond to the possible energies the system can take [19].

For some cases, the state $|\psi\rangle$ does not provide a complete description of the system. That is why a more general object called the density operator was introduced. It is defined as $\hat{\rho} : |\psi\rangle \rightarrow |\psi\rangle\langle\psi|$ and it follows an evolution according to the following Schrödinger's equation (also called the quantum Liouville equation) $i\hbar\frac{d}{dt}\hat{\rho} = [\hat{H}, \hat{\rho}]$ [22]. This density operator as defined above corresponds to a pure state. However, it is possible to have a probabilistic mixture of pure states $|\psi_k\rangle$ each associated with a probability p_k ($0 \leq p_k \leq 1$, $\sum_k p_k = 1$). This notion of mixture is called a mixed state and is also captured by the density operator as such $\hat{\rho} = \sum_k p_k \underbrace{|\psi_k\rangle\langle\psi_k|}_{=\hat{\rho}_k}$ [22].

Group theory (which we will develop in the next section) also plays an important role in quantum mechanics. It is in fact impossible to fully formulate quantum mechanics with references to symmetry. In physics, we are interested in how the groups act on physical systems [23]. Group theory led to major symmetry tools that helped the development (and are still widely used today) of the latter such as parity, energy, and momentum conservation. We can define a symmetry group as such [21]: "Finite or infinite set of symmetry operations with an associative composition law, involving a neutral element (identity operation $\mathcal{I}$) and an inverse $\mathcal{S}^{-1}$ for each operation $\mathcal{S}$ ($\mathcal{S}\mathcal{S}^{-1} = \mathcal{S}^{-1}\mathcal{S} = \mathcal{I}$)." Moreover, a symmetry group is a "Commutative or abelian group $\iff$ it satisfies the commutative law." We can think of discrete (parity or permutation groups) or continuous groups (such as translation, boost, or fixed axis rotations) [21].

Furthermore, to any symmetry operation, $\mathcal{S}$ corresponds an operator S defined as $S : |\psi\rangle \rightarrow$

$|\psi_S\rangle = S|\psi\rangle$. This operator has several properties such as the following: it is generally supposed unitary i.e. scalar product preserving. This can be written as $S^\dagger S = SS^\dagger = 1 \iff \langle\phi_S|\psi_S\rangle = \langle\phi|S^\dagger S|\psi\rangle = \langle\phi|\psi\rangle\, \forall|\phi\rangle, |\psi\rangle \in \mathcal{H}$ [21].

Up to now we have been talking about what we call Schrödinger's perspective where "The state vector changes with time, but the operator remains constant with time" which is to be distinguished from the Heisenberg's perspective where "The operator changes with time, while the state vector remains constant with time" [20]. These perspectives are represented (in a more general way) in the following Figure 2.1 where the unitary transformation S may correspond to the time evolution operator. In the rest of this work, we will choose (as shown in the following Figure 2.1) the convention of an operator transformation by S as $A_S = S^\dagger A S$, A being the operator.

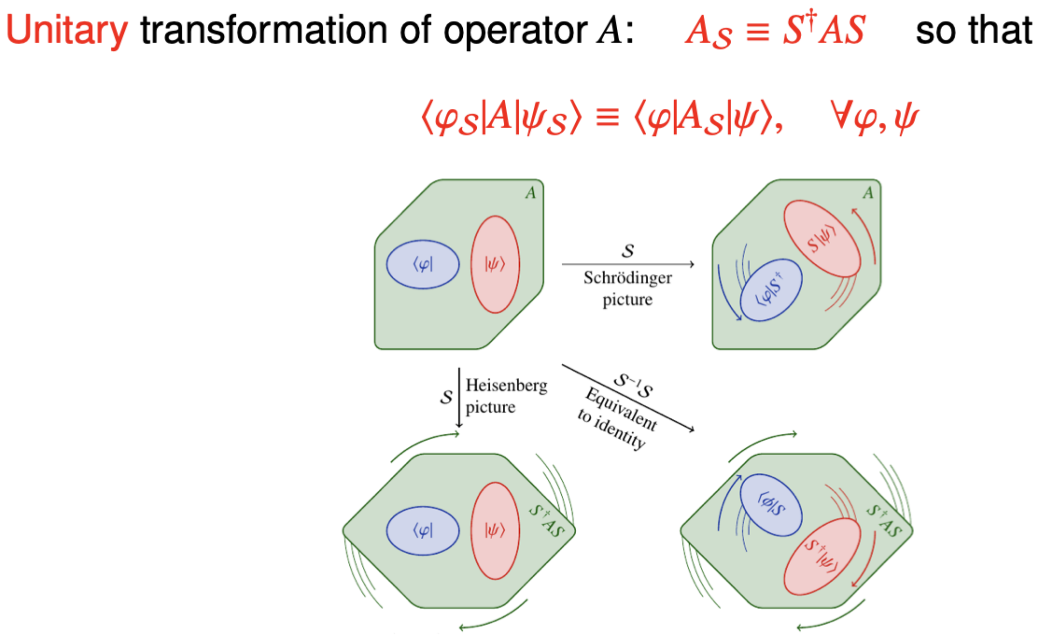

Figure 2.1: Schrödinger vs Heisenberg's picture [21]

2.2 Group theory

Group theory is a broad, technical and abstract field of mathematics. Hence, to improve the readability of this Section, we will display and list in an explicit and formal way the relevant definitions, lemma and theorem. These concepts will provide us the basis on which we will explain in Section 3.2 the framework developed in [17].

Compared to what we have done in the reminder of quantum mechanics, let us now define more formally the group as follows [23]:

Definition 1. *A group is a set G equipped with a binary operation (i.e. a direct mapping) called the group multiplication* $\circ : G \times G \to G$*, such that the following holds*

1. *Closure: if* a *&* $b \in G$*, then so is* $a \circ b \in G$*,*
2. *Associative:* $(a \circ b) \circ c = a \circ (b \circ c), \forall a, b, c \in G$*,*
3. *Identity element:* $\exists$ *an element* e *such that:* $a \circ e = a, \forall a \in G$*,*
4. *Inverses:* $\forall a \in G : \exists$ *an element* $a^{-1} \in G$ *known as the inverse of* a *such that* $a \circ a^{-1} \in G$*.*

Now that we have defined what a group G is, another important concept we should define is the notion of Abelian group. Indeed [23]:

Definition 2. *If all the elements of* G *commute, then* G *is Abelian.*

We will also need the definition of subgroups [23]:

Definition 3. *Suppose* G *is a group and* $H \subset G$ *such that* H *forms a group under the same multiplication of* G*. Then* H *is a subgroup (which verifies the four properties of a group and in particular it has the same identity element).*

Now, to make sure we are dealing with non-trivial things, we will define the notion of proper subgroups. Indeed [23]:

Definition 4. *Both* $\{e\}$ *&* G *are trivially subgroups. Such groups are known to be improper subgroups. If* $H \subset G$ *but* $H \neq \{e\}$ *and* $H \neq G$*, then we say* H *is a proper subgroup.*

Let us now introduce the concept of group homomorphisms and isomorphisms which will be needed for the quantum reference frame transformation. We have [23]:

Definition 5. *A group homomorphism f between two groups* G_1 *and* G_2 *is a map f:* $G_1 \to G_2$ *which preserves the group structure i.e. such that* $\forall a, b \in G_1$*: f(a · b) = f(a) · f(b). If it is a one-to-one map, this mapping becomes an isomorphism.*

Let us also redefine more formally what we have called previously a symmetry group. Suppose we have a set of n objects (in n places). Hence, this set can have $n!$ configurations. We define the symmetry group S_n as follows [24]:

Definition 6. *A symmetry group* S_n *is the group of all permutations of* n *objects.*

Let us introduce the concept of conjugate elements relevant for future definitions as follows:

Definition 7. *Two elements a & b of a group G are conjugate to each other if there exists some element g (not necessarily unique) such that $a = g \cdot b \cdot g^{-1}$. g is called the conjugating element.*

Is is also useful to define the direct product of groups [25]:

Definition 8. *A group G is said to be the direct product of two subgroups A & B i.e. $A \otimes B$ if:*

1. *All elements of A commute with all elements of B*
2. *Every element of G can be written in a unique way as $g = a \cdot b$ with $a \in A$, $b \in B$.*

Moreover, it will be useful to also define the direct sum of groups [26]:

Definition 9. *Suppose that V is a vector space with two subspaces U and W such that for every $v \in V$,*

1. *There exists vectors $u \in U, w \in W$ such that: $v = u + w$*
2. *If $v = u_1 + w_1$ and $v = u_2 + w_2$ where $u_1, u_2 \in U, w_1, w_2 \in W$ then $u_1 = u_2$ and $w_1 = w_2$.*

Then V is the direct sum of U and W and we write $V = U \bigoplus W$.

Now, in quantum mechanics we are interested in how states evolve under some symmetry transformations. These states are actually vectors, and according to Representation theory, these symmetry transformations are represented by linear operators or finite-dimensional matrices [25]. The set of operators associated with a symmetry group is called the representation of the group [21]. More formally, a representation is defined as follows [27]:

Definition 10. *If $GL(n,\mathbb{C})$ is the group of $n \times n$ matrices with complex entries and non-zero determinant, then a representation of the group G is defined as the homomorphism D: $G \to GL(n,\mathbb{C})$ such that the group structure is preserved i.e. if $g_1, g_2 \in G$: $D(g_1 g_2) = D(g_1)D(g_2)$.*

Furthermore, let us introduce the concept of reducibility of representations [27]:

Definition 11. *A representation D (of a group G) of dimension n+m is said to be reducible if D(g) takes the form:* $D(g) = \begin{pmatrix} A(g) & C(g) \\ 0 & B(g) \end{pmatrix}$ $\forall g \in G$, *where A, B and C are* $n \times n$, $n \times m$ *&* $m \times m$ *matrices respectively. If a representation is not reducible, then we say that the representation is an irreducible representation.*

Moreover, how can we go further without mentioning the famous Schur's lemma [27]:

Lemma 1. *If D is an irreducible representation of a group G and B is a matrix such that* $[B, D(g)] = 0 \; \forall g \in G$, *then* $B = \lambda \mathbb{1}$ *i.e. B is proportional to the identity matrix.*

We say a representation is irreducible if there is no subspace of that representation which is also invariant under all symmetry operations of the group [21]. For example, an irreducible representation of dimension 1 corresponds to a single state which will always be sent onto itself under all symmetry operations of a given group [21].

Just before we introduce a fundamental group theory theorem for mathematical physics, let us just briefly introduce some relevant properties: an operator is unitary if its inverse is equal to its adjoint [19]. Moreover, we can define an anti-linear operator as follows [28]:

Definition 12. *An operator* θ *acting on a complex-linear space is called "antilinear" if for any two vectors* ϕ_1, ϕ_2 *and complex numbers* c_1, c_2 *we have:*

$$\theta(c_1\phi_1 + c_2\phi_2) = c_1^*\theta(\phi_1) + c_2^*\theta(\phi_2). \tag{2.1}$$

Now because we consider quantum mechanics, let us state one of the fundamental Theorem that links it with Group theory, which has been demonstrated by Wigner [27]:

Theorem 1. *The group of symmetries of a quantum system is represented by linear, unitary or anti-unitary (unitary+anti-linear) operators acting on a Hilbert space.*

Finally, to finish our (brief!) introduction on group theory relevant to our work, let us now introduce Lie groups, Lie Algebras and associated useful notions such as the left/right regular representations [29]:

Definition 13. *A Lie group G is a group which is also a manifold. There are continuous parameters labelling the group elements and these parameters are coordinates on some curved space. In addition to the group structure* $G \times G \to G$, *a Lie group inherits all the additional structure of a manifold such as continuity and differentiability.*

Before we continue, let us also define an algebra (more precisely an algebra over a field) which is according to [30] "a vector space with a bilinear multiplication" which is "not necessarily associative."

Let us now establish the link between the Lie group and the Lie Algebras [29]:

Definition 14. *An element of a Lie group is also a point on the same manifold. We can parametrise these elements by the coordinates α_i, $i = 1, ..., n$, on the manifold: $g = g(\vec{\alpha})$, where $\vec{\alpha}$ is an n-tuple.*

We can choose the coordinates/a parametrisation such that $g(\vec{\alpha})|_{\vec{\alpha}=\vec{0}} = e$ and such that the corresponding irreducible representation is $D_n(g(\vec{\alpha}))|_{\vec{\alpha}=\vec{0}} = \mathbb{1}_n$. We can also show that the form of a finite element of the Lie group is $e^{i\alpha_i X_i}$ for small $\delta\alpha_i$ around e and where the X_i are the generators for $\vec{\alpha} = \vec{0}$ [29].

We could also show [29] that for a general group element $e^{i\theta_i X_i}$ the form of the algebra satisfied by $\{X_i\}$ is $[X_i, X_j] = i f_{ij}^k X_k$ (i.e. the commutator of two generators must be proportional to another generator) where f_{ij}^k is called the structure constant. Hence, the Lie algebra captures the non-commutative nature of the Lie group multiplication in the neighbourhood of e [29].

To conclude this part, we can say that on a Lie group G, there is a natural notion of vector field [29].

Definition 15. *We can define the left action (or left regular representation) as follows:*

$$f : G \to G : g \to f_h(g) = hg, \tag{2.2}$$

and the right action (or right regular representation) as follows

$$f' : G \to G : g \to f'_h(g) = gh^{-1} \tag{2.3}$$

Chapter 3

State-of-the-art

The field of study of quantum reference frames is a relatively recent subject in the study of quantum mechanics. This state-of-the-art will provide us with a better overview of the current problems with which the scientific community is confronted. Indeed, this field of research has already been treated in a lot of articles, we can mention [31] which underlines the need for taking into account internal degrees of freedom, [32] which proposes a symmetry-based approach by studying a translational-invariant toy model of three free particles. Some other previous works have also treated the case of non-inertial quantum reference frames [33] [34] [35]. Also, a broad amount of work already exists on quantum reference frames transformations such as for translations [36] [37] [38] [39], boosts [36] [38] [39] and for the Galilei group [38] [40] [41]. Here, however, we will make a particular focus on two recent works [17] [42].

3.1 Current limitations in literature

In this section we will focus on a fundamental distinction between previous approaches such as [17] and [42]. One of the former's limitation is that this model seems to be restricted only to a vanishing total momentum. In particular, to be able to derive reversible transformations between two QRFs, previous approaches required that their description contained the whole universe. This non-locality of the description (that what we observe locally should depend on all objects of the universe) is what motivated the approach taken by [17].

Let's describe the problem we might encounter using this internal approach of [42]. Let's consider 2 particles A and B. Initially let us say we're in A perspective and A measures B (which is in a superposition of 2 states B and B' with position x_1 and x_2 respectively). This situation is represented in Figure 3.1 and mathematically this state can be written as the following linear combination:

$$|x\rangle_{B|A} = (\alpha|x_1\rangle^B + \beta|x_2\rangle^B)|0\rangle^A. \tag{3.1}$$

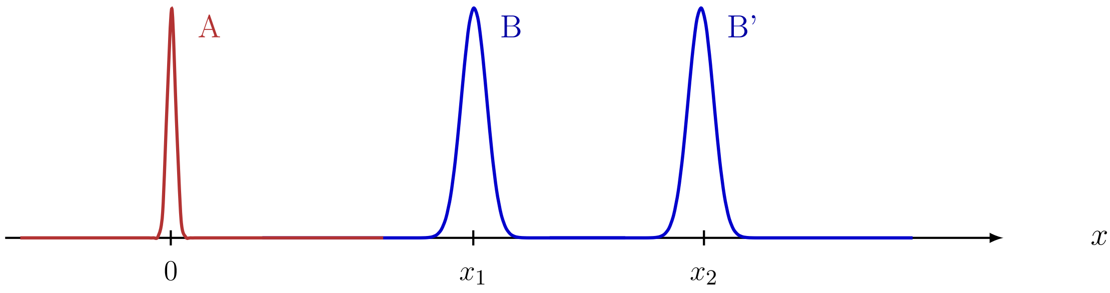

Figure 3.1: Particle B in a superposition of two states observed from a localised particle A

If we now go to B's perspective, by applying a translation towards each of those tight packets B and B' as they would do in [42] (naturally this is possible since A particle is localised), we observe an A which is now in a superposition of states. This new state can be written as the following superposition:

$$|x\rangle_{A|B} = (\alpha'|-x_1\rangle^A + \beta'|-x_2\rangle^A)|0\rangle^B. \tag{3.2}$$

The problem occurs when we have a new particle C (cf paradox of the third particle [43] [37]). Indeed, let us say that A and C are localised and are separated with a distance x_3.

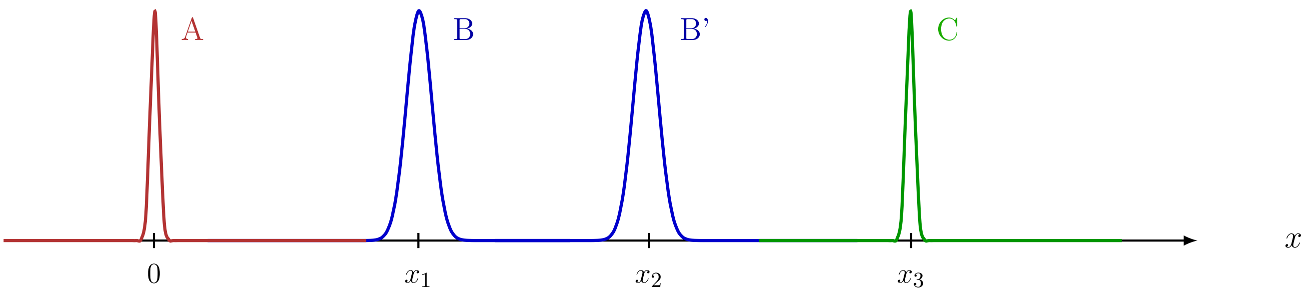

Figure 3.2: Particle B in a superposition of two states and a localised particle C observed from a localised particle A

Again, let us say we observe the situation from A's perspective, we have the following:

$$|x\rangle_{BC|A} = |0\rangle^A(\alpha|x_1\rangle^B + \beta|x_2\rangle^B)|x_3\rangle^C. \tag{3.3}$$

But now if we jump according to [42] to B's perspective, we get

$$|x\rangle_{AC|B} = |0\rangle^B(\alpha'|-x_1\rangle^A|-x_1+x_3\rangle^C + \beta'|-x_2\rangle^A|-x_2+x_3\rangle^C). \tag{3.4}$$

When we go from a description of a pure state to one which is in a mixed state, we get descriptions of pure states which are now entangled i.e. if we trace out C (by taking the partial trace over C of the quantum state of this bipartite system AC we actually discard C), we see that A is mixed [22] i.e. A and B are not related by a reversible transformation (i.e there is a loss of information going from one RF to the other [44]).

