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Characterization of Higher-order Quantum Processes

When projective methods recover a model of logic

Thèse présentée par Timothée HOFFREUMON

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FREEDOM TO RESEARCH

Thèse de doctorat en sciences de l'ingénieur

Characterization of Higher-order Quantum Processes

When projective methods recover a model of logic

Timothée Hoffreumon

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Colophon
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I am grateful to my friend Pierre Vandenhove for suggesting the template.
The figures were drawn using draw.io.

Cassandra
Je ne te ferai pas l'affront de te la dédier,
vu tout le temps qu'elle nous a dérobé.

Résumé

Les transformations de transformations, également appelées processus d'ordres supérieurs, forment un concept courant en informatique et en traitement de l'information. De tels processus apparaissent dès qu'il est question de manipulations sur l'opération à appliquer aux données, plutôt que sur les données elles-mêmes. Par exemple, lorsque l'on veut représenter un protocole informatique avec des boucles de rétroaction d'opérations, comme des boucles "for" imbriquées, ou lorsque l'on veut représenter un protocole de communication avec un contrôle dynamique des opérations, comme lorsqu'un adversaire agit sur les données d'entrée et de sortie d'une autre partie afin de la tromper, on utilisera des processus d'ordre supérieur.

Ce paradigme appliqué à l'informatique quantique a récemment suscité un grand intérêt, tant au niveau pratique que fondamental. D'une part, il a été démontré que certains processus quantiques d'ordre supérieur permettaient de réduire le nombre d'opérations nécessaires à la réalisation de certains protocoles. D'autre part, ces processus présentent parfois des relations causales *indéfinies* au sens quantique du terme ; l'ordre des événements A et B peut se superposer entre A puis B *et* B puis A. Ce comportement est d'un grand intérêt fondamental car il remet en question certaines idées préconçues que d'aucun pensent incompatibles avec une théorie quantique de la gravité.

Un cadre général pour représenter les transformations quantiques d'ordres supérieurs est dès lors nécessaire pour pleinement exploiter les améliorations qu'elles apportent et, en parallèle, pour étudier les relations causales quantiques singulières qu'elles présentent. Pareil cadre est développé dans cette thèse. Plus précisément, un ensemble d'outils pour caractériser les processus quantiques d'ordre supérieurs valides reposant sur la dualité canal-état ainsi que l'utilisation de projecteurs superopératoires est présenté. Il est montré que les manières possibles de définir un ensemble donné de transformations d'un même ordre sont homomorphes à une algèbre de ces projecteurs superopératoires, qui sont à leur tour homomorphes aux relations de signalisation que les objets de cet ensemble peuvent permettre. De plus, il est démontré que cette algèbre est très proche d'un modèle de logique linéaire appelé BV. Ainsi, la définition d'une transformation au moyen de ces projecteurs se réduit à l'élaboration de formules logiques, tandis que l'énumération des relations causales qu'elle comporte se réduit à des manipulations symboliques sur ces formules.

Abstract

Transformations of transformations, also called higher-order processes, is a commonly occurring concept in computing and information processing. Such processes arise in situations involving manipulations of the operation applied to the data, rather than of the data itself. For example, when one wants to represent a computing protocol with feedback loops of operations, like nested 'for loops', or when one wants to represent a communication protocol with dynamical control over operations, like where an adversary party is acting on the input and output data of some other party so to deceive her, higher-order processes will be used.

Applied to quantum computing, this paradigm has recently attracted significant interest both at the practical and fundamental levels. On the one hand, specific higher-order quantum processes were shown to decrease the number of operations needed to realize certain protocols. On the other hand, these processes sometimes feature causal relations that are 'indefinite' in the quantum sense; the ordering of events A and B can become *superposed* between A then B *and* B then A. This behavior is of great fundamental interest as it challenges some pre-conceived ideas some believe to be incompatible with a quantum theory of gravity.

A general framework to represent higher-order quantum transformations is then necessary to fully harness the improvements they provide and, in parallel, to study the puzzling quantum causal relations they feature. Such a framework is developed in this thesis. Specifically, a set of tools for characterizing valid higher-order quantum processes relying on channel-state duality and the use of superoperator projectors is presented. It is shown that the possible ways to define a given set of higher-order transformations are homomorphic to an algebra of these superoperator projectors, which are in turn homomorphic to the signaling relations that the objects in this set may allow. Moreover, this algebra is shown to be very close to a model of linear logic called BV. Whence, defining a transformation through these projectors is reduced to forming logic-like formulae, whereas tracking down the causal relations it features is reduced to symbolic manipulations of these formulae.

List of Publication and Declaration of Authorship

The present dissertation is the culmination of four years of doctoral research during which I co-authored three articles:

- [1]: Timothée Hoffreumon and Ognyan Oreshkov. 'The Multi-round Process Matrix'. In: *Quantum* 5 (Jan. 2021). arXiv: `2005.04204 [quant-ph]`;
- [2]: Timothée Hoffreumon and Ognyan Oreshkov. *Projective characterization of higher-order quantum transformations*. 2022. arXiv: `2206.06206 [quant-ph]`;
- [3]: Titouan Carette, Timothée Hoffreumon, Émile Larroque, and Renaud Vilmart. 'Complete Graphical Language for Hermiticity-Preserving Superoperators'. In: *2023 38th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)* (June 2023). arXiv: `2302.04212 [quant-ph]`.

Only the second one, [2], is part of this thesis.

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I, Timothée Hoffreumon, hereby declare that I am the sole author of this thesis. To the best of my knowledge, it contains no material previously published by any other person except where due acknowledgment has been made.
It is noteworthy to mention that I used the AI-powered tools Grammarly and Writeful to correct my spelling. At no point in the text I used them for generating content.

Acknowledgements

This thesis possesses a genuine Indefinite Causal Order that can be witnessed in a lab: with it, I was able to start a post-doc before I even defended it. Jokes aside, this provides some information about the chaotic context in which I finished this thesis. My time management made it so that I was constantly behind schedule so I am very grateful to have received help from my supervisor Ognyan, understanding from my future[*] supervisor Pablo, and patience from the reviewers. Mentioning the latter, I do am indebted to Jessica, Aleks, Stefano, and Jérémie for agreeing to review this file despite the very crude aspect of the first versions. This thesis greatly benefited from their input and comments; it must be said that a significant portion of it has been improved based on their suggestion. As for Ognyan, he did manage to supervise all of this chaos not once, but twice. As a matter of fact, he has been helping me since that time when I was an engineering student looking for a master's thesis and the idea of a pandemic caused by a beer-named virus seemed ridiculous. For that I am appreciative.