Which is in contradiction with the computation when you don't have C. The argument used in [42] is that we should have considered C initially. However, if we extrapolate this reasoning, we must argue that this frame holds only if we had initially considered all other particles in the rest of the universe.

The framework obtained in [42] using only internal reference frames switching is actually derivable from [17] when the total momentum of the system vanishes (cf. Subsection 3.3.2) and this would correspond to a system (= all objects in the universe) that is *infinitely spread* according to Heisenberg's principle [19].

The model derived in [17] starts from an external perspective E looking at a given system composed of observers A and B and a particle S. From there, we can jump to A's perspective internally to the system and then the internal transformations are self-sufficient as the model introduces the concept of this extra particle to keep track of additional information such that the transformations between all reference frames are reversible. We could see [42] as a framework in which this extra particle is 0. It is worth underlying again that the model developed in [17] does not depend and is not based on this initial external frame since what is observed in the system could be true for any external observer. The internal transformations thus become sufficient.

In addition because of this internal treatment at some point in [42] they consider 3 free particles A, B, and C and assume that from C's perspective, the Hamiltonian is given by $\hat{H}_{AB|C} = \hat{H}_A + \hat{H}_B$. However, this result is only true if the reference frame C is infinitely heavy because there is no trace of this C's mass. But then they switch RF and the mass of C suddenly appears because they consider finite mass particles. So using their assumption of the relative Hamiltonian $\hat{H}_{AB|C} = \hat{H}_A + \hat{H}_B$ they actually get to a contradiction.

Hence this Thesis is a direct and original application of [17] to a simple system considered for the translation, boost, and Galilei groups which correspond to translations in space, momentum, and combination of the two respectively. This system will allow us to study the situation when: the quantum reference frame is interacting with a part of the system, when the quantum reference frame is a composed system of both inertial and non-inertial for the Galilei group (we will have both an $\hat{x}$-frame and $\hat{p}$-frame).

In Section 3.2 we will expose and explain the important concepts of the framework in which our analysis will be made. All following explanations come from [17] from which we will recover the same notations and [6].

3.2 The Quantum reference frame transformation model

First of all, let us already say that [17] is a group theoretic generalisation that derives reversible transformations between quantum reference frames that hold for general groups [17] [6].

Let us now define a group G with its corresponding group elements g. We can define a regular representation as the following, it is a representation of the group where elements of the group are labelled by kets. And in that context, we define a reference frame as a system characterised by a Hilbert space that carries the regular representation of a group. So for example, a position operator $\hat{x}$ which lives on the Hilbert space of the reference frame can be seen as a parameter of the translation group symmetry i.e. it labels group elements because each group element g can be obtained by a translation. In particular, the regular representation of a group is such that all group elements g, h $\in$ G are orthogonal i.e. $\langle g|h\rangle = \delta(g^{-1}h)$. In other words, if we take a state $|x\rangle$ and we translate it by α those states are orthogonal to one another i.e. $\langle x|x+\alpha\rangle = 0$ i.e. we say that they break the symmetry maximally which means that they are perfectly distinguishable. When a quantum reference frame can be prepared in a basis of states which break the symmetry maximally of the group, we say that this quantum reference frame is perfect [17] [6].

Furthermore let us consider the context with which this model was developed, for that we will consider 2 observers and 1 system: A (Alice), B (Bob), and S (the system). We say that A and B carry a regular representation of the group G and S carries some unitary representation U_S of G. If A serves as a quantum reference frame, then its Hilbert space $\mathcal{H}_A$ is the span of a fully distinguishable basis of states that are labelled by group elements $|g\rangle_A$ Let us also define what we call the left regular representation which acts as $L_A(g)|h\rangle_A = |gh\rangle_A$ and right regular representation which acts as $R_A(g)|h\rangle_A = |gh^{-1}\rangle_A$. We say that $\mathcal{H}_A$ carries both regular representations. The same holds for B of course [17] [6].

Let us also assume an external observer E (Eve) which observes the full Hilbert space i.e. her description of the composite system lies in the following Hilbert space $\mathcal{H}_{ABS|E} = \mathcal{H}_{A|E} \otimes \mathcal{H}_{B|E} \otimes \mathcal{H}_{S|E}$ where we introduce the notation $|E$ which corresponds to the perspective of Eve. $\mathcal{H}_{A|E}$ for example is the Hilbert space of A relative to E. We also state that Eve sees what Alice sees. However, the main assumption is that Alice and Bob do not have access to what Eve sees (they can not access this reference frame). We say that Alice and Bob only have access to (i.e. they can only measure) invariant observables under the group. For example, Alice and Bob both have access to momenta measurement under the translation group since $|p\rangle$ is invariant under translations [17] [6].

There comes the link between [42] and [17]. In the former we have a rather internal description, whereas in the latter we start from Eve which has a quantum mechanical description of A, B, and S. However according to [17] both approaches agree when projecting into the invariant subspace of pure states $|\psi\rangle$ which is defined as $U(g)|\psi\rangle = |\psi\rangle \ \forall g \in G$ where U is the global action of G on the total Hilbert space (i.e. $\mathcal{H}_{ABS|E}$ here). It is the subspace where we apply a global transformation to the global system composed of A, B, and S. In the case of [42] they actually restrict to the trivial subspace where the global state has a total momentum = 0 as we will see in 3.3.2 [17] [6].

As it is reminded in Section 2.1, we know that the density operator $\hat{\rho}$ is a more general concept than $|\psi\rangle$, hence it can be easily seen than the approach taken by [17] is a less restrictive one than [42] since in the former, observers that do not have access to the external observer Eve have access to the invariant density operators which are invariant under the action group G i.e. the $\hat{\rho}$ such that $U^{\dagger}(g)\rho U(g) = \rho$ (according to the convention chosen in 2.1). In particular, this allows for [17] framework to not even have to precise the value of the total momentum in the case of the translation group. In fact, the latter is (in more general group theory terms) what they call in [17] the "total charge" and is a GLOBAL invariant quantity [1]. As we will see the transformations which have been derived in [17] are indeed consistent with [42] for the restricted case where we consider a vanishing total momentum (i.e a zero charge sector) as in Section 3.3.

For the sake of completeness, we can also mention the use of the G-twirl operation on an operator T that lives on $\mathcal{H}_{A|E} \otimes \mathcal{H}_{B|E} \otimes \mathcal{H}_{S|E}$. Indeed for such operators T, as we already said, A and B only have access to their invariant part hence, to remove all the redundant part i.e. the part which is not invariant, we apply on T the G-twirl defined as such:

$$\mathcal{G}[T] := \int dg V^{\dagger}(g)\, T\, V(g), \tag{3.5}$$

where $V(g) = L_A(g) \otimes L_B(g) \otimes U_S(g)$ with respectively the left regular representation of A and B and the representation of S [17] [6].

Now that we have introduced some of the key concepts of [17] let us now associate rather algebraic concepts to what we have seen. For that, we can simplify our initial configuration and consider only one observer A, and a system S, both being observed by an external observer E. If Alice measures an observable T (whether it is the position, momentum, or anything else) of the system S, she applies to the system the operator $T_{S|A}$. These operators form an algebra We said earlier that Eve observes Alice and the system but also sees what Alice sees, hence we can express in the perspective of Eve this operator $T_{S|A}$ which is:

$$T_{(S|A)|E} = \int dg |g\rangle\langle g|_A \otimes U_S^{\dagger}(g) T_{S|A} U_S(g). \tag{3.6}$$

[1] Global invariant quantities under the group are accessible from other external observers E, E', etc. This is to be distinguished with local invariant quantities of S accessible by A and B. This is just to show that the principle is consistent no matter the system for which we consider the invariant subspace.

These operators can be written more easily as the following (by dropping the $|E$ but also the $|A$ for the operator in the integral, however, T_S does not lose its initial meaning)

$$T_{S|A} = \int dg|g\rangle\langle g|_A \otimes U_S^\dagger(g) T_S U_S(g). \tag{3.7}$$

These operators $T_{S|A}$ (previously $T_{(S|A)|E}$) form an algebra called $S|A$ [17] [6].

Let us now state an important concept which is illustrated in Figure 3.4 and which we will show right now: A quantum reference frame corresponds to a certain tensor factorisation of the invariant subspace and a quantum reference frame transformation corresponds to a change of the factorisation. To show this, let us come back to the Hilbert space of Alice and the system S relative to Eve and let us drop the $|E$ (as we did with 3.6) we have: $\mathcal{H}_{AS} = \mathcal{H}_A \otimes \mathcal{H}_S$. As we said previously, $\mathcal{H}_A$ carries both right and left regular representation hence we can rewrite the Hilbert space relative to E as:

$$\mathcal{H}_{AS} = [\bigoplus_q \mathcal{H}_{A_L}^{(q)} \otimes \mathcal{H}_{A_R}^{(q)}] \otimes \mathcal{H}_S. \tag{3.8}$$

In the above expression, we have a direct sum over q which labels a specific irreducible representation and is called the charge. Then we have those $\mathcal{H}_{A_L}^{(q)}$ and $\mathcal{H}_{A_R}^{(q)}$ which correspond to the Hilbert space carrying the left and right regular representation of the group respectively. The former is analog to the centre of mass (and is called colour) whereas the latter is analog to the relative coordinates between the things that compose the regular representation (and is called the flavour). $\mathcal{H}_{A_R}^{(q)}$ does not transform under the action of the group on the system (Alice+S) indeed for example: if E translates the full system composed of A and S, the center of mass will move but not the relative coordinates i.e. the group acts only on $\mathcal{H}_{A_L}^{(q)}$ and $\mathcal{H}_S$. In other words, the subsystem on which the group acts are composed of operators of the form $\bigoplus_q T_{A_L}^{(q)} \otimes \mathbb{1}_{A_R}^{(q)} \otimes T_S$. And because these transform under the group, they are the redundant part of the full Hilbert space i.e. there are not part of the GLOBAL INVARIANT subspace. However, operators of the form $\bigoplus_q \mathbb{1}_{A_L}^{(q)} \otimes T_{A_R}^{(q)} \otimes \mathbb{1}_S$ (which describe the relative degrees of freedom) are invariant under the action of the group on the global system. Hence, 3.7 (which is a relative observable) is invariant under a global transformation i.e. using 3.5 with $V(g) = L_A(g) \otimes U_S(g)$ we have: $\mathcal{G}[T_{S|A}] = T_{S|A}$. Moreover, we can say that $\mathcal{H}_{A_L}^{(q)}$ is the commutant of $\mathcal{H}_{A_R}^{(q)}$ (intuitively, if we have $\mathcal{H}_{A_R}^{(q)}$, then $\mathcal{H}_{A_L}^{(q)}$ is the "rest" of $\mathcal{H}_A$). And as $\mathcal{H}_{A_R}^{(q)}$ is the invariant part, we call its commutant the gauge subsystem [17] [6].

Now we want to define an operator/a reference frame transformation that will change the factorisation of the Hilbert space. Indeed, 3.8 is a factorisation that is natural to Eve but one may ask: what does a tensor product factorisation natural to Eve would look like if we wanted to write things with respect to Alice? For her, it is natural that what she calls the system S relative to her occupies a given tensor factor in the full Hilbert space [2]. Thus

[2]Note that this tensor factor $\mathcal{H}_{S|A}$ overlaps the invariant subspace but the latter is actually bigger than the former [17]

we can define an operator (analog to 3.7) which allows us to perform this change of tensor product factorisation of the Hilbert space seen from E i.e. a rearrangement of this Hilbert space to make it natural to A [17] [6]:

$$U_S^\dagger(\hat{g}_A) = \int dg |g\rangle\langle g|_A \otimes U_S^\dagger(g_A). \tag{3.9}$$

To make things more concrete, we can give the example of the Translation Group (which we will further study in Section 4.2.1) where if we were to translate from Eve to Alice we would simply replace all the g by x [3]. So if I take an operator which is $\in \mathcal{H}_{AS}$ (tensor factorisation natural to E) as defined in 3.8 and I apply the following isomorphic transformation :

$$\mathcal{V}_{E \to A} : T_{AS|E} \to V_{E\to A}^\dagger T_{AS|E} V_{E\to A}, \tag{3.10}$$

where $V_{E\to A} = F_{E\to A} \circ U_S^\dagger(\hat{g}_A)$ and $F_{E\to A}|g\rangle_{A|E}|\alpha\rangle_{S|E} = |g\rangle_C|\alpha\rangle_{S|A}$ we obtain an operator which is now natural to A's perspective. In particular, we could now write 3.8 as:

$$\mathcal{H}_{AS} \cong \mathcal{H}_{C,S|A} := \mathcal{H}_C \otimes \mathcal{H}_{S|A}. \tag{3.11}$$

To make things clear: 3.9 makes the transformation and $F_{E\to A}$ relabels the indices.

Now because $U_S^\dagger(\hat{g}_A)$ is a unitary operator on $\mathcal{H}_{AS}$, $\mathcal{H}_{S|A}$ carries a unitary representation $U_{S|A}$ of G [17] and $\mathcal{H}_C$ (where C is what we call the commutant of algebra $S|A$) carries the left and right regular representation of G [17] we can rewrite 3.11 as

$$\mathcal{H}_{C,S|A} = [\bigoplus_q \mathcal{H}_{C_L}^{(q)} \otimes \mathcal{H}_{C_R}^{(q)}] \otimes \mathcal{H}_{S|A}. \tag{3.12}$$

What is interesting here is that we know there are redundancies in the operators that live on $\mathcal{H}_{C,S|A}$. Indeed [17] shows that we can map the gauge subsystem to $\mathcal{H}_{C_L}$. Hence we conclude that C_R and $S|A$ form together the complete invariant algebra. The invariant operators (the only ones measurable by Alice) can be written as [17] [6]:

$$T_{inv} = \bigoplus_q \mathbb{1}_{C_L}^{(q)} \otimes T_{C_R,S|A}^{(q)}. \tag{3.13}$$

Because C_R is the complement of $S|A$ in the full invariant subspace of $\mathcal{H}_{C,S|A}$ (which we define as $\mathcal{B}_{inv}(\mathcal{H}_{C,S|A})$), we define C_R as $\overline{S|A}$ such that [17] [6]

$$\mathcal{B}_{inv}(\mathcal{H}_{C,S|A}) = \overline{S|A} \otimes S|A. \tag{3.14}$$

This algebra $\overline{S|A}$ is called the extra-particle [17] [6]. This algebra contains information that is necessary for the unitarity of the quantum reference frame transformations between quantum observers observing a quantum system.

[3] An important comment to make is that 3.9 has to be seen as a controlled transformation from E to A. In other words, we make the transformation of group parameter $\hat{g}_A$ (it is controlled in this sense) but we have not yet jumped in that quantum reference frame. This will be taken care of by the isomorphic relabelling $F_{E\to A}$ which will be defined just after 3.10.

Going back to the previous configuration with an additional observer Bob, we can derive an analogous procedure to get to the same conclusions for him. And using back the transformation we have used to go from E to A 3.10, we can actually define a Quantum Reference Transformation that allows us to go from A to B as the following:

$$\mathcal{S}_{A \to B} = \mathcal{V}_{E \to A} \circ \mathcal{V}^{\dagger}_{E \to B}, \tag{3.15}$$

where $\mathcal{V}^{\dagger}_{E \to B}$ where is defined in the same way as 3.10.

Figure 3.3 shows an intuitive picture of what we have been discussing in this Section. As we see, we have an external observer, Eve, our two observers Alice and Bob, and a system S. The transformation that allows us to go from Eve to Alice is 3.10 (corresponding to the grey arrow). If we would like to change from Alice to Bob's perspective, we apply the transformation 3.15 (the big arrow) which is a composition of a transformation going from Alice to Eve and from Eve to Bob's perspective.

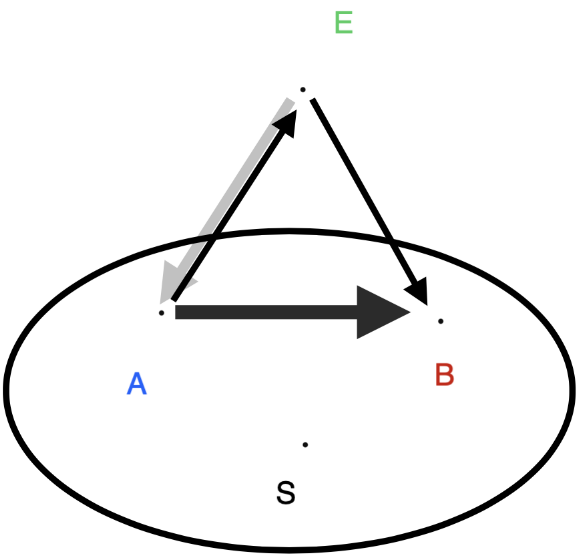

Figure 3.3: The situation in practice

On Figure 3.4 [17] is represented how this change of quantum reference frame perspectives corresponds to a tensor factorisation change. In particular, the green lattices correspond to Bob's perspective, whereas the orange ones correspond to Alice's. As we see, the algebras $S|A$ and $S|B$ partially overlap, as well as $BS|A$ and $AS|B$ do too. However, those are not equal to one another i.e. they are not unitarily related. This is why we need the extra particle for each quantum reference frame. Indeed, for example, $\overline{SB|A}$ needs to be there to complete the full invariant subsystem with S and B relative to A. Hence, it will provide us the remaining information to make the quantum reference transformations unitary. This Figure 3.4 shows a configuration for two observers, but it can be extended for an arbitrary number of quantum reference frames.