The people at QuIC were also a great source of help and comfort during these four years. I am thankful to all the people who shared room 214A with me and who coincidentally had the courage to listen to my ramblings: Chrysoula, Atul, Joseph, Nicola, and Benoît. It was also a pleasant honor to be the colleague of all the people present in QuIC during that time; I am especially grateful to Esteban, Lin-Qing, Julian, Hamed, and Ravi for their technical help, especially towards the end of the redaction. I am also thankful for the discussions with my former classmates at ULB, in particular Célia (whom I happened to plagiarize by doing a thesis at QuIC as well), Eliott, Emmanuel, Jonathan, Henri, and the esteemed 'thugs assembly' think tank: Casimir, Denis, Enea, and Philippe.

During my thesis, I also learned that the world is bigger than the ULB. I had the chance to meet many inspiring people I pride myself on calling my friends. At the very least, I must mention Titouan, Pablo, Jan, Augustin, Natália, and Lefteris for their help... The list of all the people I wish to mention is still very long, so I will try to sum it up by mentioning all the participants of the Sejny Summer Institute 2021, of the Kefalonia Foundations Conference 2022, of the QISS spring school 2023, as well as all members of the Basic Research Community for Physics (BRCP).

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Naturally, help did not only come from experts. I am lucky enough to have been supported by the Belgian government through the École Polytechnique de Bruxelles (EPB) and the Fonds de la Recherche Scientifique (F.R.S.-FNRS) so I could dream about abstracting the abstraction without any financial concerns. My parents were actually my first patrons in this endeavor, but their support was way broader than just monetary. It is them who put me on the path of science, they helped me develop my curiosity as well as my love for science since my early days and they were always supportive of my choices; I value that above all. When things were not going well, I was also lucky to count on my brothers Louis and Florian as well as the 'commu' to cheer

[*] Where the word 'future' is used here in the retrocausal sense to mean November.
[†] Also known under the names of Satan and Belzebuth.
[‡] I recommend reading his manifesto.

me up. Finally, all stories must contain a bad guy and a love interest. I will not talk about the former here[§], but I saved the last words of these acknowledgements for the latter: While I cannot pretend I found definitive answers during my Ph.D., at least I found love. Cass, without you broadening my perspectives and without your unfailing support, I have no idea where I would be today.

P.S.: If your name did not appear in these too-short acknowledgments but you feel it should have, I will be happy to buy you a drink at any time to listen to the reason why!

[§] However, the interested reader can refer to Footnote †.

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Introduction

> Indeed, one can wonder what kind of quantum computer the universe is: It could be a gigantic quantum circuit where information is encoded in the state of many qubits and is processed in time from a spacelike surface to the next, or it could be a quantum Turing machine, or also be a higher-order computer, that processes information encoded in transformations (e.g. in scattering amplitudes) rather than in states.
>
> Chiribella et al. (2009), *Quantum computations without definite causal structure* [4]

> One of the striking features of quantum mechanics is that it challenges the view that physical properties are well defined prior to and independent of their measurement. (...) Is it possible that, in some circumstances, even causal relations would be 'uncertain', similarly to the way other physical properties of quantum systems are ?
>
> Oreshkov et al. (2012), *Quantum correlations with no causal order* [5]

The same way that a quantum channel describes the most general transformation mapping an input quantum state to an output quantum state [6, 7], a *quantum supermap* describes the most general transformation mapping an input quantum channel to an output quantum channel [8]. Interpreting the quantum channel as a transformation between states, the supermap is then a transformation of transformations. For that reason, it is called a *higher-order* transformation. Since nothing forbids *a priori* to nest transformations of transformations in quantum theory, one can consider successive nestings to recursively build whole hierarchies of higher-order quantum transformations [9–11].

Fragments of a quantum circuit are a concrete instance of the use of a higher-order hierarchy. A circuit fragment that 'goes around' a channel is a supermap: it takes a channel as input and outputs a channel [12, 13]. This supermap itself can be seen as the input for some super-supermap that will output a channel, and so on. This ensuing hierarchy has been defined under the name *quantum comb formalism* [9], which has proven to be a valuable tool in the field of quantum information theory. It is used to model the successive operations of a single party in a multipartite quantum protocol, in order to optimize her strategy as a whole but independently of the other parties' actions.

With a different goal than modeling circuit fragments, supermaps with multiple inputs were subsequently studied. First, the *quantum switch* was proposed as a supermap that takes two channels and outputs them in an order that depends on a control qubit [4]. Soon after, the *Process Matrices* (PM) were proposed as a general framework of supermaps that take a fixed number of quantum instruments [14] and map them to a joint probability for their outcomes [5]. Both concepts led to the identification of *Indefinite Causal Order* (ICO) as a feature of supermaps.

Indefinite causal order is the idea that, in some circumstances, even causal relations can become 'uncertain', similarly to how other physical properties of quantum systems can be. It was conceptualized in the early 2000s by Hardy as a prerogative for a quantum description of gravity [15, 16]. Later, he also proposed using this property to extend the quantum circuit formalism [17], which led to the quantum switch and the process matrix.

Soon after this theoretical proposition, experiments were conducted to demonstrate the feasibility of a quantum switch (see e.g., Reference [18]). Since then, harnessing the new possibilities offered by ICO has been a very active subject of research. These have been proposed as a means to improve the efficiency of protocols in many different areas of quantum information technologies, with examples having been found in e.g., quantum computing [19, 20], communications [21, 22], thermodynamics [23–25], metrology [26, 27], etc. and the list keeps growing.

Yet, to this date, the experimental realizations of the quantum switch are still debated [28–31]. And this is only one aspect of the general lack of systematic study of the theory of higher-order quantum processes. On a fundamental level, a lot is still not understood about them. In particular, there are no ways to determine if a process will even have a *non-fixed* causality, albeit this information is imperative to even speak about the phenomenon of *indefinite* causality. Another striking shortcoming is that almost all processes studied in the literature so far are assuming quantum channels as their input. Nevertheless, a general study of higher-order quantum processes ought to go beyond this assumption: to infer general properties, processes must be abstracted as *any* transformation between *any* transformations; if the switch is a valid process, then a switch whose inputs are quantum switches must be considered as well. The first example of such a truly nested process was considered to model the evolution of process matrices [32]: this evolution is indeed interpretable as a process matrix on process matrices. In parallel, Perinotti [10] and Bisio [11] started considering the theory as a whole. As mentioned in the first paragraph, they theorized and defined the hierarchy of higher-order processes, and provided a type system to classify the different kinds of nested transformations. Inspired by Perinotti's approach, Kissinger and Uijlen obtained a similar characterization but using the framework of category theory instead of type theory [33].

In their formalisms, it becomes possible to treat process matrices and quantum combs on the same footing, as two different classes of transformation within the whole hierarchy of *admissible* transformations. Still, one can then wonder what differentiates the switch from a comb, or the PM from a comb. Especially, why certain maps and hierarchies of maps may feature non-fixed causal orders while others will not. While these works provide partial answers, they fail to provide a general explanation. Motivated by these considerations, the ambition of this thesis is to present a framework that formalizes and characterizes higher-order quantum transformations. This framework uses the notion of *signaling* as a primitive for sorting the different processes. Briefly summarized, the two main questions answered through these pages are 'What is a higher-order quantum process?' and 'When does such a process feature a non-fixed causal order?'. In a more technical form, the first question is 'Given an operator on a set of input and output Hilbert spaces, what kind(s) of a higher order processes does it represent? Conversely, how can I represent a higher-order process as an operator?' while the second is 'What is the underlying signaling structure(s) of such a process? Does it feature more than one fixed direction of signaling?'.

This dissertation is based on the following preprint:

[2]: Timothée Hoffreumon and Ognyan Oreshkov. *Projective characterization of higher-order quantum transformations*. 2022. arXiv: `2206.06206 [quant-ph]`.

It generalizes the characterization method developed in a previous article about the *Multi-round Process Matrix* (MPM) [1]. Note that some of its results on MPMs are used in the thesis although the work on itself is not presented within it. This article is itself based on the technical part of my master thesis [34], which is in turn based on a method of Araújo and coworkers developed to define causal witnesses for process matrices [35]. Around the same time as the preprint [2] was completed, similar characterizations were independently derived by Simmons and Kissinger [36] and then by Milz and Quintino [37].

The thesis is organized as follows: in the first part, a somewhat original review of the process formalism is conducted in parallel with setting up the notation. In Chapter 1, the formalism is first presented as a statistical model for local interventions so as to present the concepts of signaling and causal correlations. Then it is specialized for interventions on quantum systems exchanged with an environment. The dichotomy of local interventions/global environment is subsequently explored in Chapter 2, in which the process formalism is discussed in the multipartite case. Multiple local parties interacting with a global environment are argued to necessarily result in a picture admitting *higher-order transformations* when comparing the interventions of the local parties. This means that some local parties' intervention may happen to be everything happening in the vicinity of another local party. From the point of view of this latter party, the former then acts as her environment, which is effectively represented as a higher-order intervention. The Choi-Jamiołkowski (CJ) correspondence is then reviewed to represent these *admissible* higher-order transformations as the same kind of objects as interventions. Using this correspondence, the representation of the environment, called the process matrix, is introduced at the end of the chapter, along with two special cases mentioned above: the quantum comb formalism and the quantum switch.

After this background review, the whole concept of 'a class of admissible higher-order transformations' is abstracted into a formal structure called *state structure* in Chapter 3. Under the CJ correspondence, any class, like maps, maps on maps, etc., is indeed identified with a state structure. Under the further observation that the higher-order maps represent the deterministic interventions of parties, the statistical structure of the theory is adapted at the level of state structures: the concepts of a *resolution* of a state structure, which represents probabilistic interventions, as well as the state structure of *functionals*, which represent the measurements and the action of the global environment, are introduced. Following this translation of process heuristics into state structures, it is then shown that defining a given set of admissible transformations is but a special way of defining the composition of an input state structure with an output state structure. This supplants the characterization of higher-order processes by the one of composite state structures. At that point, signaling is reintroduced in the theory as a guiding principle to define the relevant compositions to consider. By doing so, defining a set of admissible higher-order transformations is shown to define a two-way signaling composition. In addition, this section also makes the point that the state structures are in one-to-one correspondence with specific projectors. Therefore, defining functionals or composite state structures amounts to applying an operation on projectors, further supplanting the characterization of state structures by the one of their projectors. To conclude this chapter, a toy model based on a state structure that does not correspond to a class of higher-order transformation is introduced under the name *biased quantum theory*. It is used as a demonstrating example for the methods developed in the chapter.

In Chapter 4, concrete examples of the utilization of these methods are presented. Several objects that appeared in the literature, like the no-signaling bipartite channel or the bipartite process matrix, will be recovered from these methods. In particular, the type theory of Perinotti [10] and Bisio [11] is reviewed at the end of this chapter and then interpreted in terms of projectors. This small bridging chapter aims to highlight certain peculiar behaviors when utilizing the methods on concrete quantum objects rather than abstract state structures. This is done so as to motivate the study of the projectors using abstract algebra.

This is what is done in Chapter 5. Just as Chapter 3 is about supplanting the characterization of higher-order transformations by the one of state structures and then the one of state structures by the one of projectors, Chapter 5 is about supplanting the study of relations between higher-order transformations by the one of the compositions of state structures and then by the one of the compositions of projectors. The various ways of composing state structures, which encode the signaling relations between the subsystems in the bipartite state structures, are shown to correspond to various ways of composing the projectors. In particular, these various compositions taken together will be shown to form a certain kind of lattice, which can almost be interpreted as a model of logic. The assessment of the signaling structure in a class of higher-order transformation is then reduced to symbolic manipulation of projectors under the simple rules of this Boolean-logic lookalike. With this result, assessing the signaling structure of a class of higher-order transformations is reduced to decomposing the projector associated with its state structure into a normal form presented by the end of the chapter. Using this normal form, it is then proven why quantum combs have a fixed signaling direction by algebraic manipulations on projectors only.