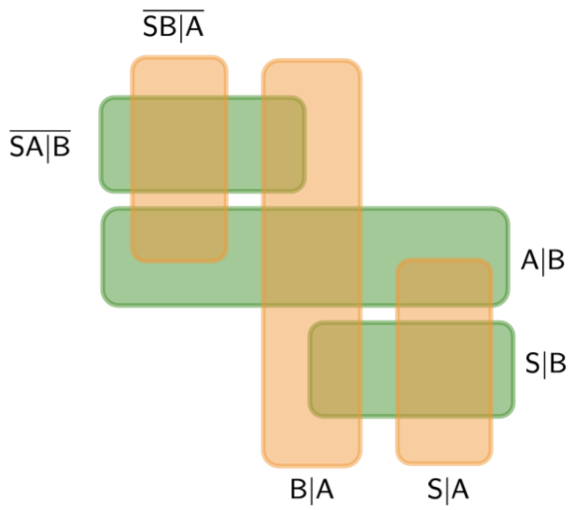

Figure 3.4: Hilbert space seen from the perspective of A and B with the extra particles required to get a unitary relation from A to B's factorisation [17]

Sections 3.4.1 and 3.4.2 will allow us to identify some important points from [42] on which this Thesis will not rely and on which it will actually disagree. Sections 3.3 and 3.3.2 will allow us to compare articles [17], [42], and [45]. In particular, Sections 3.3 and 3.3.2 will show how the transformations derived in [17] are consistent with the transformations of two other previous works provided we restrict [17]'s framework in the zero charge sector (i.e. a vanishing total momentum in the case of the Translation group).

We will now recover results from [42], [45], and [17]. This will allow us to show how [42] and [45] are particular cases of [17].

3.3 Link between different works

3.3.1 Transformations from one observer to another using three distinct approaches

1. First, let us see how [45] has achieved the change of reference frame in [42].

Consider 3 particles A, B, and C and the corresponding state of the joint system on $\mathbb{R}$ relative to A is [45]:

$$|\psi\rangle_{ABC} = |0\rangle \otimes \int dx \int dy\, \psi(x,y)|x\rangle_B \otimes \int dy\, |y\rangle_C. \tag{3.16}$$

If the reference system is not explicitly included in the state of the joint system: then the state equivalent to $|\psi\rangle_{ABC}$ is:

$$|\psi\rangle_{BC} = \int dx \int dy\, \psi(x,y)|x\rangle_B \otimes \int dy\, |y\rangle_C. \tag{3.17}$$

To go from A's point of view to B's we apply the transformation:

$$\hat{S}^{A\to B} = \hat{\mathcal{P}}_{AB} e^{\frac{i}{\hbar} x_B \hat{p}_C}. \tag{3.18}$$

Thus, we have

$$|\psi\rangle_{AC} = \hat{S}^{A\to B}|\psi\rangle_{BC} = \int dx \int dy\, \psi(x,y)|{-x}\rangle_A \otimes |y-x\rangle_C. \tag{3.19}$$

2. We will now consider a change of reference frame in [45].

To go from A's point of view to B's, we apply the operator [45]

$$U^{A\to B} = SWAP_{A,B} \circ \int dxdy|{-x}\rangle\langle x|_B \otimes \mathbb{1}_A \otimes |y-x\rangle\langle y|_C \tag{3.20}$$

to the ket $|\psi\rangle_{ABC}$ (here the reference system A is explicitly included in the joint system state) i.e.

$$\begin{aligned} U^{A\to B}|\psi\rangle_{ABC} &= SWAP_{A,B} \circ (|0\rangle_A \int dx'dy'dxdy\psi(x,y) \left\langle -x'|x'\right\rangle |x\rangle|y'-x'\rangle \left\langle y'|y\right\rangle) \\ &= |0\rangle_B \int dx \int dy\psi(x,y)|{-x}\rangle_A \otimes |y-x\rangle_C \end{aligned}$$

$$\implies |\psi\rangle_{AC} = \int dx \int dy\, \psi(x,y)|{-x}\rangle_A \otimes |y-x\rangle_C. \tag{3.21}$$

Hence we see that transformation 3.20 is equivalent to 3.18.

3. Now we will show how [17] makes a change of reference frame for states in the zero charge sector.

Consider 2 quantum reference frames A and B and a system S. The total Hilbert space decomposes into a sum of charge sectors cf. Subsection 3.2. Consider $|\psi\rangle$ in the zero charge sector such that

$$L_A(g) \otimes L_B(g) \otimes U_S(g)|\psi\rangle = |\psi\rangle \quad \forall g \in G, \tag{3.22}$$

with the left-regular representation L_A, acting as defined in Subsection 3.2.

The quantum state $|\psi\rangle$ can be obtained as such:

$$\implies |\psi\rangle = \int dg L_A(g) \otimes L_B(g) \otimes U_S(g)|\phi\rangle, \tag{3.23}$$

where $|\phi\rangle$ is an arbitrary state defined as

$$|\phi\rangle = \int dg_A dg_B |g_A\rangle_A \otimes |g_B\rangle_B \otimes |\phi(g_A, g_B)\rangle_S. \tag{3.24}$$

In the tensor factorisation natural to A, this state can be written as:

$$F_{E\to A} U^\dagger_{BS}(\hat{g}_A)|\psi\rangle = |\Omega\rangle_C \otimes \int dg\, L^\dagger_{B|A}(g) \otimes U^\dagger_{S|A}(g)|\phi(g)\rangle_{B|A,S|A}, \tag{3.25}$$

where C is the commutant of BS|A, $|\Omega\rangle = \int dg|g\rangle$, $|\phi(g)\rangle_{B|A,S|A} = \int dg'|g'\rangle_{B|A} \otimes |\phi(g,g')\rangle_{S|A}$ and $\int dg\, L^\dagger_{B|A}(g) \otimes U^\dagger_{S|A}(g)|\phi(g)\rangle_{B|A,S|A}$ is the state of B and S relative to A. Now to go from A to B's perspective we apply

$$S_{A\to B} = U^\dagger_{AS}(\hat{g}_B) U_{BS}(\hat{g}_A) \tag{3.26}$$

to the joint system state seen from A i.e. $U^\dagger_{BS}(\hat{g}_A)|\psi\rangle$ i.e.

$$\begin{aligned} S_{A\to B} U^\dagger_{BS}(\hat{g}_A)|\psi\rangle &= U^\dagger_{AS}(\hat{g}_B)|\psi\rangle \\ &= \int dg dg' L_{A|B}(g)^\dagger \otimes \mathbb{1}_D \otimes U^\dagger_{B|S}(g)|g'\rangle_{A|B} \otimes |\Omega\rangle_D \otimes |\phi(g',g)\rangle_{S|B} \end{aligned} \tag{3.27}$$

where D is the commutant of AS|B. Now we will show that applying $S_{A\to B}$ to $U^\dagger_{BS}(\hat{g}_A)|\psi\rangle$ is equivalent to the transformation done in [45] with $U^{A\to B}$. Indeed [17] has defined the operator

$$\hat{D} = SWAP_{AB} \circ \mathbb{1}_C \otimes \int dh |h^{-1}\rangle\langle h|_{B|A} \otimes U^\dagger_{S|A}(h), \tag{3.28}$$

which is the operator found in [45] up to an arbitrary exchange of the roles between the left and the right regular representations and where $SWAP_{AB}$ is the operator that swaps A and B's Hilbert spaces. Furthermore, it has been shown that:

$$S_{A\to B} U^\dagger_{BS}(\hat{g}_A)|\psi\rangle = \hat{D} U^\dagger_{BS}(\hat{g}_A)|\psi\rangle, \tag{3.29}$$

hence $S_{A\to B}$ and $\hat{D}$ coincide in the zero charge subspace. So $S_{A\to B} = U^\dagger_{AS}(\hat{g}_B)U_{BS}(\hat{g}_A)$ applied to states in the zero charge subspace is equivalent to $\hat{D}$ in [45] which is itself equivalent to $\hat{S}^{A\to B}$ in [42].

3.3.2 Vanishing total momentum and the zero charge sector

In [42] they have an internal treatment, they allow a jump from a quantum reference frame to another without using an external observer. Moreover, they treat separately 1D translations and Galilean boosts which correspond to specific quantum reference frame transformations. Indeed they use a single particle model i.e. a single particle of finite mass m can be a perfect reference frame for the translation group or for Galilean boosts but not for both simultaneously.

Whereas in [17], they consider that in a fixed mass sector m there is a system of 2 particles for which one could be used as a perfect quantum reference frame for position and the other for momentum. In the limit of mass sector m $\to \infty$, a single particle can be used as a perfect QRF for both position and velocity as we will see for the external reference frame in Subsection 4.2.1 (cf. Section 6 for additional discussions on this topic).

Hence if we introduce in the latter model an external reference frame for momentum which is aligned with the momentum of centre-of-mass we get:

$$\hat{p}_{CM}|\psi\rangle = 0. \tag{3.30}$$

In this equation, the constraint state $|\psi\rangle$ can be interpreted as a state whose centre-of-mass momentum vanishes when seen from the external observer. So there are N+1 particles, the N+1$^{\text{th}}$ particle is used for the momentum reference frame and the 1$^{\text{st}}$ is the position reference frame. So if the extra particle is ignored (as in [42]) and the total momentum of the N particles is assumed to be 0 with respect to the particle N+1, then we get the special case corresponding to article [42].

3.4 Small digression regarding covariance

This Section is about a small discussion regarding a result obtained in [42] where they suppose that the transformation that allows to go from one perspective to another is time dependent.

3.4.1 Schrödinger's equation in the case of a change in reference frames: Consequence for the Hamiltonian

Consider 3 particles A, B and C and let us consider the QRF unitary transformation that allows us to move from C's perspective to A's [42]:

$$\hat{S}_{C\to A} : \mathcal{H}_{A|C} \otimes \mathcal{H}_{B|C} \to \mathcal{H}_{B|A} \otimes \mathcal{H}_{C|A}. \tag{3.31}$$

We also know the Schrödinger's equation for the density matrix [22] in C's perspective is the following:

$$i\hbar \frac{d\hat{\rho}_{AB|C}}{dt} = [\hat{H}_{AB|C}, \hat{\rho}_{AB|C}]. \tag{3.32}$$

When applying the transformation

$$\hat{\rho}_{BC|A} = \hat{S}^{\dagger}_{C\to A} \hat{\rho}_{AB|C} \hat{S}_{C\to A}, \tag{3.33}$$

we obtain the following Schrödinger equation relative to the quantum reference frame A:

$$i\hbar \frac{d\hat{\rho}_{BC|A}}{dt} = [\hat{H}_{BC|A}, \hat{\rho}_{BC|A}], \tag{3.34}$$

where the new Hamiltonian is expressed as an expected term and an inhomogeneous term [42]:

$$\hat{H}_{BC|A} = \hat{S}_{C\to A}\hat{H}_{AB|C}\hat{S}^{\dagger}_{C\to A} + i\hbar\frac{d\hat{S}_{C\to A}}{dt}\hat{S}^{\dagger}_{C\to A}. \tag{3.35}$$

This is a result which has been considered in [42] for which the proof is given in Appendix 8.1. However, in the context of the model introduced in Subsection 3.2, we will use a quantum reference frame transformation that does not depend on time.

3.4.2 Explanation for why operator $\hat{S}$ is generally time dependant

According to [42], if we consider a system with two moving potential observers A and B and a system S, then the transformation that allows us to go from one observer to another is time dependent because the distance between the two observers evolves in time. The following reasoning comes from [42] and tries to justify the time dependency of S in (3.35), however as mentioned at the end of the previous section and as we will show at the end, this thesis will use a different approach. Let us define a symmetry transformation as a map that leaves the functional form of the Hamiltonian H invariant i.e. H of A and B is the same function of operators as H of C and B i.e.

$$\hat{H}(\{m_i, \hat{q}_i, \hat{\pi}_i\}_{i=B,C}) = i\hbar\frac{d\hat{S}}{dt}\hat{S}^{\dagger} + \hat{S}\hat{H}(\{m_i, \hat{x}_i, \hat{p}_i\}_{i=A,B})\hat{S}^{\dagger}, \tag{3.36}$$

where the position and parity operators and the mass of A are replaced by the ones of C. If 3.20 is satisfied, then $\hat{S}$ allows to define a map between dynamical conserved quantities in RF C to the ones in RF A such that they keep the same form.

To go from one RF to the other (when they are considered abstract (as defined in [45])) we apply the transformation

$$\hat{U}_i = \prod_n e^{\frac{i}{\hbar}f^n(t)\hat{O}^n_B}, \tag{3.37}$$

where $f^n(t)$ is a function which links the first RF to the second. It depends on the displacement X(t) between the two and its time derivatives. The number n is the number of observables and the operator $\hat{O}^n_B$ acting on states from $\mathcal{H}_B$. A general transformation cannot always be decomposed into the product of a function of A and an operator of B and thus we have the product $\prod_n$ and we can replace the functions $f^n(t)$ into operators $\hat{f}^{(n)}_A(t)$. This transformation is defined in A's operators i.e. we're considering Heisenberg's point of view where transformations are applied to observables. As such, we establish a clear distinction between the approach of [42] and [17].

However we want to apply the transformation $\hat{S}$ to states of A and B at time t to obtain states B and C at time t. So we want an expression which corresponds to Schrödinger's point of view where transformations are applied to states. Hence, we have:

$$\hat{S} = e^{-\frac{i}{\hbar}\hat{H}_C t}\hat{\mathcal{P}}_{AC}^{(i)} \prod_n e^{\frac{i}{\hbar}\hat{f}_A^{(n)}(t)\hat{O}_B^n} e^{\frac{i}{\hbar}\hat{H}_A t}, \tag{3.38}$$

which corresponds to in order of application (from right to left):

1. the mapping of A's state to Heisenberg picture by evolving towards the past with $\hat{H}_A$,
2. the application of a classical transformation's generalisation using operators $\hat{f}_A^{(n)}(t)$,
3. the application of the "generalised parity operator" $\hat{\mathcal{P}}_{AC}^{(i)}$ to switch equations of motion of A and C,
4. the mapping of C's state back to Schrödinger's picture by evolving towards the future via $\hat{H}_C$,

To summarize according to [42] displacement X(t) between 2 reference frames which evolve differently will vary over time and as the transformation that leads from one to another depends on this displacement, it also depends on time. However, even though, the framework developed in [42] seems coherent regarding this time dependency of the quantum reference frame Transformation, the framework developed in [17] as shown in Subsection 3.2 does not seem to make this time dependency explicit. Rather this time dependency seems to "appear" in the equation of motions as we express velocity and acceleration in Section 4. We have not found a reason for why precisely there is no time dependency in [17]. But we suspect it may simply be linked to the fact that in [17] they are using Schrödinger's perspective rather than Heinsenberg's.

3.5 Covered topics

The purpose of this work is to study the Hamiltonian dynamics of a system under change of different quantum reference frames and interpret the resulting consequences. In particular, the main objective is to understand: how to apply pragmatically Oreshkov's model [17] to a simple system for Translation, Boost, and Galilei groups transformations using this new formalism to switch reference frames, to see the differences in results between an inertial and a non-inertial reference frame and to seek for a link between the fictitious force obtained in the non-inertial reference frame and the extra particle. The latter point is of fundamental interest since it would provide for example some hints of quantum gravitation implications. More generally, the purpose of this Thesis is to gain a deeper understanding of this extra-particle's role in practice for the different groups.

Chapter 4

Non-inertial quantum reference frames and the covariance under three different groups

This Section corresponds to the core of this work where we will apply on a concrete system (cf. Figure 4.1) the model developed in [17] and which was explained in Subsection 3.2. The chosen system is relevant for the study of quantum reference frames because it involves the use of a free particle but also accelerating ones. Hence, it will allow us to write equations of motion relative to both an inertial and non-inertial quantum reference frames. This Section will be structured as follows, first, we will have a brief introduction to fundamental principles in physics that lead to great discoveries, we will then study our system of interest under the framework of [17] for Translation, Boost, and Galilei groups. We will leave our interpretations of the results for Section 5.

4.1 Reminder of some important concepts

What is covariance and invariance and why do those matter?

In theoretical physics, covariance corresponds to the invariance of the form of physics laws under arbitrary coordinate transformations [46].

Invariance however, designates the fact that a quantity (or an observable in quantum mechanics) remains the same under a certain operation. In particular, in quantum mechanics, we say that an operator A is invariant for a given symmetry operation $\mathcal{S}$ (with its corresponding operator S) $\iff$ $[A, S] = 0$ $\iff$ $S^\dagger A S = A$. This property is particularly important because it can lead to conservation laws [21].

Now because this Thesis attempts to describe a system that is composed of both non-accelerating and accelerating objects which will be used as reference frames, it is therefore relevant to remind ourselves of another important concept.

What is an inertial and a non-inertial reference frame?

An inertial reference frame is simply a frame which is not accelerating and where according to Newton: "The Laws of Physics are the same in all inertial frames of reference". Whereas, a non-inertial one is accelerating. When someone places themselves in an accelerating frame, all observed bodies will appear to experience a "fictitious acceleration" of equal magnitude to the observer's but with opposite direction.

4.2 A toy model with interacting quantum reference frames

Consider a quantum system consisting of three particles A (Alice), B (Bob), and C (Charlie). Both of the denominations shall be used in this Thesis without losing meaning. We know Charlie is inertial whereas Alice and Bob are accelerating each other because of the spring linking them. The whole configuration is observed by an external inertial observer E as shown in Figure 4.1.

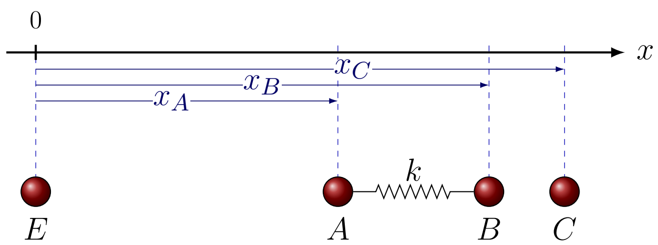

Figure 4.1: The system composed of a mass C and a two mass-spring seen from an external observer E

We will now describe this system using [17]. The interest of this system is to study the behaviour of the configuration from perspectives of both inertial and non-inertial quantum reference frames. Indeed, in the latter, we should expect that a fictitious force appears in the equations as we have in classical mechanics. Hence, we shall proceed to a study of relative Hamiltonians such that we satisfy the principle of covariance. One of the expectations of this study is to understand the link between this fictitious force and the extra-particle.