Finally, Chapter 6 discusses the future developments envisioned for the formalism. First, the quantum super-supermap (the third order of nested quantum channels) is characterized as a proof of concept. This class is shown to be equivalent to a specific subclass of tripartite Multi-round Process Matrices. Afterward, the generalization of the concepts of *causal separability* [5, 38–40] as well as of a *causal witness* [35] to higher-order processes are briefly formulated as future research directions. In particular, the projective constraints developed in the previous chapters to characterize higher-order processes are shown to lead to the formulation of causal witnesses as Semi-Definite Programming (SDP) problems [41]. This generalizes what was done for the case of the process matrix [35, 40, 42, 43] as well as general supermaps [44–46] so to open the path for computer-assisted search of higher-order processes with interesting causal structures. In addition, a preliminary discussion about the difficulties encountered when (de)composing projectors for more than two parties is presented.

Introduction to Higher-Order Processes

The Process Formalism | 1.

If there is no God, anything would be permitted.

Dostoevsky (1880), *The Brothers Karamazov*[*]

God is dead.

Nietzsche (1882), *The Gay Science*

This section provides an overview of the process formalism and its applications. This formalism aims to predict the correlation that local parties can achieve through the exchange of systems. It is first presented as a probabilistic model alone, in a manner distinguishing between local and probabilistic *interventions* from global and deterministic *environments*. Some of the hypotheses underlying this model are detailed, but the first part aims primarily to define the notion of *signaling correlations*, around which the rest of the thesis will be built.

The second part of the section specializes the formalism to the case of the parties exchanging quantum systems as a means for information transfer. Interventions on quantum systems are reviewed in a manner to set up the notation used throughout the thesis. Once interventions are defined as quantum operations, the question of representing the environment is addressed. This will result in a few observations and assumptions underlying the notion of an *admissible higher-order quantum transformation*, a notion that will be further refined in Chapter 3.

1.1. Process Formalism

The process formalism is an instance of a probabilistic model used to represent an experiment involving a group of local parties sharing a common global resource called their *environment*. The goal of such a model is to predict the joint distribution of the *outcomes* that these parties will see according to their *interventions*, i.e. what they saw according to what they did. It was developed by Oreshkov and collaborators during the 2010s [5, 38, 47, 48], and it can be understood as a special instance of a generalized probabilistic theory [49] or operational probabilistic theory [50].

This framework assumes that the parties are in 'closed laboratories' that can only be related to each other through their interactions with their shared environment. After each party has interacted with the environment, a distribution of outcomes given the interventions and the environment is obtained. Phrasing this in a notation following References

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.
[38]: Oreshkov et al. (2016), *Causal and causally separable processes*.
[47]: Oreshkov et al. (2016), *Operational quantum theory without predefined time*.
[48]: Oreshkov et al. (2015), *Operational formulation of time reversal in quantum theory*.
[49]: Barrett (2007), *Information processing in generalized probabilistic theories*.
[50]: Chiribella et al. (2010), *Probabilistic theories with purification*.

[*] The exact citation is "*(...) without God and Immortal life? All things are lawful then (...)*" (appearing in Part IV, Book XI, Chapter 4 of the 1912 translation by C. Garnett). The quoted version, which is more common, is due to Jean-Paul Sartre's 1946 essay *L'existentialisme est un humanisme* (p.39 in Ed. Folio essais, Paris, France: Gallimard, 1996).

[51]: Shrapnel et al. (2018), *Causation does not explain contextuality*.

[38] and [51], three operational primitives underly the description of an experiment: first, the local laboratories are abstracted into local regions which are referred to using alphabetical labels $A, B, C, \ldots$. These can be thought of as locally flat patches of spacetime for instance. Each local region is under the control of the local parties Alice, Bob, Charlie, ... each of which performing an *intervention* noted $\widetilde{\mathcal{M}}^A, \widetilde{\mathcal{M}}^B, \widetilde{\mathcal{M}}^C, \ldots$. Each party's intervention encapsulates everything that they have chosen to do in their local lab during their interaction with the environment. At the end of each of the parties A(lice), B(ob), C(harlie), ...'s interactions with the environment, they each obtain an outcome, respectively noted $a, b, c, \ldots$. These outcomes represent the locally observed consequences of their interventions. Each of these is modeled as the realization of a random variable sampled from an outcome set whose specification depends on the choice of intervention. Second, everything else happening outside of one of the local regions is represented as the *environment*, noted $\widetilde{\mathcal{W}}^{ABC\cdots}$. Third, there exists a joint probability distribution of these variables given all possible choices of interventions and for all environments,

$$\begin{aligned} &\forall a, b, c, \ldots, \forall \widetilde{\mathcal{M}}^A, \widetilde{\mathcal{M}}^B, \widetilde{\mathcal{M}}^C, \ldots, \forall \widetilde{\mathcal{W}}^{ABC\cdots}, \\ &\qquad \exists\, p(a, b, c, \ldots | \widetilde{\mathcal{M}}^A, \widetilde{\mathcal{M}}^B, \widetilde{\mathcal{M}}^C, \ldots, \widetilde{\mathcal{W}}^{ABC\cdots}) \in [0, 1] \,. \end{aligned} \tag{1.1}$$

The exact nature of these interventions $\widetilde{\mathcal{M}}$ is a parameter of the formalism; their description as well as the set of all interventions accessible to a given party may vary according to the physical theories used to describe them. Once fixed, it in turn constrains the description of the possible environments $\widetilde{\mathcal{W}}$ by *logical consistency*: for all allowed interventions of every party, the obtained joint distribution must be well-defined under certain assumptions that will be defined in the following.

This requirement is a specificity that differentiates the process framework from the usual circuit model[1] as well as from the more common quantum network model [54, 55]. In these models, it is indeed presupposed that the causal relations between the parties are fixed a priori by the environment. In other words, one specificity of the process formalism is that it does not assume the local parties to be embedded in a pre-determined causal order, only that the interventions can be associated with the environment in a way that results in valid probability distributions.

1: Without assuming the *quantum* circuit model as in e.g. [52], see in particular the works of Hardy [15, 16, 53] that inspired the process formalism.

[52]: Nielsen et al. (2009), *Quantum Computation and Quantum Information*.

[15]: Hardy (2005), *Probability Theories with Dynamic Causal Structure: A New Framework for Quantum Gravity*.

[16]: Hardy (2007), *Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure*.

[53]: Hardy (2001), *Quantum Theory From Five Reasonable Axioms*.

[54]: Fritz (2012), *Beyond Bell's theorem: correlation scenarios*.

[55]: Fritz (2016), *Beyond Bell's Theorem II: Scenarios with Arbitrary Causal Structure*.

Like the circuit model and other generalized probabilistic theories, the purpose of this abstract formalism is to study the correlations achievable by a set of parties according to the assumed local description. While of great foundational interest, it is also the appropriate framework for studying communication protocols independently of the theoretical description of the systems. Accordingly, the obtained correlations will often be interpreted as communication protocols in which the parties send and receive *systems* as a means of communication.

Applied to quantum theory, the formalism is about a set of parties acting on quantum systems exchanged with their common background environment. A typical instance of tasks that can be modeled as such are the Bell-kind experiments [56]. In these, two spacelike separated (hence local) parties share a bipartite quantum system. They are interested in the joint probability distribution of the outcomes obtained when each one has measured their share of the system. Treated as a process,

[56]: Brunner et al. (2014), *Bell nonlocality*.

the *environment* provides each party with their share of the bipartite quantum system, and the *interventions* are the measurements of the local parties. The representation compatible with local quantum theory then consists of representing the environment by a bipartite quantum state and the measurements as a pair of POVM in a tensor product [57]. The conjunction of these two then yields the joint probability distribution through the *Born rule*.

[57]: Barnum et al. (2010), *Local Quantum Measurement and No-Signaling Imply Quantum Correlations*.

Without assuming spacelike separation as the Bell scenario does, the process formalism generalizes the notion of a quantum state into a 'spacetime state', called the *process functional* in this thesis. This functional encodes the environment, meaning that it represents everything outside the parties' control that is relevant for representing their interactions while not contradicting logical consistency, regardless of what it is. For instance, a multipartite state shared by the parties is a process functional, but a set of quantum channels connecting some of the parties can also be one.

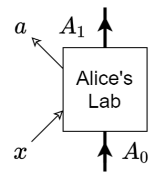


Figure 1.1.: Graphical representation of the intervention by party Alice. (Diagrams are read from bottom to top.) The box represents Alice's local lab in which her overall intervention $\widetilde{\mathcal{M}}^A$ takes place, the incoming thick wire (bottom) represents the system A_0 entering her lab and the outgoing one the system A_1 she outputs, the thin wires represent her choice of setting as well as the outcome she recorded during the intervention so that they specify which actual map $\mathcal{M}_{a|x}$ happened.

Formally, the *local intervention* of a party Alice as is depicted in Figure 1.1 is the mathematical representation of everything she can do when acting on a received *input system*[2] A_0 and preparing an *output system* A_1 to be sent to the environment at the end of her intervention. An extra assumption, therefore, is that the action realized by Alice during her intervention is represented as a collection of mappings from the input system to the output system. Each of the maps in the collection is indexed by its corresponding outcome. For example, if Alice chooses to do nothing to the input system and passes it on, her intervention is a single-element collection $\widetilde{\mathcal{M}}^A = \{\mathcal{M}_{a=0}\}$ constituted of a map that identically sends A_0 to A_1 indexed by the outcome 0. In such a case, Alice is said to make a *deterministic* intervention. If, however, she measures the system according to a procedure resulting in two possible outcomes and passes on the resulting system, her different intervention $\widetilde{\mathcal{N}}^A \neq \widetilde{\mathcal{M}}^A$ is now represented as a two-elements collection $\widetilde{\mathcal{N}}^A = \{\mathcal{N}_{a=0}, \mathcal{N}_{a=1}\}$, each corresponding to one possible outcome. In that case, Alice is said to make a *probabilistic* intervention, as each element of the collection has a certain probability of happening.

2: Systems are noted with the same letter as the party they are associated to and a numeral index. By convention and whenever it is possible, the input systems, i.e. those received, will be noted with an even number, whereas the output systems, i.e. those sent, will be noted with an odd number.

3: For simplification purposes, all random variables in this thesis are assumed discrete, but the random variables could be continuous in the general case.

Thus, the outcomes model the random experimental behavior that the parties have possibly no control over. An outcome a is the realization of a random variable $\hat{a}$ which takes values in an n-valued[3] set $\Omega_{a|\widetilde{\mathcal{M}}^A} = \{0, 1, \ldots, n-1\}$ (see Appendix A.1.4 for a brief recap of the theory of probabilities needed in this thesis). The notation $\cdot|\widetilde{\mathcal{M}}^A$ in the index is there as a reminder that the values the outcome can take are defined based on which intervention the party performed. The party's intervention then consists of a collection of mappings $\{\mathcal{M}_a^A\}_{a\in\Omega_{a|\widetilde{\mathcal{M}}^A}}$ from the representation of the input system to the representation of the output system and concisely noted $\{\mathcal{M}_a^A\}$. This collection represents what Alice has control over: she can freely choose which specific collection she wants to use, so this collection represents what *might happen* during the intervention. Contrastingly, it is only during the intervention –once $\hat{a}$ has been realized– that it can be told which actual element $\mathcal{M}_a^A$ has been used as the actual mapping between the input system A_0 and the output system A_1, i.e. which map represents what *actually happened*.