It is important to note that we will be considering quantum reference frames associated to Groups as mentioned in Subsection 3.2. In particular, we will consider quantum reference frames for Translation, Boost, and the Galilei groups. As such, we will consider the following:

one particle can serve as a quantum reference frame only for position or momentum. It can not measure both at the same time. This is a consequence of the fact that the Hilbert space of one quantum reference frame carries the regular representation of one group. Furthermore, if one quantum observer measures both the position and momentum, the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ would not be satisfied. This is a fundamentally interesting topic that has been discussed in [17] and which will be further elaborated in Section 6.

4.2.1 Quantum reference frames for translations

As we said, we will start our study with quantum reference frames translations. From E's perspective, the Hamiltonian of the quantum system can be written as such:

$$\hat{H}_{ABC|E} = \frac{\hat{p}_{A|E}^2}{2m_A} + \frac{\hat{p}_{B|E}^2}{2m_B} + \frac{\hat{p}_{C|E}^2}{2m_C} + \frac{1}{2}k(\hat{x}_{B|E} - \hat{x}_{A|E})^2, \tag{4.1}$$

where we have introduced subscripts to the relational observables for mass i: $i|E$ refers to the measurement of the observable relative to Eve. For the latter, we may simplify the notation to i as we did in Section 3.2, hence (4.1) becomes

$$\boxed{\hat{H}_{ABC} = \frac{\hat{p}_A^2}{2m_A} + \frac{\hat{p}_B^2}{2m_B} + \frac{\hat{p}_C^2}{2m_C} + \frac{1}{2}k(\hat{x}_B - \hat{x}_A)^2.} \tag{4.2}$$

Indeed, as Eve is an external observer cf Figure 3.3 (for example: a laboratory Frame), we may actually suppose that it is infinitely heavy. This statement allows us to see Eve as a classical observer observing quantum systems. As a classical observer, Eve can measure every observables of the system (whether it is $\hat{x}, \hat{p}$ or others) she wants and from her perspective the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ is satisfied. Hence in this case the Hamiltonian of ABC is written as (4.2).

However, in the case where we apply a translation (in position space) towards C, then the frame of C would only be used as a quantum reference frame for position (from which Charlie would only be able to measure positions). We may call this an $\hat{x}$-frame. And in parallel Eve would then be measuring momenta. In that case, the Hamiltonian of A and B relative to C would be written with an $(\hat{x})$ as an upper index as such: $\hat{H}_{AB|C}^{(\hat{x})}$. If however we had applied a boost (so a translation in momentum space), we would have ended up in a quantum reference frame for momentum (from which we would only be able to measure momenta). We may call this a $\hat{p}$-frame. And in parallel Eve would then be measuring positions. In that case, the Hamiltonian of A and B relative to C would be written with a $(\hat{p})$ as an upper index as such: $\hat{H}_{AB|C}^{(\hat{p})}$.

To ease the reading of this work, this subsection will be structured as follows, we will first compute the Hamiltonian of Alice and Bob relative to Charlie (as a position quantum reference frame) and then we will be able to compute Alice and Bob's associated relative velocity and acceleration. After that, we will jump from Charlie to Alice's quantum reference frame (for position), from which we will get the Hamiltonian of Bob and Charlie relative to her. From there, we may be able to compute Bob and Charlie's associated relative velocity and acceleration.

Relative to Charlie

Let us now switch to Charlie's position reference frame. This transformation is done in two steps (as described in Subsection 3.2): first, the translation, then, the relabelling. When we speak of a translation, it has to be seen as a "controlled translation" relative to Eve of parameter x_C. Whereas the relabelling is really about jumping into the perspective of Charlie. Hence we will first apply the transformation $\hat{U}^\dagger_{AB}(\hat{x}_C)$ so by definition (3.9) we have:

$$\hat{U}^\dagger_{AB}(\hat{x}_C)\hat{H}_{ABC}\hat{U}_{AB}(\hat{x}_C) = e^{i\hat{x}_C\hat{p}_S}\hat{H}_{ABC}e^{-i\hat{x}_C\hat{p}_S} \tag{4.3}$$

where $\hat{p}_S$ is the momentum of the system we'll be looking at from the perspective of C i.e. $\hat{p}_S = \hat{p}_A + \hat{p}_B$.
To evaluate the above transformation, we need to use the following identity for operators (demonstrated in Appendix 8.2):

$$e^{i\hat{A}}f(\hat{B})e^{-i\hat{A}} = f(\hat{B} + i[\hat{A},\hat{B}]). \tag{4.4}$$

Hence, we can write 4.3 as:

$$\begin{aligned} e^{i\hat{x}_C\hat{p}_S}\hat{H}_{ABC}e^{-i\hat{x}_C\hat{p}_S} &= \frac{1}{2m_A}(\hat{p}_A + i[\hat{x}_C\hat{p}_S,\hat{p}_A])^2 + \frac{1}{2m_B}(\hat{p}_B + i[\hat{x}_C\hat{p}_S,\hat{p}_B])^2 \\ &+ \frac{1}{2m_C}(\hat{p}_C + i[\hat{x}_C\hat{p}_S,\hat{p}_C])^2 + \frac{1}{2}k(\hat{x}_B - \hat{x}_A + i[\hat{x}_C\hat{p}_S,\hat{x}_B - \hat{x}_A])^2. \end{aligned} \tag{4.5}$$

Furthermore, by applying the following property for commutators [19] :

$$[\hat{A}\hat{B},\hat{C}] = [\hat{A},\hat{C}]\hat{B} + \hat{A}[\hat{B},\hat{C}], \tag{4.6}$$

and since positions and momenta which live on different Hilbert spaces commute, (4.5) becomes:

$$\begin{aligned} e^{i\hat{x}_C\hat{p}_S}\hat{H}_{ABC}e^{-i\hat{x}_C\hat{p}_S} &= \frac{1}{2m_A}(\hat{p}_A + i\underbrace{[\hat{x}_C\hat{p}_S,\hat{p}_A]}_{=0})^2 + \frac{1}{2m_B}(\hat{p}_B + i\underbrace{[\hat{x}_C\hat{p}_S,\hat{p}_B]}_{=0})^2 \\ &+ \frac{1}{2m_C}(\hat{p}_C + i[\hat{x}_C\hat{p}_S,\hat{p}_C])^2 + \frac{1}{2}k(\hat{x}_B - \hat{x}_A + i\underbrace{[\hat{x}_C\hat{p}_S,\hat{x}_B - \hat{x}_A]}_{=0})^2 \\ &= \frac{1}{2m_A}\hat{p}_A^2 + \frac{1}{2m_B}\hat{p}_B^2 + \frac{1}{2m_C}(\hat{p}_C - \hat{p}_S)^2 + \frac{1}{2}k(\hat{x}_B - \hat{x}_A)^2. \end{aligned} \tag{4.7}$$

The transformation is not yet complete and without the relabelling defined in Subsection 3.2, it would make no sense because we would have a new tensor factorisation of the full Hilbert space but we would have kept the old labels, and this is not correct. Thus, the next step in the quantum reference frame transformation is to make the direct mapping $F_{E\to C}|g\rangle_C \otimes |\alpha\rangle_B \to |g\rangle_D \otimes |\alpha\rangle_{B|C}$. This allows us to have this new tensor factorisation of the full Hilbert space that includes a Hilbert subspace on which operators that make sense from Charlie's perspective live. By doing so we make the observable corresponding to the extra particle appear such that (4.7) becomes:

$$\boxed{\hat{H}^{(\hat{x})}_{AB|C} = \frac{1}{2m_A}\hat{p}^2_{A|C} + \frac{1}{2m_B}\hat{p}^2_{B|C} + \frac{1}{2m_C}(\hat{p}_D - \hat{p}_{A|C} - \hat{p}_{B|C})^2 + \frac{1}{2}k(\hat{x}_{B|C} - \hat{x}_{A|C})^2,} \tag{4.8}$$

where the momenta are measured from Eve and the positions from Charlie. This is tricky because we say that momenta are measured relative to Eve (because we have applied a translation hence we are in an $\hat{x}$-frame), but we write the momenta in (4.8) with the subscript $|C$ as if they were measured relative to C. Actually, this ambiguous notation, in the Quantum Reference Transformation procedure, is necessary to get the correct equations of motion (or at least the same as in classical mechanics). Indeed, as such, when we compute velocity or acceleration, it is as if $\hat{x}_{A|C}$ and $\hat{p}_{A|C}$, for example, lived on the same Hilbert space and we could apply the commutation relation $[\hat{x}_{A|C}, \hat{p}_{A|C}] = i$ (if we consider the convention $\hbar = 1$). This may be trivial, as it may be interesting to further dig into this peculiarity in future research. We also see the extra particle $\hat{p}_D$ appearing however we will discuss and interpret the results in Section 5.

From $\hat{H}_{AB|C}$, we can then obtain the velocity and the acceleration of Alice and Bob relative to Charlie. Indeed we have these formulas (which are demonstrated in Appendix 8.4) for which $\hbar = 1$:

$$\dot{\hat{x}}(t) = \frac{1}{i}[\hat{x}(t), \hat{H}^{(\hat{x})}], \tag{4.9}$$

and

$$\ddot{\hat{x}}(t) = \frac{1}{i}[\dot{\hat{x}}(t), \hat{H}^{(\hat{x})}]. \tag{4.10}$$

Let us start by considering these quantities for Alice and to simplify notations let's define $\dot{\hat{x}}(t) := \dot{\hat{x}}$. We have:

$$\dot{\hat{x}}_{A|C} = \frac{1}{i}[\hat{x}_{A|C}, \hat{H}_{AB|C}] = \frac{1}{2im_A}[\hat{x}_{A|C}, \hat{p}^2_{A|C}] + \frac{1}{2im_C}[\hat{x}_{A|C}, -2\hat{p}_D\hat{p}_{A|C} + \hat{p}^2_{A|C} + 2\hat{p}_{A|C}\hat{p}_{B|C}], \tag{4.11}$$

which gives us the velocity:

$$\dot{\hat{x}}_{A|C} = \frac{1}{m_A}\hat{p}_{A|C} + \frac{1}{m_C}(\hat{p}_{A|C} + \hat{p}_{B|C} - \hat{p}_D). \tag{4.12}$$

Furthermore, we obtain the acceleration:

$$\begin{aligned}\ddot{\hat{x}}_{A|C} &= \frac{1}{i}[\dot{\hat{x}}_{A|C}, \hat{H}_{AB|C}] \\ &= \frac{1}{i}[\frac{\hat{p}_{A|C}}{m_A} + \frac{\hat{p}_{A|C} + \hat{p}_{B|C} - \hat{p}_D}{m_C}, \frac{k}{2}(\hat{x}_{B|C} - \hat{x}_{A|C})^2].\end{aligned} \tag{4.13}$$

In the last expression, we only kept the terms of the Hamiltonian that depend on position since all momenta commute with each other. From there, we deduce that:

$$\ddot{\hat{x}}_{A|C} = \frac{k}{2}\{2(\frac{1}{m_A} + \frac{1}{m_C})(\hat{x}_{B|C} - \hat{x}_{A|C}) - \frac{1}{m_C}2(\hat{x}_{B|C} - \hat{x}_{A|C})\}, \tag{4.14}$$

where the $\frac{1}{m_C}$ terms simplify. Hence, we obtain:

$$\boxed{\ddot{\hat{x}}_{A|C} = \frac{1}{i}[\dot{\hat{x}}_{A|C}, \hat{H}_{AB|C}] = \frac{k}{m_A}(-\hat{x}_{A|C} + \hat{x}_{B|C}).} \tag{4.15}$$

Moreover, it is easy to see that the velocity and the acceleration of Bob are symmetric with respect to the ones of Alice.

Relative to Alice

We can now switch to Alice's quantum reference frame (for position). When we went from Eve to Charlie we applied the transformation $\hat{U}^\dagger_{AB}(\hat{x}_C)\hat{H}_{ABC}\hat{U}_{AB}(\hat{x}_C)$ as in (4.3). If we now want to go to Alice, we must apply the opposite transformation to (4.3) to do $C \to E$ i.e. $\hat{U}_{AB}(\hat{x}_C)\hat{H}_{AB|C}\hat{U}^\dagger_{AB}(\hat{x}_C)$, composed with another one which allows us to do $E \to A$ which is the transformation $\hat{U}^\dagger_{BC}(\hat{x}_A)\hat{H}_{ABC}\hat{U}_{BC}(\hat{x}_A)$. Hence let us now switch from Charlie to Alice's perspective as such:

$$\begin{aligned}\hat{U}^\dagger_{BC}(\hat{x}_A)\hat{H}_{ABC}\hat{U}_{BC}(\hat{x}_A) &= e^{i\hat{x}_A\hat{p}_S}\hat{H}_{ABC}e^{-i\hat{x}'_A\hat{p}_S} \\ &= \frac{1}{2m_A}(\hat{p}_A + i[\hat{x}_A\hat{p}_S, \hat{p}_A])^2 + \frac{1}{2m_B}(\hat{p}_B + i[\hat{x}_A\hat{p}_S, \hat{p}_B])^2 \\ &+ \frac{1}{2m_C}(\hat{p}_C + i[\hat{x}_A\hat{p}_S, \hat{p}_C])^2 + \frac{1}{2}k(\hat{x}_B - \hat{x}_A + i[\hat{x}_A\hat{p}_S, \hat{x}_B - \hat{x}_A])^2 \\ &= \frac{1}{2m_A}(\hat{p}_A - \hat{p}_B - \hat{p}_C)^2 + \frac{1}{2m_B}\hat{p}_B^2 + \frac{1}{2m_C}\hat{p}_C^2 + \frac{1}{2}k\hat{x}_B^2\end{aligned} \tag{4.16}$$

since $\hat{p}_S = \hat{p}_B + \hat{p}_C$.
After applying the direct mapping, we can write the Hamiltonian of Bob and Charlie relative to Alice as:

$$\boxed{\hat{H}^{(\hat{x})}_{BC|A} = \frac{1}{2m_A}(\hat{p}_D - \hat{p}_{B|A} - \hat{p}_{C|A})^2 + \frac{1}{2m_B}\hat{p}^2_{B|A} + \frac{1}{2m_C}\hat{p}^2_{C|A} + \frac{1}{2}k\hat{x}^2_{B|A}.} \tag{4.17}$$

We can now compute Bob's and Charlie's velocities and accelerations relative to Alice as we did for the case relative to Charlie. Hence the velocity of Bob relative to Alice gives us:

$$\dot{\hat{x}}_{B|A} = \frac{1}{i}[\hat{x}_{B|A}, \hat{H}_{BC|A}] = \frac{1}{m_B}\hat{p}_{B|A} + \frac{1}{m_C}(\hat{p}_{B|A} + \hat{p}_{C|A} - \hat{p}_D), \tag{4.18}$$

and for the acceleration we get:

$$\boxed{\ddot{\hat{x}}_{B|A} = \frac{1}{i}[\dot{\hat{x}}_{B|A}, \hat{H}_{BC|A}] = -\frac{k}{m_A}\hat{x}_{B|A} - \frac{k}{m_B}\hat{x}_{B|A} = -k\hat{x}_{B|A}(\frac{1}{m_A} + \frac{1}{m_B})} \tag{4.19}$$

Now, following the same reasoning as previously we obtain for Charlie's velocity and acceleration the following:

$$\dot{\hat{x}}_{C|A} = \frac{1}{i}[\hat{x}_{C|A}, \hat{H}_{BC|A}] = \frac{1}{m_A}(\hat{p}_{B|A} + \hat{p}_{C|A} - \hat{p}_D) + \frac{1}{m_C}\hat{p}_{C|A}, \tag{4.20}$$

and

$$\boxed{\ddot{\hat{x}}_{C|A} = \frac{1}{i}[\dot{\hat{x}}_{C|A}, \hat{H}_{BC|A}] = -\frac{k}{m_A}\hat{x}_{B|A}.} \tag{4.21}$$

4.2.2 Quantum reference frames for boosts

After having considered the quantum reference frames for translations, we will now study quantum reference frames for boosts. Our approach will remain the same, except that now, the group elements will be labelled by momenta and will be related to one another by boosts. Let us start back from Equation (4.2) which is reminded here:

$$\hat{H}_{ABC} = \frac{\hat{p}_A^2}{2m_A} + \frac{\hat{p}_B^2}{2m_B} + \frac{\hat{p}_C^2}{2m_C} + \frac{1}{2}k(\hat{x}_B - \hat{x}_A)^2. \tag{4.22}$$

Now, according to [17] [42] [48] [49] we know that a general transformation for the Galilean Group is

$$U(a, v) = e^{-i(a\hat{p} + v\hat{k})}. \tag{4.23}$$

Indeed we can distinguish two separate parts in (4.23), one corresponding to a translation: $e^{-ia\hat{p}}$ and the other to a boost $e^{-iv\hat{k}}$ but as we only consider a boost transformation, we will remove the translation part hence, (4.23) becomes:

$$U(v) = e^{-iv\hat{k}}, \tag{4.24}$$

where $\hat{k} = \hat{p}t - m\hat{x}$. The operator $\hat{k}$ characterises the observed system. Furthermore, the framework developed in [17] uses Schrödinger's picture i.e. the operator does not have a time dependency. Hence, as this Thesis applies this framework, we can simplify the operator 4.24 by stating that t=0 such that:

$$U(v) = e^{iv m\hat{x}}, \tag{4.25}$$

where $m\hat{x}$ can be seen as the position of the center of mass times the mass of the observed system (for example: if we are in Charlie's perspective $m\hat{x} = m_{AB}\hat{x}_{AB}$=$m_A\hat{x}_A + m_B\hat{x}_B$).

Let us make a small reminder of Section 4.2 and Subsection 4.2.1: now that we will use boosts as quantum reference frame transformations i.e. a translation in momentum space, we will end up in a quantum reference frame for momentum from which we will only be able to measure momenta. This is what we call a $\hat{p}$-frame. And in parallel, Eve will be measuring positions. We will also do as for the translation case, where we will first jump into Charlie's perspective from which we will deduce the relative Hamiltonian and the corresponding equations of motion. We will then do the same but from an inertial perspective i.e. Alice.