To faithfully represent general tasks like communication protocols, it is sometimes necessary that the choice of intervention depends on a parameter that is only fixed *during* the intervention. For example, Alice may want to change the system she outputs depending on a specific message she wants to pass to the other parties, and this message is not fixed *a priori*. In such case, Alice is said to *act randomly*, and her behavior becomes conditioned by a second random variable $\hat{x}$, called her *setting*. This variable is typically pictured as "given by an external referee" to represent the potential random behavior of Alice: whether she flips a coin to decide which intervention to perform or she makes up her mind on which message she wishes to send among a set of possible options she fixed in the past, these cases are represented as if the decision was randomly sampled from a set Ω_x representing possible values like "heads or tails" in the case of a coin flipping. When Alice's choice of course of action is independent of the value of the setting (either because she ignores the setting or because it was taking value in a single outcome set), she is said to *act deterministically*. The intervention of a party acting randomly is then *randomized* between several choices $\widetilde{\mathcal{M}}^A, \widetilde{\mathcal{N}}^A, \dots$ depending on the value of the setting $x = 0, 1, \dots$. In that case, the different choices are noted with the same letter, but indexed by the value of x, i.e. $\widetilde{\mathcal{M}}^A = \widetilde{\mathcal{M}}^A_{x=0}; \widetilde{\mathcal{N}}^A = \widetilde{\mathcal{M}}^A_{x=1}; \dots$. Accordingly, the reference to a specific choice of intervention can be made by a specific value of the setting: the collection of mappings forming it are now also indexed by the setting, i.e. $\widetilde{\mathcal{M}}^A_{x=0} = \{\mathcal{M}_{a|x=0}\}_{a\in\Omega_{a|x=0}}$, where the shorthand notation $\Omega_{a|x=0}$ replaces $\Omega_{a|\widetilde{\mathcal{M}}^A}$. The overall intervention then becomes a collection of collections of mappings, i.e. $\widetilde{\mathcal{M}}^A = \left\{ \left\{ \mathcal{M}^A_{a|x} \right\}_{a\in\Omega_{a|x}} \right\}_{x\in\Omega_x} = \left\{ \left\{ \mathcal{M}^A_{a=0|x=0}, \mathcal{M}^A_{a=1|x=0}, \dots \right\}, \left\{ \mathcal{M}^A_{a=0|x=1}, \mathcal{M}^A_{a=1|x=1}, \dots \right\}, \dots \right\}$, and shortly noted as $\{\mathcal{M}^A_{a|x}\}$ where this set should be implicitly understood as running over all values of a and x. Representing the interventions as maps indexed by random variables and introducing the settings to the picture allows to write the probability distribution (1.1) as a distribution of the outcomes conditional on the settings,

$$p(a, b, c, \dots | \widetilde{\mathcal{M}}^A, \widetilde{\mathcal{M}}^B, \widetilde{\mathcal{M}}^C, \dots, \widetilde{\mathcal{W}}^{ABC\dots}) \mapsto$$
$$p(a, b, c, \dots | x, y, z, \dots; \{\mathcal{M}^A_{a|x}\}, \{\mathcal{M}^B_{b|y}\}, \{\mathcal{M}^C_{c|z}\}, \dots, \widetilde{\mathcal{W}}^{ABC\dots}), \quad (1.2)$$

thus allowing the interaction of a party with the environment to be modeled as a pair of random variables.

Figure 1.2.: Graphical representation of a bipartite process. This process concerns two local parties, Alice and Bob, using settings x and y and obtaining outcomes a and b according to the joint distribution $p(a, b|x, y)$ as defined in Equation (1.3). In such a diagram, Alice's box is the application of the map $\mathcal{M}^A_{a|x}$, Bob's of $\mathcal{M}^B_{b|y}$, and the I-beam-shaped box surrounding them is the environment $\widetilde{\mathcal{W}}^{AB}$. When no thick wire is dangling in a diagram, it can be interpreted as a conditional probability distribution of the outgoing thin wires, here a and b, given the incoming ones, x and y.

With respect to the above picture of interventions as conditioned by settings and outcomes, the global environment shared by a collection of parties can be seen as a way to assign a probability of outcomes given interventions and, by extension, given settings. The abstraction of it, $\widetilde{\mathcal{W}}^{ABC\dots}$, is thus an assignment from all possible maps in all possible interventions to a probability.

For example, consider the situation depicted in Figure 1.2. This is a bipartite case where party Alice chooses to do an intervention $\{\mathcal{M}^A_{a|x}\}$ conditioned by setting x and yielding outcome a, whereas Bob[4] chooses to do $\{\mathcal{M}^B_{b|y}\}$ conditioned on y and yielding b. For a given environment represented by process functional $\widetilde{\mathcal{W}}^{AB}$, the resulting joint distribution of outcomes is $p(a, b|x, y; \{\mathcal{M}^A_{a|x}\}, \{\mathcal{M}^B_{b|y}\}, \mathcal{W}^{AB})$; this function output different probabilities depending on the values taken by the settings and

4: Unless said otherwise, the labeling will follow the alphabetical order when passing from one party to the next, in the example, Bob's setting is labeled by y because Alice's is labeled by x. Also note that in the superscripts denoting the parties will sometimes be dropped, in which case the letter representing the mapping can also be changed: if Alice's intervention were noted $\{\mathcal{M}_{a|x}\}$ then Bob's would be noted as $\{\mathcal{N}_{b|y}\}$.

outcomes. As such, it can be concisely thought of as a distribution of outcomes given settings, i.e.,

$$p(a,b|x,y;\{\mathcal{M}^A_{a|x}\},\{\mathcal{M}^B_{b|y}\},\widetilde{\mathcal{W}}^{AB}) =: p(a,b|x,y)\;, \tag{1.3}$$

where the shorthand notation $p(a,b|x,y)$ has been introduced in the above. The reader should however keep in mind that this is only a shortening used to highlight the theory-independent character of the process formalism: it is the choice of maps that truly amounts to the intervention of the parties; the choice of settings is but a way to label these choices. Therefore, the above notation $p(a,b|x,y)$ will be used only where there is no risk of confusion about the interventions and the environment.

Using the process formalism, an experiment is consequently represented as an environment, encompassing everything given without being controlled, and interventions, encompassing locally controllable interactions. The set of all distributions that can be achieved through these two elements constitutes the *process*.

Definition 1.1.1 (Process) *A **process** is the collection*

$$\begin{aligned}
&\forall\{\mathcal{M}^A_{a|x}\},\forall\{\mathcal{M}^B_{b|y}\},\ldots,\\
&\forall a\in\Omega_{a|x},\forall x\in\Omega_x,\forall b\in\Omega_{b|y},\forall y\in\Omega_y,\ldots,\\
&\Big\{p(a,b,\ldots|x,y,\ldots;\{\mathcal{M}^A_{a|x}\},\{\mathcal{M}^B_{b|y}\},\ldots,\widetilde{\mathcal{W}}^{AB\ldots})\Big\}
\end{aligned} \tag{1.4}$$

of all conditional probability distributions for all outcomes $a,b,\ldots$ given all settings $x,y,\ldots$ that a finite set of local parties $A,B,\ldots$ can obtain for all choices of interventions $\{\mathcal{M}^A_{a|x}\}$, $\{\mathcal{M}^B_{b|y}\},\ldots$ and when acting on a given shared environment $\widetilde{\mathcal{W}}^{AB\cdots}$.

This definition is inspired by References [38, 51]. It relies on the non-trivial hypothesis that a joint probability distribution exists for all environments and interventions. Following the literature on non-locality [56], it is standard to add some extra assumptions to this definition. First, the model should be a description at the level of equivalence classes: any two interventions with no experimentally distinguishable consequences are represented by the same collection of mathematical objects. That is, for any two interventions of Alice $\widetilde{\mathcal{M}}^A=\{\mathcal{M}_{a|x}\}$ and $\widetilde{\mathcal{N}}^A=\{\mathcal{N}_{\tilde{a}|\tilde{x}}\}$, if there exists a way to map $\widetilde{\mathcal{M}}^A$ to $\widetilde{\mathcal{N}}^A$ so that the probability distributions are unchanged under the replacements $a\to\tilde{a}$ and $x\to\tilde{x}$ for all interventions of the other parties as well as for all environments, then the two interventions are equivalent and should be represented by the same collection of maps (up to a permutation of the outcomes and settings)[5]. This assumption is called *operational equivalence*: any two interventions that always result in similar distributions of outcomes given settings can never be operationally distinguished and, therefore, are equivalent for all intents and purposes. The other assumption to be added is *freedom of choice*: the value of a setting and the choice of intervention made by a party cannot be influenced by the setting nor the choice of intervention of any other party, and neither can they be influenced by the process itself. That is to say, if Alice obtains a specific value of x and chooses to do a specific intervention $\widetilde{\mathcal{M}}^A$, there is *a priori* no hidden 'superdeterminisit

[38]: Oreshkov et al. (2016), *Causal and causally separable processes*.
[51]: Shrapnel et al. (2018), *Causation does not explain contextuality*.
[56]: Brunner et al. (2014), *Bell nonlocality*.

5: This collection should be the one whose description uses the minimal number of settings and outcomes. For example, if the color of Alice's measurement device has no influence on the joint probability distribution but its orientation does, then the settings can always be mapped from x=(orientation, color) to $\tilde{x}$ = (orientation) (which will then be substituted by numbers), so that several descriptions of the intervention like $\{\mathcal{M}_{a|x=(\text{orientation, color = blue})}\}$ and $\{\mathcal{M}'_{a'|x'=(\text{orientation, color = red})}\}$ are mapped to some minimal description $\{\mathcal{N}_{\tilde{a}=a=a'|\tilde{x}=(\text{orientation})}\}$.

mechanism' outside of the process that forces Bob to obtain a specific value of y, or that constrains him into choosing to do a specific intervention $\widetilde{\mathcal{M}}^B$. Remark in passing that this model is device-dependent: the parties 'trust' their interventions, like $\widetilde{\mathcal{M}}^A$, to be exactly represented as a specific ensemble of maps, like $\{\mathcal{M}^A_{a|x}\}$. In other words, the parties know which mathematical objects correspond to which given course of actions performed on their experimental apparatuses. These assumptions are just mentioned for the sake of completeness; they will be mentioned sometimes in margin notes but are not necessary for the discussion in the following. An interested reader should consult the discussion in Reference [5].

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

Yet, Definition 1.1.1 is still not quite close to the original definition of Reference [5]. Without assuming quantum theory still, the definition amounts to the following.

Definition 1.1.2 (Process Functional) *A process involving parties $A, B, \ldots$ and featuring a given environment $\widetilde{\mathcal{W}}^{AB\cdots}$ as in Definition 1.1.1 is uniquely characterized by its* ***process functional*** *$\mathcal{W}^{AB\cdots}$. It is the map from the elements of the interventions of the parties to a probability, i.e. the map defined by the relation*

$$\mathcal{W}^{AB\cdots}(\mathcal{M}^A_{a|x}, \mathcal{M}^B_{b|y}, \ldots) = \\ p(a, b, \ldots | x, y, \ldots; \{\mathcal{M}^A_{a|x}\}, \{\mathcal{M}^B_{b|y}\}, \ldots, \widetilde{\mathcal{W}}^{AB\cdots}) \,. \quad (1.5)$$

Hence, the environment variable $\widetilde{\mathcal{W}}^{AB\cdots}$ can be replaced by a functional $\mathcal{W}^{AB\cdots}$ which is uniquely characterized by its action on all possible maps representing all possible interventions of the parties for all possible outcomes. Several extra hypothesis must have been added to the model to arrive at this formulation. One in particular is worth mentioning: that it obeys an analog assumption to Spekken's measurement non-contextuality [58, 59], on the interventions [51, 60]. This means that the probability distribution only depends on the actual maps corresponding to the specific realizations of the outcomes and settings. Equation (1.5) is independent of which interventions were chosen, i.e. it does not depend on the collections of maps representing the parties' action, but only on the specific element within these collections that is associated with the *actual* value of a and x. A comprehensive discussion on these assumptions can be found in References [47, 51, 60].

[58]: Spekkens (2005), *Contextuality for preparations, transformations, and unsharp measurements*.
[59]: Spekkens (2014), *The Status of Determinism in Proofs of the Impossibility of a Noncontextual Model of Quantum Theory*.
[51]: Shrapnel et al. (2018), *Causation does not explain contextuality*.
[60]: Shrapnel et al. (2018), *Updating the Born rule*.
[47]: Oreshkov et al. (2016), *Operational quantum theory without predefined time*.
[51]: Shrapnel et al. (2018), *Causation does not explain contextuality*.
[60]: Shrapnel et al. (2018), *Updating the Born rule*.

With such assumptions, there is no danger when confusing the collections of probabilities, the representation of the environment, and the functional mapping the interventions to a probability: they are essentially the same thing. Thus, any of these three concepts will be implied interchangeably when using the word *process*.