Relative to Charlie

Now, if we want to apply a boost to get to Charlie's quantum reference frame, we will follow the same reasoning as we did for translations (cf Subsection 4.2.1). In particular we will apply an analogous transformation as (4.3), and according to the identity (3.9) our general boost transformation becomes:

$$\begin{aligned}\hat{U}^\dagger_{AB}(\hat{v}_C)\hat{H}^{(\hat{p})}_{ABC}\hat{U}_{AB}(\hat{v}_C) &= e^{-i\hat{v}_C m_{AB}\hat{x}_{AB}}\ \hat{H}_{ABC}\ e^{i\hat{v}_C m_{AB}\hat{x}_{AB}} \\ &= \frac{1}{2m_A}(\hat{p}_A + i[-\hat{v}_C m_{AB}\hat{x}_{AB}, \hat{p}_A])^2 \\ &+ \frac{1}{2m_B}(\hat{p}_B + i[-\hat{v}_C m_{AB}\hat{x}_{AB}, \hat{p}_B])^2 \\ &+ \frac{1}{2m_C}(\hat{p}_C + i\underbrace{[-\hat{v}_C m_{AB}\hat{x}_{AB}, \hat{p}_C]}_{=0})^2 \\ &+ \frac{1}{2}k(\hat{x}_B - \hat{x}_A + i\underbrace{[-\hat{v}_C m_{AB}\hat{x}_{AB}, \hat{x}_B - \hat{x}_A]}_{=0})^2,\end{aligned} \tag{4.26}$$

with

$$m_{AB}\hat{x}_{AB} = m_A\hat{x}_A + m_B\hat{x}_B \tag{4.27}$$

and where we have used the same argument as in Subsection 4.2.1 regarding the commutators that vanish.

From 4.26 we can deduce :

$$\begin{aligned}\hat{H}^{(\hat{p})}_{AB|C} &= \frac{1}{2m_A}(\hat{p}_A + m_A\hat{v}_C)^2 \\ &+ \frac{1}{2m_B}(\hat{p}_B + m_B\hat{v}_C)^2 + \frac{\hat{p}_C^2}{2m_C} + \frac{1}{2}k(\hat{x}_B - \hat{x}_A)^2.\end{aligned} \tag{4.28}$$

However, the transformation is not finished as it weren't for (4.7). Indeed we have only applied the controlled boost of parameter $\hat{v}_C$ yet. There are still two things to be done: first, we will make sense of this $\hat{v}_C$ which has appeared in our Hamiltonian, then we will apply the direct mapping we had seen in Subsection 4.2.1 to finish the quantum reference frame boost.

Let us now use the definition of the velocity (cf. 4.9) for Charlie relative to the classical and external observer Eve. The Hamiltonian $\hat{H}_C$ for a free moving particle C is given by $\hat{H}_C = \frac{\hat{p}_C^2}{2m_C}$. Hence, we can say that

$$\begin{aligned} \hat{v}_C &= \frac{1}{i}[\hat{x}_C, \hat{H}_C] \\ &= \frac{\hat{p}_C}{m_C}. \end{aligned} \tag{4.29}$$

We can thus rewrite the Hamiltonian (4.28) as:

$$\begin{aligned} \hat{H}^{(\hat{p})}_{AB|C} = \frac{1}{2m_A}(\hat{p}_A + \frac{m_A}{m_C}\hat{p}_C)^2 \\ + \frac{1}{2m_B}(\hat{p}_B + \frac{m_B}{m_C}\hat{p}_C)^2 + \frac{\hat{p}_C^2}{2m_C} + \frac{1}{2}k(\hat{x}_B - \hat{x}_A)^2. \end{aligned} \tag{4.30}$$

And if now use the direct mapping defined in Subsection 4.2.1, we get:

$$\boxed{\begin{aligned} \hat{H}^{(\hat{p})}_{AB|C} = \frac{1}{2m_A}(\hat{p}_{A|C} + \frac{m_A}{m_C}\hat{p}_D)^2 \\ + \frac{1}{2m_B}(\hat{p}_{B|C} + \frac{m_B}{m_C}\hat{p}_D)^2 + \frac{\hat{p}_D^2}{2m_C} + \frac{1}{2}k(\hat{x}_{B|C} - \hat{x}_{A|C})^2 \end{aligned}} \tag{4.31}$$

where the positions are actually measured from Eve and the momenta from Charlie. Again we have to be careful with this ambiguous notation. Here we have applied a boost from Eve to Charlie. This means that Charlies serves as a $\hat{p}$-frame and indeed we see the momenta with a subscript $|C$. The momenta ARE measured relative to Charlie. However, the positions are also written with the subscript $|C$ as if they were also measured by Charlie. This is not the case, they are in fact measured relative to Eve. But this notation is necessary to make sure we get the correct equations of motion (in the sense that they are the same as in classical mechanics). As such we treat $\hat{x}_{A|C}$ and $\hat{p}_{A|C}$ for example as if they lived on the same Hilbert space such that the commutation relation $[\hat{x}_{A|C}, \hat{p}_{A|C}] = i$ are satisfied (if we consider the convention $\hbar = 1$).

This being said, in a momentum frame we can only measure $\hat{p}_{A|C}$, hence there are no calculations to be made for the "velocity". We can only make sense of an acceleration defined relative to this $\hat{p}$-frame which is: $\dot{\hat{p}}_{A|C}$. We will compute the latter using the same type of definition as (4.10) (except here we will be using momenta observables):

$$\dot{\hat{p}}_{A|C} = \frac{1}{i}[\hat{p}_{A|C}, \hat{H}^{(\hat{p})}_{AB|C}]. \tag{4.32}$$

As previously, we have considered the $\hbar = 1$ convention. We can thus compute the acceleration of Alice relative to Charlie which gives us:

$$\dot{\hat{p}}_{A|C} = k(\hat{x}_{B|C} - \hat{x}_{A|C}). \tag{4.33}$$

To make things clearer, we may define the operator $\hat{a}_{A|C} := \frac{\dot{\hat{p}}_{A|C}}{m_A}$ so that we can write:

$$\boxed{\hat{a}_{A|C} = \frac{k}{m_A}(\hat{x}_{B|C} - \hat{x}_{A|C}).} \tag{4.34}$$

We can follow an analogous reasoning for Bob and derive:

$$\boxed{\hat{a}_{B|C} = \frac{k}{m_B}(\hat{x}_{A|C} - \hat{x}_{B|C}).} \tag{4.35}$$

Now that we have considered the perspective of the quantum reference frame Charlie, will now apply a quantum reference frame boost towards Alice following an analogous reasoning as in Subsection 4.2.1 i.e. going from $C \to E \to A$.

Relative to Alice

As we want Alice's perspective, we can start from (4.2) and simply apply the following boost towards Alice's $\hat{p}$-frame:

$$\hat{U}^\dagger_{BC}(\hat{v}_A) = e^{i\hat{v}_A m_{BC}\hat{x}_{BC}}, \tag{4.36}$$

where

$$m_{BC}\hat{x}_{BC} = m_B\hat{x}_B + m_C\hat{x}_C. \tag{4.37}$$

We thus get the following:

$$\begin{aligned}
&\hat{U}^\dagger_{BC}(\hat{v}_A)\hat{H}_{ABC}\hat{U}_{BC}(\hat{v}_A) \\
&= \frac{1}{2m_A}(\hat{p}_A + i\underbrace{[-\hat{v}_A m_{BC}\hat{x}_{BC}, \hat{p}_A]}_{=0})^2 + \frac{1}{2m_B}(\hat{p}_B + i\underbrace{[-\hat{v}_A m_{BC}\hat{x}_{BC}, \hat{p}_B]}_{*})^2 \\
&+ \frac{1}{2m_C}(\hat{p}_C + i\underbrace{[-\hat{v}_A m_{BC}\hat{x}_{BC}, \hat{p}_C]}_{**})^2 \\
&+ \frac{1}{2}k(\hat{x}_B - \hat{x}_A + i\underbrace{[-\hat{v}_A m_{BC}\hat{x}_{BC}, \hat{x}_B - \hat{x}_A]}_{***})^2,
\end{aligned} \tag{4.38}$$

where

$$
\begin{aligned}
* &= -i\frac{m_B}{m_A}\hat{p}_A, \\
** &= -i\frac{m_C}{m_A}\hat{p}_A, \\
*** &= m_{BC}\hat{x}_{BC}[-\underbrace{\hat{v}_A}_{\frac{1}{i}[\hat{x}_A,\hat{H}_A]=\frac{\hat{p}_A}{m_A}}, -\hat{x}_A] \\
&= -i\frac{m_{BC}}{m_A}\hat{x}_{BC}.
\end{aligned}
$$

Here we have used the fact that, in general, the moving particle A can be described as a Hamiltonian cf. (4.39), which has a kinetic term and a potential term that only depends on the position of Bob relative to Alice:

$$\hat{H}_A = \frac{\hat{p}_A^2}{2m_A} + \hat{V}(\hat{x}_B - \hat{x}_A). \tag{4.39}$$

As this relative position commute with $\hat{x}_A$ we get $\frac{1}{i}[\hat{x}_A, \hat{H}_A] = \frac{\hat{p}_A}{m_A}$. Hence, we obtain from the above that:

$$*** = -i\frac{m_B\hat{x}_B + m_C\hat{x}_C}{m_A}. \tag{4.40}$$

Thus, we can deduce that equation (4.38) gives us:

$$
\begin{aligned}
\hat{U}^\dagger_{BC}(\hat{v}_A)\hat{H}^{(\hat{p})}_{ABC}\hat{U}_{BC}(\hat{v}_A) &= \frac{1}{2m_A}\hat{p}_A^2 + \frac{1}{2m_B}(\hat{p}_B + \frac{m_B}{m_A}\hat{p}_A)^2 \\
&+ \frac{1}{2m_C}(\hat{p}_C + \frac{m_C}{m_A}\hat{p}_A)^2 \\
&+ \frac{1}{2}k(\hat{x}_B - \hat{x}_A + \frac{m_B\hat{x}_B + m_C\hat{x}_C}{m_A})^2.
\end{aligned} \tag{4.41}
$$

And by applying the direct mapping/the relabeling $F_{E\to A}|g\rangle_A \otimes |\alpha\rangle_B \to |g\rangle_D \otimes |\alpha\rangle_{B|A}$, the previous expression becomes:

$$
\boxed{
\begin{aligned}
\hat{H}^{(\hat{p})}_{BC|A} = \frac{\hat{p}_D^2}{2m_A} + \frac{1}{2m_B}(\hat{p}_{B|A} + \frac{m_B}{m_A}\hat{p}_D)^2 + \frac{1}{2m_C}(\hat{p}_{C|A} + \frac{m_C}{m_A}\hat{p}_D)^2 \\
+\frac{1}{2}k(\hat{x}_{B|A} - \hat{x}_D + \frac{m_B\hat{x}_{B|A} + m_C\hat{x}_{C|A}}{m_A})^2.
\end{aligned}
} \tag{4.42}
$$

Again here, the notation is ambiguous. However as previously, momenta are measured relative to Alice, whereas positions are measured relative to Eve. We can already notice that the extra particle has appeared in the expression of the Hamiltonian. As a reminder, we will be focusing on the interpretation of the results in Section 5.

Let us now derive the expressions for the accelerations of Bob and Charlie relative to A i.e. $\hat{p}_{B|A}$ and $\hat{p}_{C|A}$. We have by definition:

$$\dot{\hat{p}}_{C|A} = \frac{1}{i}[\hat{p}_{C|A}, \hat{H}^{(\hat{p})}_{BC|A}], \tag{4.43}$$

and because all momenta commute, we only consider the commutator between $\hat{p}_{C|A}$ and $\frac{1}{2}k(\hat{x}_{B|A} - \hat{x}_D + \frac{m_B\hat{x}_{B|A}+m_C\hat{x}_{C|A}}{m_A})^2$. This gives us:

$$\dot{\hat{p}}_{C|A} = -\frac{m_C}{m_A}k(\hat{x}_{B|A} - \hat{x}_D + \frac{m_B}{m_A}\hat{x}_{B|A} + \frac{m_C}{m_A}\hat{x}_{C|A}). \tag{4.44}$$

As previously, and to make things clearer, we may define the operator $\hat{a}_{C|A} := \frac{\dot{\hat{p}}_{C|A}}{m_C}$ so that we can write:

$$\boxed{\hat{a}_{C|A} = -\frac{k}{m_A}(\hat{x}_{B|A} - \hat{x}_D + \frac{m_B}{m_A}\hat{x}_{B|A} + \frac{m_C}{m_A}\hat{x}_{C|A}).} \tag{4.45}$$

Now that we have obtained the acceleration of C relative to A for the boosts, let us now apply an analogous reasoning to get $\dot{\hat{p}}_{B|A}$. By definition, we have:

$$\dot{\hat{p}}_{B|A} = \frac{1}{i}[\hat{p}_{B|A}, \hat{H}^{(\hat{p})}_{BC|A}], \tag{4.46}$$

which gives us:

$$\dot{\hat{p}}_{B|A} = -(1 + \frac{m_B}{m_A})k(\hat{x}_{B|A} - \hat{x}_D + \frac{m_B}{m_A}\hat{x}_{B|A} + \frac{m_C}{m_A}\hat{x}_{C|A}). \tag{4.47}$$

Again, by defining an operator $\hat{a}_{B|A} := \frac{\dot{\hat{p}}_{B|A}}{m_B}$ we can write (4.47) as:

$$\boxed{\hat{a}_{B|A} = -(\frac{1}{m_A} + \frac{1}{m_B})k(\hat{x}_{B|A} - \hat{x}_D + \frac{m_B}{m_A}\hat{x}_{B|A} + \frac{m_C}{m_A}\hat{x}_{C|A}).} \tag{4.48}$$

As we see, the accelerations obtained (4.45) and (4.48) are quite laborious. However we will show in Section 5 that with an appropriate choice for the extra particle, we retrieve the correct form for the equations of motion.

Now, let us explore quantum reference frame transformations for a combination of translations and boosts i.e. for the Galilei group.

4.2.3 Quantum reference frame for Galilei group

We can now combine the results we have obtained for translation and boost groups and use them to explore the consequences for the Galilei group. Indeed the composition rule of the latter (a',v') · (a,v) = (a'+a, v'+v) [17]. Now let's analyze the original (and quite exotic!) case where we apply a translation to a particle 1 followed by a boost towards a particle 2. That means we place a reference frame for position on particle 1 and a reference frame for momentum on particle 2. Let us start back from the initial Hamiltonian seen from an external observer:

$$\boxed{\hat{H}_{ABC} = \frac{\hat{p}_A^2}{2m_A} + \frac{\hat{p}_B^2}{2m_B} + \frac{\hat{p}_C^2}{2m_C} + \frac{1}{2}k(\hat{x}_B - \hat{x}_A)^2} \tag{4.49}$$

A translation towards Alice followed by a boost towards Charlie

Now, we can apply a translation towards Alice. We naturally get back 4.17. From there, we can apply a boost towards Charlie using a similar reasoning as in 4.26 with the same transformation. As such that we obtain:

$$\boxed{\begin{aligned}\hat{H}_{BC|A^{(\hat{x})},AB|C^{(\hat{p})}} &= \frac{1}{2m_A}(\hat{p}_D - \hat{p}_{B|A} - \hat{p}_{C|A} - \frac{m_B}{m_C}\hat{p}_F)^2 \\ &+ \frac{1}{2m_B}(\hat{p}_{B|C} + \frac{m_B}{m_C}\hat{p}_F)^2 + \frac{\hat{p}_F^2}{2m_C} + \frac{1}{2}k\hat{x}_{B|A}^2.\end{aligned}} \tag{4.50}$$

We observe two different extra-particles which corresponds to the two different transformations.

Now, we can compute the velocities and the accelerations. If we consider the quantum reference frame of Alice, as it measures positions, we should, in principle, use the definitions (4.9) and (4.10) for velocities and accelerations of Bob and Charlie. Let us start with Charlie, and as we did in Section 4.2.1, we obtain:

$$\dot{\hat{x}}_{C|A} = -\frac{1}{m_A}(\hat{p}_D - \hat{p}_{B|A} - \hat{p}_{C|A} - \frac{m_B}{m_C}\hat{p}_F). \tag{4.51}$$

We can take this result, and inject it in (4.10) from which we get:

$$\boxed{\ddot{\hat{x}}_{C|A} = -\frac{k}{m_A}\hat{x}_{B|A}.} \tag{4.52}$$

Now, we can do the same type of reasoning for Bob such that we obtain:

$$\dot{\hat{x}}_{B|A} = -\frac{1}{m_A}(\hat{p}_D - \hat{p}_{B|A} - \hat{p}_{C|A} - \frac{m_B}{m_C}\hat{p}_F). \tag{4.53}$$

Once gain, we can take this result and inject it in (4.10), from which we get:

$$\boxed{\ddot{\hat{x}}_{B|A} = -\frac{k}{m_A}\hat{x}_{B|A}.} \tag{4.54}$$

Here we have considered Alice's perspective, but let us now consider Charlie's which measures momenta. Because we are now in a $\hat{p}$-frame, we will consider an acceleration as previously defined (4.32). We easily see that:

$$\boxed{\dot{\hat{p}}_{A|C} = 0.} \tag{4.55}$$

This acceleration is quite surprising. But we will further comment on this in Section 5.

Now, let us try the case where we will translate towards Charlie and boost towards Alice.

A translation towards Charlie followed by a boost towards Alice

By first applying a translation towards Charlie, we get back 4.8. We can then apply a boost towards Alice as we did in 4.38 such that we obtain:

$$\boxed{\begin{gathered}\hat{H}_{AB|C^{(\hat{x})},BC|A^{(\hat{p})}} = \frac{\hat{p}_F^2}{2m_A} + \frac{1}{2m_B}(\hat{p}_{B|A} + \frac{m_B}{m_A}\hat{p}_F)^2 \\ + \frac{1}{2m_C}(\hat{p}_D - \hat{p}_{A|C} - \hat{p}_{B|C} - \frac{m_B}{m_A}\hat{p}_F)^2 + \frac{1}{2}k(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A} + m_C\hat{x}_{C|A}}{m_A})^2\end{gathered}} \tag{4.56}$$

where the positions are relative to C and the momenta are relative to A. Again we observe two different extra-particles that correspond to the two different transformations.

Now we can compute the velocities and the accelerations. First, we consider the quantum reference frame Charlie, and as it measures positions, we will use the definitions (4.9) and (4.10) for the velocities and accelerations of Alice and Bob. As we did in Subsection 4.2.1, let us first compute Alice's velocity as such:

$$\dot{\hat{x}}_{A|C} = -\frac{1}{m_C}(\hat{p}_D - \hat{p}_{A|C} - \hat{p}_{B|C} - \frac{m_B}{m_A}\hat{p}_F). \tag{4.57}$$

We can take this result, and inject it in (4.10) from which we get:

$$\boxed{\ddot{\hat{x}}_{A|C} = 0.} \tag{4.58}$$

An analogous reasoning holds for Bob.

Let us now consider Alice as a $\hat{p}$-frame and compute the accelerations for Bob and Charlie. Using (4.32), we have for Bob :

$$\dot{\hat{p}}_{B|A} = -\frac{m_B}{m_A}k(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A} + m_C\hat{x}_{C|A}}{m_A}), \tag{4.59}$$

where if we define the operator $\hat{a}_{B|A} = \frac{\dot{\hat{p}}_{B|A}}{m_B}$, we can write the acceleration of Bob as the following:

$$\boxed{\hat{a}_{B|A} = -\frac{k}{m_A}(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A} + m_C\hat{x}_{C|A}}{m_A}).} \tag{4.60}$$

Now, for Charlie we make an analogous reasoning such that we obtain:

$$\dot{\hat{p}}_{C|A} = -\frac{m_C}{m_A}k(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A} + m_C\hat{x}_{C|A}}{m_A}), \tag{4.61}$$

and thus,

$$\boxed{\hat{a}_{C|A} = -\frac{k}{m_A}(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A} + m_C\hat{x}_{C|A}}{m_A}).} \tag{4.62}$$

Chapter 5

Discussions

In this section, we will interpret and discuss the physical meaning of the results obtained in the previous section. For each group, we will: summarise our results, compare them to what we would have in classical mechanics, compare them with one another, and discuss the role of the extra particle regarding the covariance of our results. As in the previous section, we will start with the quantum reference frames translations.

5.1 Translation case

Let us first recover and summarize our findings from Charlie's perspective.

Relative to Charlie

These are the results we have obtained in summary:

$$\boxed{\begin{gathered}\ddot{\hat{x}}_{A|C} = \frac{k}{m_A}(\hat{x}_{B|C} - \hat{x}_{A|C}),\\ \ddot{\hat{x}}_{B|C} = \frac{k}{m_B}(\hat{x}_{A|C} - \hat{x}_{B|C}),\\ \hat{H}^{(\hat{x})}_{AB|C} = \frac{1}{2m_A}\hat{p}^2_{A|C} + \frac{1}{2m_B}\hat{p}^2_{B|C} + \frac{1}{2m_C}(\hat{p}_D - \hat{p}_{A|C} - \hat{p}_{B|C})^2 + \frac{1}{2}k(\hat{x}_{B|C} - \hat{x}_{A|C})^2.\end{gathered}}$$

As Charlie is not subject to any external forces, it is, by definition, an inertial quantum reference frame. That means that no fictitious force should appear in the equations of motion, and the accelerations of Alice and Bob should only depend on their relative position between one another. This is indeed what we have obtained, and this result is consistent with the classical case.

Here, the extra-particle appears only in a kinetic term of the Hamiltonian as $\hat{p}_D$. In particular, we see that $\hat{p}_C$ is replaced by $\hat{p}_D - \hat{p}_{A|C} - \hat{p}_{B|C}$. As previously mentioned in the section on translations, momenta are measured from Eve's perspective despite the ambiguous notation. Also, as we saw in Section 3.2, the extra particle is part of the invariant algebra under translation. However, applying a quantum reference frame translation corresponds to refactorise the whole Hilbert space relative to E, which means that this extra particle must contain information that is: invariant under translation and concerns the whole system. The corresponding quantity is the total momentum.

Before we continue, let us open a small parenthesis regarding Hamiltonian mechanics. In the latter, it is well known that the Hamiltonian H which characterises the energy of a system is sufficient to get the equations of motion for the system. The latter are obtained by applying the so-called Poisson Brackets $\{,\}$ which are equivalent to the commutator in quantum mechanics. In particular, if we call x the position of an object, the velocity, and the acceleration can be obtained as follows $\dot{x} = \{x, H\}$ and $\dot{p} = \{p, H\}$ respectively [47]. And equivalently, as $p = m\dot{x}$, the acceleration could also be obtained using $\ddot{x} = \{\dot{x}, H\}$. In other words, whether we use the velocity obtained using the position $\dot{x}$ or the momentum p, the result of the acceleration is the same. Therefore, we could be tempted to suppose this situation would be the same in quantum mechanics (using the commutator). And we can show that it is for an inertial quantum reference frame. We showed, however, that there is a clear discrepancy in a non-inertial frame. As we are still considering Charlie, let us start there and compute now $\dot{\hat{p}}_{A|C} = \frac{1}{i}[\hat{p}_{A|C}, \hat{H}^{(\hat{x})}_{AB|C}]$ and this gives us $\dot{\hat{p}}_{A|C} = k(-\hat{x}_{A|C} + \hat{x}_{B|C})$, which is the same as (4.15) if we define the following operator: $\frac{\dot{\hat{p}}_{A|C}}{m_A}$.

Let us now analyse the results we have obtained from the non-inertial quantum reference frame Alice for translations. But first, let us remind ourselves what we had obtained in the previous section.

Relative to Alice

These are the results we get in summary:

$$\ddot{\hat{x}}_{B|A} = -k\hat{x}_{B|A}\left(\frac{1}{m_A} + \frac{1}{m_B}\right),$$
$$\ddot{\hat{x}}_{C|A} = -\frac{k}{m_A}\hat{x}_{B|A},$$
$$\hat{H}^{(\hat{x})}_{BC|A} = \frac{1}{2m_A}(\hat{p}_D - \hat{p}_{B|A} - \hat{p}_{C|A})^2 + \frac{1}{2m_B}\hat{p}^2_{B|A} + \frac{1}{2m_C}\hat{p}^2_{C|A} + \frac{1}{2}k\hat{x}^2_{B|A}.$$

Now, from Alice's perspective, we observe both an acceleration of Charlie and an additional term in the acceleration of Bob. These terms are fictitious accelerations that apply on Bob and Charlie, and they correspond to the acceleration Alice feels. This is consistent with what we observe in classical mechanics where this fictitious force appears as an ad-hoc potential term in the Hamiltonian. Indeed, we have obtained the correct accelerations without having to introduce this additional potential in the Hamiltonian as we usually do in classical mechanics.

The same interpretation, as the one provided for the case relative to Charlie, holds for the extra particle.

An important conclusion is that those Hamiltonians remain invariant under translation whether we are in an inertial or a non-inertial reference frame since all momenta are observed externally, and the potential term has its variables changed, but not its value. Hence according to Subsection 3.2, any $\hat{x}$-frame observers who do not have access to the external frame, will have access to the full Hamiltonian since it is invariant under translation. And as we will see now, this is not necessarily the case for boosts.

Now, before we get to boosts, let us make this parallel with classical mechanics as we did in the final paragraph of Charlie's perspective. By following an analogous reasoning, we can compute $\dot{\hat{p}}_{C|A} = \frac{1}{i}[\hat{p}_{C|A}, \hat{H}^{(\hat{x})}_{BC|A}]$ and $\dot{\hat{p}}_{B|A} = \frac{1}{i}[\hat{p}_{B|A}, \hat{H}^{(\hat{x})}_{BC|A}]$. For the former, we clearly see that we obtain $\dot{\hat{p}}_{C|A} = 0$, whereas for the latter we get $\dot{\hat{p}}_{B|A} = -k\hat{x}_{B|A}$. Again, if we define the operator $\frac{\dot{\hat{p}}_{B|A}}{m_B}$, we find back (4.21). Interestingly, we see that when we consider quantum reference frames for translations in a non-inertial frame and if we compute the accelerations in a way that uses the $\hat{x}$ variable we get a result that does take into account the acceleration of the frame as in 4.19 and 4.21. However, if we compute the accelerations using the $\hat{p}$ variable, then we obtain a result that does not take into account the fact that the frame is accelerating. This would not be the case in classical mechanics: as we have discussed in the previous page, this would not be the case if we don't distinguish translation and boost frame in classical mechanics. We will see that the opposite conclusion holds for boosts.

5.2 Boost case

First, let us summarize our findings for boosts from Charlie's perspective.

Relative to Charlie

These are the results we have obtained in summary:

$$
\boxed{
\begin{aligned}
\hat{a}_{A|C} &= \frac{k}{m_A}(\hat{x}_{B|C} - \hat{x}_{A|C}),\\
\hat{a}_{B|C} &= \frac{k}{m_B}(\hat{x}_{A|C} - \hat{x}_{B|C}),\\
\hat{H}^{(\hat{p})}_{AB|C} &= \frac{1}{2m_A}(\hat{p}_{A|C} + \frac{m_A}{m_C}\hat{p}_D)^2 + \frac{1}{2m_B}(\hat{p}_{B|C} + \frac{m_B}{m_C}\hat{p}_D)^2 + \frac{\hat{p}_D^2}{2m_C} + \frac{1}{2}k(\hat{x}_{B|C} - \hat{x}_{A|C})^2.
\end{aligned}
}
$$

As for the translation case, we get back results in the perspective of an inertial quantum reference frame consistent with the classical case. We can conclude that the equations of motion are independent of the type of reference frame chosen. This is true even though we are in a $\hat{p}$-frame i.e. the accelerations of Alice and Bob depend on relative positions relative to Eve. That means that relative to an inertial frame, the accelerations of Alice and Bob are both invariant under translations and boosts.

Furthermore, as we see here, the extra-particle which appears in the Hamiltonian as $\hat{p}_D$ can be interpreted as Charlie's momentum measured relative to Eve.

As we did for translations, we can now make a parallel to classical mechanics. For that, instead of computing the accelerations as we did using (4.32), let us rather compute them with the definition for $\hat{x}$-variable (4.9) and (4.10). As for the translation case, because Charlie is an inertial quantum reference frame, we should get the same result as (4.32). And as for the translations, this will not be the case when relative to a non-inertial quantum reference frame. So indeed, using (4.9) we have $\dot{\hat{x}}_{A|C} = [\hat{x}_{A|C}, \hat{H}^{(\hat{p})}_{AB|C}]$ we obtain $\dot{\hat{x}}_{A|C} = \frac{\hat{p}_{A|C}}{m_A} + \frac{\hat{p}_D}{m_C}$. Now with with (4.10) we can compute the acceleration: $\ddot{\hat{x}}_{A|C} = [\dot{\hat{x}}_{A|C}, \hat{H}^{(\hat{p})}_{AB|C}]$, because $\hat{p}_D$ commutes with all the terms we get that $\ddot{\hat{x}}_{A|C} = \frac{k}{m_A}(\hat{x}_{B|C} - \hat{x}_{A|C})$. As expected, we get back (4.34). The same holds for the acceleration of Bob.

Let us now express the results from the non-inertial quantum reference frame Alice for boost.

Relative to Alice

These are the results we have obtained in summary:

$$\boxed{\begin{gathered}\hat{a}_{B|A} = -k(\hat{x}_{B|A} - \hat{x}_D + \frac{m_B}{m_A}\hat{x}_{B|A} + \frac{m_C}{m_A}\hat{x}_{C|A})(\frac{1}{m_A} + \frac{1}{m_B}),\\ \hat{a}_{C|A} = -\frac{k}{m_A}(\hat{x}_{B|A} - \hat{x}_D + \frac{m_B}{m_A}\hat{x}_{B|A} + \frac{m_C}{m_A}\hat{x}_{C|A}),\\ \hat{H}^{(\hat{p})}_{BC|A} = \frac{\hat{p}_D^2}{2m_A} + \frac{1}{2m_B}(\hat{p}_{B|A} + \frac{m_B}{m_A}\hat{p}_D)^2 + \frac{1}{2m_C}(\hat{p}_{C|A} + \frac{m_C}{m_A}\hat{p}_D)^2\\ +\frac{1}{2}k(\hat{x}_{B|A} - \hat{x}_D + \frac{m_B\hat{x}_{B|A} + m_C\hat{x}_{C|A}}{m_A})^2.\end{gathered}}$$

Here, we will first start by understanding the role of the extra particle quantities $\hat{x}_D$ and $\hat{p}_D$. As for Charlie's perspective, $\hat{p}_D$ also plays the role of the momentum of Alice relative to Eve. Furthermore, this $\hat{x}_D$ has appeared since Alice is a non-inertial quantum reference frame. So the question is: what could it be? We want to recover invariant accelerations. Indeed the value of accelerations should not depend on the type of reference frames we choose. Hence to act according to this, we want to find back the same accelerations as we had for the translation group from Alice's perspective. To do so, we can choose the value of this extra particle $\hat{x}_D$ as follows:

$$\boxed{\hat{x}_D = \frac{m_A\hat{x}_A + m_B\hat{x}_B + m_C\hat{x}_C}{m_A},} \tag{5.1}$$

we actually do get back to the accelerations obtained for translation. In other words, we retrieve:

$$\boxed{\begin{gathered}\hat{a}_{B|A} = -k\hat{x}_{B|A}(\frac{1}{m_A} + \frac{1}{m_B}),\\ \hat{a}_{C|A} = -\frac{k}{m_A}\hat{x}_{B|A},\\ \hat{H}^{(\hat{p})}_{BC|A} = \frac{\hat{p}_D^2}{2m_A} + \frac{1}{2m_B}(\hat{p}_{B|A} + \frac{m_B}{m_A}\hat{p}_D)^2 + \frac{1}{2m_C}(\hat{p}_{C|A} + \frac{m_C}{m_A}\hat{p}_D)^2 + \frac{1}{2}k\hat{x}_{B|A}^2.\end{gathered}}$$

These results are consistent with the translation group and the classical case. However, we can also say that the Hamiltonian is not invariant nor covariant under boost. Or at least there is a part of it that is not. In particular, the potential does remain unchanged. That means that if Alice does not have access to Eve, then she will not fully observe this Hamiltonian.

Hence, to obtain these results, we had to choose the extra particle as (5.1). This can be interpreted as the center of mass of the system relative to E modulated by the ratio of masses $\frac{m_{tot}}{m_A}$.

Finally, let us make this parallel with classical mechanics again. We can first calculate the velocities of Charlie and Bob as such: $\dot{\hat{x}}_{C|A} = [\hat{x}_{C|A}, \hat{H}^{(\hat{p})}_{BC|A}]$ and $\dot{\hat{x}}_{B|A} = [\hat{x}_{B|A}, \hat{H}^{(\hat{p})}_{BC|A}]$. We get respectively: $\dot{\hat{x}}_{C|A} = \frac{\hat{p}_{C|A}}{m_C} + \frac{\hat{p}_D}{m_A}$ and $\dot{\hat{x}}_{B|A} = \frac{\hat{p}_{B|A}}{m_B} + \frac{\hat{p}_D}{m_A}$. From there, we can derive the accelerations as such: $\ddot{\hat{x}}_{C|A} = [\dot{\hat{x}}_{C|A}, \hat{H}^{(\hat{p})}_{BC|A}]$ and $\ddot{\hat{x}}_{B|A} = [\dot{\hat{x}}_{B|A}, \hat{H}^{(\hat{p})}_{BC|A}]$. And this leads to Charlie's and Bob's respectively: $\ddot{\hat{x}}_{C|A} = 0$ and $\ddot{\hat{x}}_{B|A} = -\frac{k}{m_A}\hat{x}_{B|A}$. As previously said in the translation case, the reverse conclusion holds here. Indeed for boosts i.e. $\hat{p}$-frames, when we compute the accelerations using $\hat{p}$ variables, we get a result that considers the frame's acceleration. Whereas, when we use $\hat{x}$ variables to compute accelerations, the fictitious acceleration does not appear in the equations of motion of the observed system. However, this is not what we experience in reality, nor is it what we get in classical mechanics, where in the latter, both methods (using either x or p variables) lead to the correct results.

Let us now proceed with a discussion on the Galilei case.

5.3 Galilei group case

For the Galilei group case, as we will see we get surprising results which disagree with the one we have obtained for translation and boost cases. This suggests that there was probably a mistake in the calculations after doing the translation. Nevertheless, we will still analyse the results we have obtained and remain critical regarding them. Let us first remind ourselves of those results we have obtained for the case of a translation towards Alice, followed by a boost towards Charlie.

A translation towards Alice followed by a boost towards Charlie

In summary, we have the following:

$$\boxed{\begin{gathered}\dot{\hat{p}}_{A|C} = 0,\\ \ddot{\hat{x}}_{B|A} = -\frac{k}{m_A}\hat{x}_{B|A},\\ \ddot{\hat{x}}_{C|A} = -\frac{k}{m_A}\hat{x}_{B|A},\\ \hat{H}_{BC|A^{(\hat{x})},AB|C^{(\hat{p})}} = \frac{1}{2m_A}(\hat{p}_D - \hat{p}_{B|A} - \hat{p}_{C|A} - \frac{m_B}{m_C}\hat{p}_F)^2\\ +\frac{1}{2m_B}(\hat{p}_{B|C} + \frac{m_B}{m_C}\hat{p}_F)^2 + \frac{\hat{p}_F^2}{2m_C} + \frac{1}{2}k\hat{x}^2_{B|A}.\end{gathered}} \tag{5.2}$$

These results display surprising characteristics. First of all, we see that the only relative acceleration which provides a result equivalent to the one obtained in classical mechanics (as well as with the translation and boost groups), is the acceleration of Charlie relative to Alice. On the other hand, the acceleration of Bob relative to Alice solely takes into account the fictitious force due to the acceleration of Alice. However, it does not capture the "intrinsic" acceleration of Bob due to its relative movement of Alice. Also, we have to mention that in the perspective of Charlie which serves as an $\hat{x}$-frame, the acceleration of Alice is 0. This appears, naturally, when we compute the commutator of $\hat{p}_{A|C}$ with the Hamiltonian since the former commute with every observable in the Hamiltonian, but in practice, this is not what we measure. However, we would see the correct acceleration for Alice (and Bob) if we had computed it using the Hamiltonian just after we had jumped into Charlie's perspective using boosts. More research is needed to investigate the discrepancies between observed results and theoretical predictions for this group. Indeed there is no clear reason why we should not be able to obtain all correct accelerations from the final Hamiltonian.

Moreover, we can make an additional comment on the Hamiltonian where two extra-particles appear, each of those corresponding to one transformation: $\hat{p}_D$ and $\hat{p}_F$ due to the translation and boost respectively. As previously mentioned, we could argue that the former should be interpreted as the total momentum for the same reason we have used in the translation case. However, in this case, both extra-particles appear in the same kinetic term, which could lead one to believe in some link between the two. This is an intriguing feature that could benefit from further examination in future research. Moreover, we notice the absence of an extra-particle position as we had in the boost case when we considered Alice as the $\hat{p}$-frame. As we will later see, for the case where we first translate to Charlie's point of view and then boost to Alice's, we will obtain additional positions in the potential term.

As we did for the translation and boost cases, let us link what we did with classical mechanics where we could either use (4.9) or (4.32) (with the Poisson brackets), we would get the same result. Let us check this for the Galilei group with $\hat{H}$ being the Hamiltonian 4.50. To do so, we will compute the accelerations as follows: $\dot{\hat{p}}_{C|A} = \frac{1}{i}[\hat{p}_{C|A}, \hat{H}] = 0$, $\dot{\hat{p}}_{B|A} = \frac{1}{i}[\hat{p}_{B|A}, \hat{H}] = -k\hat{x}_{B|A}$. Furthermore, if we define the operator $\hat{a}_{B|A} = \frac{\dot{\hat{p}}_{B|A}}{m_B}$ we get that $\hat{a}_{B|A} = -\frac{k}{m_B}\hat{x}_{B|A}$, and finally $\ddot{x}_{A|C} = \frac{1}{i}[\dot{\hat{x}}_{A|C}, \hat{H}] = 0$ since $\dot{x}_{A|C} = \frac{1}{i}[\hat{x}_{A|C}, \hat{H}] = 0$.

Let us now discuss what happens when we first translate towards Charlie and then boost towards Alice.

A translation towards Charlie followed by a boost towards Alice

Here are the results we have obtained in summary:

$$
\boxed{\begin{aligned}
&\ddot{\hat{x}}_{A|C} = 0,\\
&\hat{a}_{B|A} = -\frac{k}{m_A}(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A} + m_C\hat{x}_{C|A}}{m_A}),\\
&\hat{a}_{C|A} = -\frac{k}{m_A}(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A} + m_C\hat{x}_{C|A}}{m_A}),\\
&\hat{H}_{AB|C^{(\hat{x})},BC|A^{(\hat{p})}} = \frac{\hat{p}_F^2}{2m_A} + \frac{1}{2m_B}(\hat{p}_{B|A} + \frac{m_B}{m_A}\hat{p}_F)^2\\
&+\frac{1}{2m_C}(\hat{p}_D - \hat{p}_{A|C} - \hat{p}_{B|C} - \frac{m_B}{m_A}\hat{p}_F)^2\\
&+\frac{1}{2}k(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A} + m_C\hat{x}_{C|A}}{m_A})^2.
\end{aligned}} \tag{5.3}
$$

First, let us comment on the acceleration terms. The acceleration of Alice relative to Charlie does not correspond to what we observe in practice. However (and as in the previous case for the Galilei group) we would find the correct acceleration if we had calculated Alice's (and Bob's) accelerations using the Hamiltonian we obtained after doing the translation. But again, future research on this should be done since there are no clear reasons why we should not obtain the correct accelerations using the final Hamiltonian. Now regarding the accelerations relative to Alice, both of them have this relative distance term in them which again corresponds to this fictitious force because Alice accelerates. This is something we have observed in the previous case for the Galilei group. However, they also have this center of mass term which vanishes in the boost case for a proper choice of the extra-particle. But here there a no extra particle in the potential term of the Hamiltonian.

And as usual, we can check if the property we have in classical mechanics with the Poisson brackets also prevails in the quantum case. In other words, if $\hat{H}$ is 4.56, let us compute $\dot{\hat{p}}_{A|C} = \frac{1}{i}[\hat{p}_{A|C}, \hat{H}] = k(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A}+m_C\hat{x}_{C|A}}{m_A})$ and $\dot{\hat{p}}_{B|C} = \frac{1}{i}[\hat{p}_{B|C}, \hat{H}] = k(\hat{x}_{A|C} - \hat{x}_{B|C} - \frac{m_B\hat{x}_{B|A}+m_C\hat{x}_{C|A}}{m_A})$. We can also define the following operators $\hat{a}_{A|C} = \frac{\dot{\hat{p}}_{A|C}}{m_A}$ and $\hat{a}_{B|C} = \frac{\dot{\hat{p}}_{B|C}}{m_B}$ so that the 2 previous results become respectively $\hat{a}_{A|C} = \frac{k}{m_A}(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A}+m_C\hat{x}_{C|A}}{m_A})$ and $\hat{a}_{B|C} = \frac{k}{m_B}(\hat{x}_{B|C} - \hat{x}_{A|C} + \frac{m_B\hat{x}_{B|A}+m_C\hat{x}_{C|A}}{m_A})$. We can finally also compute $\ddot{\hat{x}}_{C|A} = \frac{1}{i}[\dot{\hat{x}}_{C|A}, \hat{H}] = 0$ since $\dot{\hat{x}}_{C|A} = \frac{1}{i}[\hat{x}_{C|A}, \hat{H}] = 0$.

In [17] it has been stated that in the case of the Galilei group: the full Hilbert space of N particles decomposes as a tensor product of the degrees of freedom (whether it is the position or the momentum) of the Centre of mass of the system (which is a 1-particle Hilbert space) and another Hilbert space which is invariant. The latter can be decomposed as a tensor product of the relative degrees of freedom of the observed particles and the extra particle necessary for a reversible transformation between observers.

Here, as we have said initially, according to the results, we may suspect that there was a mistake in the calculations. However, if we had the correct Hamiltonian with certainty, even though the Hamiltonian we obtained is not fully Galilei invariant (because it is not invariant under boosts), we could expect (according to the previous paragraph) that the Hamiltonian split nicely into two parts. First, a kinetic term for the centre of mass, plus an additional term that should be possible to express entirely in terms of invariant quantities and therefore acts entirely on the invariant subsystem. If this is true, it would mean that the centre of mass and the invariant subsystem evolve independently, each driven by its own Hamiltonian, without interaction between the two. Then, an observer who does not have access to an external frame for the Galilei group would still be able to fully describe the evolution of the invariant degrees of freedom. This, however, is to be left for future studies on this subject.

Chapter 6

Additional discussions regarding quantum foundations

Let us initiate a small discussion regarding why we use a reference frame for both position and momentum in this model. This discussion regards quantum mechanics foundations and wants to express several hints that seem to indicate that in quantum mechanics, the quantities $\hat{x}$ and $\hat{p}$ seem to be purely independent. This whole discussion directly comes from discussions Prof. Oreshkov and Dr. Lin-Qing Chen in Prof. Oreshkov's office and by mail exchanges.

In classical mechanics, it is common sense that velocity is derivable from positions. Indeed if we have two positions corresponding to two instants in time, all classical mechanics books will say that $\dot{x} = \frac{\Delta x}{\Delta t}$. From there, we can thus define the kinetic momentum as $p = m\dot{x}$ where m is the mass of the system we observe.

1. Kinetic vs. Canonical Momentum

However, in quantum mechanics, the situation is not the same. First, there is no definition of kinetic momentum as in the classical case. In fact, quantum mechanics requires that the fundamental commutation relation $[\hat{x}, \hat{p}] = i$ is satisfied (from which the Heisenberg uncertainty principle is derived) [19] where $\hat{x}$ and $\hat{p}$ are canonical quantities. In quantum mechanics, this canonical relation is postulated in the context of canonical quantization [50] and leads to theoretical results which can be observed in reality [51] (such as Heisenberg's uncertainty principle).

For this to be true, there is a choice we must make regarding $\hat{x}$ and $\hat{p}$. There is for instance this typical choice and definition of $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ [19]. This definition would require first to define a Hilbert space on which we define a $|x\rangle$ as a basis (since this $\hat{p}$ depends on this basis). But a priori, this choice for momentum is not unique. Indeed, it is easy to show that for a given position basis $|x\rangle$, the choice $\hat{p} = -i\frac{\partial}{\partial x}$ will allow us to get back to $[\hat{x}, \hat{p}] = i$ [52]. In fact, by definition, $[\hat{x}, \hat{p}] = i$ is true for any pair of canonically conjugate operators $\hat{x}$ and $\hat{p}$. Hence, it is also easy to show that any choice $\hat{p} = -i\frac{\partial}{\partial x} + constant$ will also give the correct commutation relation.

In particular, if the Hilbert space of one particle is assumed given, and if we also had the position operator $\hat{x}$ of a particle A relative to C in that single-particle Hilbert space, for example, the momentum $\hat{p}_{A|C}$ is uniquely defined merely from the canonical commutation relation. However, this cannot help us single out a choice for the operator $\hat{p}_{A|C}$ in our case, because we do not know the Hilbert space of the relative particle $A|C$. Indeed we have only been given the position operator $\hat{x}_{A|C}$ i.e. $\hat{x}_{A|C} \otimes \mathbb{1}_{rest}$ which is also equal to $\hat{x}_A - \hat{x}_C$. So we know that $\hat{p}_{A|C}$ should be a tensor factor of the full Hilbert space of A and C, but we do not know which one. There are indeed many candidates for $\hat{p}_{A|C} \otimes \mathbb{1}_{rest}$ that would satisfy the correct commutation with it, and each choice of such operator defines a different factorisation of the full Hilbert space of A and C into $A|C$ and "rest".

Furthermore, suppose that, ultimately, classical mechanics should be derivable from quantum mechanics as some sort of approximation of the latter. From that supposition, it is thus possible that the use of kinetic momentum in classical mechanics implicitly restricts the scope of the study. Therefore, it might be safer to generalize kinetic momentum to the canonical momentum. We think that these remarks should be worth exploring in future work.

2. Position vs. Momentum reference frames

Let us now show an example of how given two different changes of coordinates for positions we get two different choices of momenta. And in the light of the framework in which this Thesis is, we will show that only one of these momentum seems to be the most appropriate.

In classical mechanics, to properly say what the canonical/generalised momentum should be, one has to appeal to a Lagrangian L and derive it from there i.e. $\frac{\partial L}{\partial \dot{q}_k}$ corresponding to the coordinate q_k [53] [54]. We could use the Hamiltonian but since the two perspectives are equivalent we will allow us to change for this section.

Let us consider the same situation as in 4.1 but this time with classical particles. We can start by writing the Lagrangian of Alice, Bob, and Charlie relative to Eve:

$$L = \frac{m_A}{2}\dot{x}_A^2 + \frac{m_B}{2}\dot{x}_B^2 + \frac{m_C}{2}\dot{x}_C^2 + \frac{k}{2}(x_B - x_A)^2, \tag{6.1}$$

and we can perform a (reversible) change of coordinates towards Alice.

a) Let us make for example the following change of coordinates: $\tilde{x}_A = x_A$, $\tilde{x}_B = x_B - x_A$ and $\tilde{x}_C = x_C - x_A$ such that we get

$$\begin{cases} x_A = \tilde{x}_A \\ x_B = \tilde{x}_A + \tilde{x}_B \\ x_C = \tilde{x}_A + \tilde{x}_C \end{cases}. \tag{6.2}$$

Like that, we can write the new Lagrangian L^* as:

$$L^* = \frac{m_A}{2}\dot{\tilde{x}}_A^2 + \frac{m_B}{2}(\dot{\tilde{x}}_B^2 + 2\dot{\tilde{x}}_B\dot{\tilde{x}}_A + \dot{\tilde{x}}_A^2) + \frac{m_C}{2}(\dot{\tilde{x}}_C^2 + 2\dot{\tilde{x}}_C\dot{\tilde{x}}_A + \dot{\tilde{x}}_A^2) + \frac{k}{2}\tilde{x}_B^2. \tag{6.3}$$

Now, if we compute the canonical momentum of B for example, we get:

$$\tilde{p}_B = \frac{\partial L^*}{\partial \dot{\tilde{x}}_B} = m_B\dot{\tilde{x}}_B + m_B\dot{\tilde{x}}_A, \tag{6.4}$$

which if expressed in terms of the old variables gives us: $\tilde{p}_B = m_B\dot{x}_B = p_B$. This momentum is canonically conjugate to $\tilde{x}_B$ by construction.

b) Now, we will see that if we apply another change of coordinates, this will lead us to a different value for the momentum of B. Nevertheless, it will be, by construction, canonically conjugate to $\tilde{x}_B$. This time, let us make, for example, the following change of coordinates: $\tilde{x}_A = \frac{m_A x_A + m_B x_B + m_C x_C}{m_A + m_B + m_C}$, $\tilde{x}_B = x_B - x_A$ and $\tilde{x}_C = x_C - x_A$ where we define $M = m_A + m_B + m_C$. As such, we get:

$$\begin{cases} x_A = \tilde{x}_A - \frac{m_B}{M}\tilde{x}_B - \frac{m_C}{M}\tilde{x}_C \\ x_B = \tilde{x}_A + (1 - \frac{m_B}{M})\tilde{x}_B - \frac{m_C}{M}\tilde{x}_C \\ x_C = \tilde{x}_A - \frac{m_B}{M}\tilde{x}_B + (1 - \frac{m_C}{M})\tilde{x}_C \end{cases}. \tag{6.5}$$

Like that, we can write the new Lagrangian L^{**} as:

$$\begin{aligned} L^{**} = &\frac{m_A}{2}(\dot{\tilde{x}}_A - \frac{m_B}{M}\dot{\tilde{x}}_B - \frac{m_C}{M}\dot{\tilde{x}}_C)^2 + \frac{m_B}{2}(\dot{\tilde{x}}_A + (1 - \frac{m_B}{M})\dot{\tilde{x}}_B - \frac{m_C}{M}\dot{\tilde{x}}_C)^2 \\ &+ \frac{m_C}{2}(\dot{\tilde{x}}_A - \frac{m_B}{M}\dot{\tilde{x}}_B + (1 - \frac{m_C}{M})\dot{\tilde{x}}_C)^2 + \frac{k}{2}\tilde{x}_B^2, \end{aligned} \tag{6.6}$$

Now, if we compute the canonical momentum of B for example, we get:

$$\begin{aligned} \tilde{p}_B &= \frac{\partial L^{**}}{\partial \dot{\tilde{x}}_B} \\ &= m_A(\dot{\tilde{x}}_A - \frac{m_B}{M}\dot{\tilde{x}}_B - \frac{m_C}{M}\dot{\tilde{x}}_C)(-\frac{m_B}{M}) \\ &+ m_B(\dot{\tilde{x}}_A + (1 - \frac{m_B}{M})\dot{\tilde{x}}_B - \frac{m_C}{M}\dot{\tilde{x}}_C)(1 - \frac{m_B}{M}) \\ &+ m_C(\dot{\tilde{x}}_A - \frac{m_B}{M}\dot{\tilde{x}}_B + (1 - \frac{m_C}{M})\dot{\tilde{x}}_C)(-\frac{m_B}{M}), \end{aligned} \tag{6.7}$$

and if we express this momentum in the old coordinates and we define $p_{tot} = p_A + p_B + p_C$ we obtain the following:

$$\begin{aligned}\tilde{p}_B &= m_A\dot{x}_A(-\frac{m_B}{M}) + m_B\dot{x}_B(1-\frac{m_B}{M}) + m_C\dot{x}_C(-\frac{m_B}{M}) \\ &= m_B\dot{x}_B - \frac{m_B}{M}p_{tot},\end{aligned} \tag{6.8}$$

hence $\tilde{p}_B = m_B(\dot{x}_B - \frac{p_{tot}}{M})$. This momentum is also canonically conjugate to $\tilde{x}_B$ by construction.

Both options for the position variable $\tilde{x}_A$, whether it is x_A in scenario 'a' or the center of mass in scenario 'b', yield a momentum $\tilde{p}_B$ that is canonically conjugate to $\tilde{x}_B$. Our choice of the position variable $\tilde{x}_A$ effectively determines what we consider as a distinct system from particles B and C. In scenario 'a', we observe that the canonical momentum of B is equivalent to the kinetic momentum p_B relative to an external momentum frame, in complete accordance with our quantum prescription [55].

Conversely, in scenario 'b', we derive $\tilde{p}_B = m_B(\dot{x}B - \frac{ptot}{M})$, which also possesses the mathematical characteristics of kinetic momentum, though with respect to the center of mass momentum. Nevertheless, the center of mass is not a distinct system from A when both A and the center of mass are viewed externally. Hence, this momentum cannot be construed as a kinetic momentum relative to any momentum frame [55].

Hence, we must contemplate the following question: Why does our quantum prescription [17] advocate for the initial choice of defining relative particle positions as $\tilde{x}_B = x_B - x_A$ and $\tilde{x}_C = x_C - x_A$, and the momenta as p_B and p_C (which we would easily obtain by computing the canonical momentum of C in scenario 'a' as $\frac{\partial L^*}{\partial \dot{\tilde{x}}_C}$)? What justifies this preference over alternative choices, such as those in scenario 'b' or other possibilities [55]?

We might present the following argument: When we examine the definition of the position and momentum of the relative particle $C|A$ in [17], they represent the controlled-translated $\hat{x}_A$ and $\hat{p}_A$, where the control is determined by the position of the reference particle A. Importantly, due to the invariance of momentum under translation in quantum mechanics, the controlled translation of $\hat{p}_C$ remains equivalent to the original $\hat{p}_C$, while $\hat{x}_C$ undergoes a transformation to $(\hat{x}_C - \hat{x}_A)$. The same reasoning holds for B also. And this reasoning harmonizes with the well-accepted classical reference frame transformations applied to quantum systems [21] [55].

However, a valid inquiry surfaces: What justifies the standard definition of translation as the correct one? Why do we represent translation by an amount x on an operator T using the formula $e^{ix\hat{p}}Te^{-ix\hat{p}}$? While this formula holds a firm position in traditional quantum theory [21], it becomes a natural question given our quest to study the foundations of quantum theory. Specifically, couldn't we potentially define translation by employing the established formula followed with what is conventionally acknowledged as a boost operation? [55].

Let's consider a practical example: say we, assuming that we have the role of a classical reference frame, observe a quantum system and decide to hop onto a particle in motion within that system. One might naturally think that the momentum of the remaining particles, as perceived from our now-moving perspective, would undergo changes due to the fact that their velocities are measured relative to our frame of reference. This intuitive expectation arises because we've essentially shifted our viewpoint. This is a natural thought since this is what would happen ”classically”. Intriguingly, when we implement this additional boost operation, the position variable remains unaffected, as position operators commute with the boost generator. However, it's essential to highlight that under this boost, the momentum of particle C does indeed experience modifications. This outcome mirrors the scenario outlined at the beginning of the paragraph [55].

Consequently, the thought-provoking query that emerges is the following: What justifies the standard definition of translation as being the correct one? [55]

3. On the instantaneity of momentum

Our standpoint revolves around the fundamental separation and independence of momentum from position. This perspective finds resonance in Hamiltonian mechanics, where position and momentum are treated as a priori independent coordinates. It also extends to quantum mechanics, where momentum is regarded as an observable quantity, akin to position, that can be measured at a specific moment in time. There is a conceptual distinction between the observable quantity ”momentum” and the mass times velocity, where velocity is understood as a difference in positions at two different times (in the limit where two times become infinitesimally close). If we look at velocity as a difference of positions at different times, it cannot be measured in a single instant [55].

While this differentiation might not be apparent in the classical scenario, it becomes much more significant when we consider the quantum ones. Indeed, in the quantum context, consider the operator $\dot{\hat{x}}$ in the Heisenberg picture, which can be interpreted as velocity derived from the evolution of the operator $\hat{x}(t)$ in the Heisenberg picture (defined as 4.9 and demonstrated in Appendix 8.4). For instance, in the case of a freely moving particle with mass 'm' governed by $\hat{H} = \frac{\hat{p}^2}{2m}$, the commutation relation $[\hat{x}, \hat{p}] = i\hbar$ leads to $\dot{\hat{x}} = \frac{\hat{p}}{m}$. However, it's important to note that we cannot measure this operator by conducting quantum measurements on the system at consecutive time points. In practice, frequent measurements of the operator $\hat{x}$ forces the system to remain in its initial state at each measurement instance, illustrating the quantum Zeno effect [55] [56].

The concept can be summarized as follows: Momentum is an instantaneous property that does not entail processes akin to measuring positions at various points in time. To illustrate, consider the time-of-flight experiment, where a quantum particle traverses a narrow slit. In this setup, momentum can be deduced from the apparatus's geometry and the timing recorded upon the particle's impact on a screen, as discussed in [51]. Consequently, if we choose to align ourselves with a particle as our reference for position, the velocity at which that particle is moving should have no bearing on the outcome [55].

Moreover, in the context of quantum mechanics, where position and momentum are perceived as independent entities at a specific moment in time, it logically follows to define pure translation as an operation that solely alters position without affecting momentum. This concept bears resemblance to the scenario where, in two-dimensional orthogonal coordinates, a pure translation in one coordinate (e.g., x) does not induce changes in the other (e.g., y). Notably, this perspective aligns seamlessly with the treatment of the Galilei group detailed in [17], which delineates the reference frame into two distinct components: one dedicated to position and the other dedicated to velocity, with velocity defined in terms of momentum rather than the derivative of position [55].

4. Non-infinitely heavy reference frames

Let us add a final remark regarding classical reference frames. In classical mechanics, when we consider one single reference frame Eve that describes the position and the momentum of a particle A and if we consider the kinetic momentum $p = mv$, the only way for which x and p satisfy the commutation relation under the Poisson brackets is that the mass of the observer is implicitly infinite. Indeed, if we use the definition of the relative kinetic momentum, we have:

$$\begin{aligned}
&\{x_{A|E}, p_{A|E}\} = \{x_A - x_E, m_A(v_A - v_E)\} \\
&= \{x_A - x_E, p_A - \frac{m_A}{m_E}p_E)\} \\
&= \{x_A, p_A\} - \underbrace{\{x_E, p_A\}}_{=0} - \underbrace{\{x_A, \frac{m_A}{m_E}p_E\}}_{=0} + \{x_E, \frac{m_A}{m_E}p_E\} \\
&= 1 + \frac{m_A}{m_E} \\
&\overset{m_E \to \infty}{\to} 1
\end{aligned} \tag{6.9}$$

As quantum mechanics is a theory of the infinitesimally small, classical mechanics should be, in principle, derivable from quantum mechanics such that the former may be seen as an approximation of the latter. And indeed, the fact that we suppose that the observer's mass is infinite in the classical case is a strong hypothesis and should not be a prerequisite to get a satisfied commutation relation.

Chapter 7

Conclusions

We have studied a quantum system that allows for both an inertial and a non-inertial quantum reference frame. We further studied the cases where we “jumped” on this non-inertial quantum reference frame, either using a translation (a translation in position space) and using a boost (a translation in momentum space). Each of these cases corresponded either to an $\hat{x}$-frame (a frame that measures position) or a $\hat{p}$-frame (a frame that measures momentum). For both cases, when we transform to a non-inertial frame (without “artificially” adding a potential term to the Hamiltonian that creates the fictitious force), we have obtained the correct relative acceleration observables through quantum reference frame transformations.

Furthermore, we have studied the role that this extra-particle played in the covariance of the equations of motion, and we have discussed its interpretation in light of the invariant subspace of observables.

We have also made a link with classical mechanics when we computed the laws of motion corresponding to the relative Hamiltonians, and we have underlined the discrepancies between the classical and the quantum cases being due to the nature of the reference frame used. Indeed, in the quantum case, we have $\hat{x}$ and $\hat{p}$-frames for which the velocities and the accelerations are computed using only the commutators with the corresponding variables. Whereas, in the classical case, the nature of the reference frame does not matter and the mass of the reference frame is assumed infinite. In that case, the velocities and the accelerations can be computed using the Poisson brackets regardless of the variable (x or p) used.

Finally, we have discussed the results we have obtained to get a better understanding of the link that seems to exist between the extra-particle and the fictitious force which appears in the equations of motion when considered relative to a non-inertial quantum reference frame.

Even though this work has shown the coherence of the framework for the studied system under translation and boost transformations, it also raised several questions that should be worth exploring in future research. Indeed, for example, it is easy to show (cf. Appendix 8.5) that the system which has been studied has a center of mass that does not accelerate. It would thus be interesting to study one which features the same characteristics for which the center-of-mass does accelerate relatively to the external reference frame. Another future study could also focus on a more complicated system where for example the three masses are linked together or have additional degrees of freedom.

We have also well studied the cases for translation and boost. However, it could be insightful to further study the consequences of having simultaneously one quantum reference frame for the position and another for momentum. In particular, for the Galilei group, what are the consequences of having two extra-particles in terms of Hilbert spaces? Would those two extra-particles interfere with one another, for example? And how is the covariance of the dynamics would be expressed when we have a mixture of inertial and non-inertial quantum reference frames simultaneously? Indeed these types of questions are particularly relevant since they correspond to a genuinely new situation compared to what we have in classical mechanics.

In regards to what has been in the introduction, we could also mention that future research should focus on studying interacting and non-inertial quantum reference frames in gravity and get a more in-depth understanding of the general covariance of gravity in the quantum context.

Finally, future studies should also try to see how would this framework apply under the Lorentz and the Poincaré groups.

Chapter 8

Appendices

8.1 Appendix A: Proof of 3.35

Proof. Let us first start by developing the first member of Equation (3.34) using (3.33). To simplify notation, let us consider that :

$$\hat{S}_{C\rightarrow A} := \hat{S}. \tag{8.1}$$

This allows us to get:

$$i\hbar\frac{d\hat{\rho}_{BC|A}}{dt} = i\hbar\frac{d\hat{S}}{dt}\hat{\rho}_{AB|C}\hat{S}^{\dagger} + i\hbar\hat{S}\frac{d\hat{\rho}_{AB|C}}{dt}\hat{S}^{\dagger} + i\hbar\hat{S}\hat{\rho}_{AB|C}\frac{d\hat{S}^{\dagger}}{dt}. \tag{8.2}$$

In addition, we know that $\hat{S}$ is unitary and using (3.33) we can write Equation (8.2) as:

$$i\hbar\frac{d\hat{\rho}_{BC|A}}{dt} = i\hbar\frac{d\hat{S}}{dt}\hat{S}^{\dagger}\hat{\rho}_{BC|A} + i\hbar\hat{S}\frac{d\hat{\rho}_{AB|C}}{dt}\hat{S}^{\dagger} + i\hbar\hat{\rho}_{BC|A}\hat{S}\frac{d\hat{S}^{\dagger}}{dt}. \tag{8.3}$$

Furthermore we use the following property of an unitary operator:

$$\frac{d(\hat{S}\hat{S}^{\dagger})}{dt} = \frac{d(\hat{S}^{\dagger}\hat{S})}{dt} = 0, \tag{8.4}$$

which allows us to get:

$$\iff \frac{d\hat{S}}{dt}\hat{S}^{\dagger} = -\hat{S}\frac{d\hat{S}^{\dagger}}{dt}. \tag{8.5}$$

Using this property, Equation (8.3) can be written as:

$$i\hbar\frac{d\hat{\rho}_{BC|A}}{dt} = i\hbar\frac{d\hat{S}}{dt}\hat{S}^{\dagger}\hat{\rho}_{BC|A} + i\hbar\hat{S}\frac{d\hat{\rho}_{AB|C}}{dt}\hat{S}^{\dagger} - i\hbar\hat{\rho}_{BC|A}\hat{S}^{\dagger}\frac{d\hat{S}}{dt}. \tag{8.6}$$

Now, by using (3.32), we can write (8.6) as:

$$i\hbar\frac{d\hat{\rho}_{BC|A}}{dt} = i\hbar\frac{d\hat{S}}{dt}\hat{S}^{\dagger}\hat{\rho}_{BC|A} - i\hbar\hat{\rho}_{BC|A}\hat{S}^{\dagger}\frac{d\hat{S}}{dt} + \hat{S}\hat{H}_{AB|C}\hat{\rho}_{AB|C}\hat{S}^{\dagger} - \hat{S}\hat{\rho}_{AB|C}\hat{H}_{AB|C}\hat{S}^{\dagger}. \tag{8.7}$$

Hence, by using the unitarity of $\hat{S}$ and (3.33) we obtain:

$$i\hbar\frac{d\hat{\rho}_{BC|A}}{dt} = (i\hbar\frac{d\hat{S}}{dt}\hat{S}^\dagger + \hat{S}\hat{H}_{AB|C}\hat{S}^\dagger)\hat{\rho}_{BC|A} - \hat{\rho}_{BC|A}(i\hbar\hat{S}^\dagger\frac{d\hat{S}}{dt} + \hat{S}\hat{H}_{AB|C}\hat{S}^\dagger), \tag{8.8}$$

where we recover the definition of the commutator $[\hat{H}_{BC|A}, \hat{\rho}_{BC|A}]$. As such we conclude:

$$\hat{H}_{BC|A} = i\hbar\frac{d\hat{S}}{dt}\hat{S}^\dagger + \hat{S}\hat{H}_{AB|C}\hat{S}^\dagger \tag{8.9}$$

This concludes the proof. □

8.2 Appendix B: Proof of 4.4

Proof. Let us first make a Taylor expansion of the exponentials. By gathering the terms together we obtain:

$$e^{i\hat{A}}f(\hat{B})e^{-i\hat{A}} = f(\hat{B}) + i[\hat{A}, f(\hat{B})] + ... \tag{8.10}$$

Now by using the property (8.11) which we will be demonstrated in Appendix 8.3:

$$[\hat{A}, f(\hat{B})] = f'(\hat{B})[\hat{A}, \hat{B}]. \tag{8.11}$$

Using 8.11, we can now write (8.10) as:

$$f(\hat{B}) + i[\hat{A}, f(\hat{B})] = f(\hat{B}) + i[\hat{A}, \hat{B}]f'(\hat{B}), \tag{8.12}$$

which is by definition the Taylor expansion of f($\hat{B} + i[\hat{A}, \hat{B}]$) [21]. Hence, we can write:

$$e^{i\hat{A}}f(\hat{B})e^{-i\hat{A}} = f(\hat{B} + i[\hat{A}, \hat{B}]). \tag{8.13}$$

This concludes the proof.

□

8.3 Appendix C: Proof of 8.11

Proof. First let us follow a similar reasoning as in [57] to demonstrate by induction this property:

$$[\hat{A}, \hat{B}^n] = n\hat{B}^{n-1}[\hat{A}, \hat{B}], \tag{8.14}$$

under the assumption $[[\hat{A}, \hat{B}], \hat{B}] = 0$.

We know the above is true for n = 0, 1. Let us suppose it is true for $n-1$, we will now show it is correct $\forall n$. Hence for any operator $\hat{A}, \hat{B}$ we have by definition:

$$
\begin{aligned}
[\hat{A}, \hat{B}^n] &= \hat{A}\hat{B}^n - \hat{B}^n\hat{A} \\
&= \hat{A}\hat{B}^{n-1}\hat{B} - \hat{B}^{n-1}\hat{B}\hat{A} + \underbrace{\hat{B}^{n-1}\hat{A}\hat{B} - \hat{B}^{n-1}\hat{A}\hat{B}}_{=0} \\
&= [\hat{A}, \hat{B}^{n-1}]\hat{B} + \hat{B}^{n-1}[\hat{A}, \hat{B}] \\
&\overset{hyp}{=} (n-1)\hat{B}^{n-2}[\hat{A}, \hat{B}]\hat{B} + \hat{B}^{n-1}[\hat{A}, \hat{B}] \\
&\overset{hyp}{=} (n-1)\hat{B}^{n-2}\hat{B}[\hat{A}, \hat{B}] + \hat{B}^{n-1}[\hat{A}, \hat{B}] \\
&= n\hat{B}^{n-1}[\hat{A}, \hat{B}]
\end{aligned}
\tag{8.15}
$$

□

Proof. So if now we express a function $f(\hat{B})$ as a series:

$$
f(\hat{B}) = \sum_n c_n \hat{B}^n, \tag{8.16}
$$

we can write the commutator $[\hat{A}, f(\hat{B})]$ as follows:

$$
\begin{aligned}
[\hat{A}, f(\hat{B})] &= [\hat{A}, \sum_n c_n \hat{B}^n] \\
&= \sum_n c_n [\hat{A}, \hat{B}^n].
\end{aligned}
\tag{8.17}
$$

If we now use the result obtained in the first proof of this Appendix and the fact that $f'(\hat{B}) = \sum_n n c_n \hat{B}^{n-1}$, we naturally get:

$$
\begin{aligned}
[\hat{A}, f(\hat{B})] &= \sum_n n c_n \hat{B}^n [\hat{A}, \hat{B}^{n-1}] \\
&= f'(\hat{B})[\hat{A}, \hat{B}].
\end{aligned}
\tag{8.18}
$$

This concludes the proof.

□

8.4 Appendix D: Proofs of 4.9 and 4.10

Proof. By definition of the evolution in time of the position observable $\hat{x}$ we have:

$$
\hat{x}(t) = e^{i\frac{\hat{H}}{\hbar}t}\hat{x}e^{-i\frac{\hat{H}}{\hbar}t} \tag{8.19}
$$

Hence if we want to compute the velocity we have

$$
\begin{aligned}
\dot{\hat{x}}(t) &= \frac{d}{dt}\hat{x}(t) \\
&= \frac{d}{dt}(e^{i\frac{\hat{H}}{\hbar}t})\hat{x}e^{-i\frac{\hat{H}}{\hbar}t} + e^{i\frac{\hat{H}}{\hbar}t}\hat{x}\frac{d}{dt}(e^{-i\frac{\hat{H}}{\hbar}t}) \\
&= i\frac{\hat{H}}{\hbar}e^{i\frac{\hat{H}}{\hbar}t}\hat{x}e^{-i\frac{\hat{H}}{\hbar}t} - i\frac{1}{\hbar}e^{i\frac{\hat{H}}{\hbar}t}\hat{x}\hat{H}e^{-i\frac{\hat{H}}{\hbar}t}
\end{aligned}
\tag{8.20}
$$

because $\hat{H}$ commute with itself we get

$$\begin{aligned}\dot{\hat{x}}(t) &= i\frac{\hat{H}}{\hbar}e^{i\frac{\hat{H}}{\hbar}t}\hat{x}e^{-i\frac{\hat{H}}{\hbar}t} - i\frac{1}{\hbar}e^{i\frac{\hat{H}}{\hbar}t}\hat{x}e^{-i\frac{\hat{H}}{\hbar}t}\hat{H} \\ &= \frac{i}{\hbar}[\hat{H}, \hat{x}(t)] \\ &= \frac{1}{i\hbar}[\hat{x}(t), \hat{H}]\end{aligned} \tag{8.21}$$

For $\ddot{\hat{x}}(t)$ we just derive the previous expression and we get

$$\ddot{\hat{x}}(t) = \frac{1}{i\hbar}[\dot{\hat{x}}(t), \hat{H}] \tag{8.22}$$

This conclude the proof.

□

8.5 Appendix E: Non accelerating center of mass

In this Appendix we will simply show in classical mechanics that the study we have done restricts itself to a system ABC for which the center of mass does not accelerate.

Proof. Indeed for that let us define the position of the center of mass of the system ABC. With $m_{tot} = m_A + m_B + m_C$ we have:

$$x_{CM|E} = \frac{1}{m_{tot}}(m_A x_{A|E} + m_B x_{B|E} + m_C x_{C|E}). \tag{8.23}$$

Thus by simply deriving this position and by removing the "| E" without loss of meaning we get [58]:

$$v_{CM} = \frac{1}{m_{tot}}(m_A v_A + m_B v_B + m_C v_C). \tag{8.24}$$

If we now derive this equation again, we get the following:

$$a_{CM} = \frac{1}{m_{tot}}(m_A a_A + m_B a_B + m_C a_C). \tag{8.25}$$

If we now use the fact that for this system the accelerations of each particle are given by:

$$\begin{aligned}a_A &= \frac{1}{m_A}(x_B - x_A), \\ a_B &= \frac{1}{m_B}(x_A - x_B), \\ a_C &= 0,\end{aligned}$$

we obtain that $a_{CM} = 0$ which closes the demonstration. □

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