1.2. Signaling and Causal Correlations

General questions about the information capacities and signaling relations between parties can be asked using the process formalism. The kind of questions this thesis will primarily explore concern the signaling structure between the different parties in a process: given several parties doing local interventions for a fixed environment, the question is to infer who may *signal* to whom. That is, is there a choice of interventions that Alice and Bob can perform such that Alice can reliably send information to Bob and/or the opposite way around? The answer to this question for any chain of parties[6] in a process is what is meant by the *signaling structure* of a process. Remark that actual signaling is not guaranteed: a given environment may have a signaling structure that *allows* Alice to signal to Bob, but if they do not perform a suitable intervention, for example, if they do not interact with their environment at all, then they will never achieve signaling.

6: In the sense of 'can Alice signal to Bob and Bob to Charlie and Charlie to David etc. ?'.

Defined as such, signaling is a sufficient condition to infer causal influence: if Alice can signal to Bob, it means that her intervention can have a causal influence on the system that Bob receives. Because of that, the signaling structure is a means to infer the causal order of a process: if, for a choice of intervention, Alice can signal to Bob, then Alice is necessarily in the causal past of Bob; if, for all choices of interventions of Alice and Bob, only Alice can signal to Bob, they are in a *fixed causal order*. The in-between situation is also of interest: a given environment may allow Alice to be in the causal past of Bob for certain interventions but it can be the other way around for other interventions. In such cases, the signaling structure is not fixed and neither is the causal order.

The interest in using the process formalism to assess signaling is that it can discuss theory-independent bounds on the ability to signal. Considering settings and outcomes alone, the ability to signal can be witnessed in the correlations between the two variables, independently of any commitment to a specific theory used to describe the interventions and the environment. If the setting x of Alice is correlated with the outcome b of Bob, then one can infer that there has been signaling.

Conversely, for a given local theory describing the interventions, upper bounds on the achievable correlations can be derived so that two local descriptions can be compared. The epitomical example of such a bound has been formulated in terms of a bipartite process in which local interventions are described using the rules of quantum theory [5]. This example showed from a theory-independent bound that, while all processes compatible with a classical description of the local interventions are *causal*, there exist *non-causal* processes consistent with a quantum description of the interventions. But what does *causal* mean in this context?

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

The situation is as in Figure 1.2: Alice and Bob are two local parties sharing an environment; Alice's outcome is a and her setting is x; Bob's outcome is b and his setting is y; and the probability associated with the diagram is $p(a, b|x, y; \{\mathcal{M}^A_{a|x}\}, \{\mathcal{M}^B_{b|y}\}, \widetilde{\mathcal{W}}^{AB})$, which is shortened into $p(a, b|x, y)$ as discussed around Equation (1.3).

Some preliminary considerations: first, remember that the parties are assumed to have freedom of choice. That is, how Alice associates her settings and outcomes (x, a) to a choice of intervention $\{\mathcal{M}^A_{a|x}\}$ is independent of how Bob is associating (y, b) to $\{\mathcal{M}^B_{b|y}\}$, as well as from the environment $\widetilde{\mathcal{W}}^{AB}$.

Second, in the literature the word causal is often used (e.g., [61, 62]) to convey the idea of no-signaling from one party to another [63, 64]. A

[61]: Beckman et al. (2001), *Causal and localizable quantum operations*.
[62]: Eggeling et al. (2002), *Semicausal operations are semilocalizable*.
[63]: Popescu et al. (1994), *Quantum nonlocality as an axiom*.
[64]: Piani et al. (2006), *Properties of quantum nonsignaling boxes*.

consequence of that is that through this thesis, the terms 'fixed causal order' will be used interchangeably with 'with a single and fixed signaling direction' in accordance with the utilization made in References [4, 5]. On the contrary, they use 'indefinite causal order' (ICO) as a more subtle notion than 'with several fixed signaling directions'; ICO requires several signaling directions but in a manner that is not fixed, neither *a priori* nor dynamically. This means that what happens in a process with ICO cannot be understood as a convex sum (a mixture) of several scenarios with fixed signaling directions. Nor can it be understood solely as a process in which the action of a party has an influence on the signaling structure of the parties in her causal future. The theory-dependent notion of 'process with ICO' is defined in the process formalism with quantum interventions under the term *causal non-separability* [5] (this notion was subsequently refined in a series of follow-up works [1, 38, 39, 45]).

[4]: Chiribella et al. (2013), *Quantum computations without definite causal structure.*
[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order.*

[1]: Hoffreumon et al. (2021), *The Multi-round Process Matrix.*
[38]: Oreshkov et al. (2016), *Causal and causally separable processes.*
[39]: Wechs et al. (2019), *On the definition and characterisation of multipartite causal (non)separability.*
[45]: Bavaresco et al. (2019), *Semi-device-independent certification of indefinite causal order.*

The word *causal* alone is used in the ICO literature as an in-between: it is weakening of fixed causal order to mean something in the lines of 'with a single direction at once'. However, when there is more than one local party, a process with a single signaling direction does not guarantee that this direction is fixed. If Alice signals to Bob or Bob signals to Alice, depending on their settings, the process presents a single signaling direction, although it is not fixed.

As a consequence, stating that a process is causal in this thesis refers to the correlations; it does not mean that the physical process it models has to be causal, let alone deterministic. This postulate is not an assumption about the inner workings of the intervention: the probability distribution may have been obtained by a non-causal theory, like a de Broglie-Bohm pilot wave, or it can even have been obtained by a superdeterministic theory. Here, the word *causality* refers to the distribution of outcomes given a setting; it is a statement about the impossibility of gaining knowledge of a classical variable from another. It does not imply that the underlying physical process producing outcomes out of settings mediated no interaction even though, on average, this interaction cannot be detected as a correlation on the joint distribution (see in particular Reference [65]). See Reference [66] for disambiguation between causality and signaling and [67] for one between causality and determinism.

[65]: D'Ariano et al. (2011), *No Signaling, Entanglement Breaking, and Localizability in Bipartite Channels.*
[66]: Perinotti (2021), *Causal influence in operational probabilistic theories.*
[67]: D'Ariano et al. (2014), *Determinism without causality.*

These notions of no-signaling and causal correlations are now formally defined.

No-signaling is a constraint inspired by relativity: the interventions of two spacelike separated parties in a process should not allow the transmission of a message from one party to the other. Consequently, the process should not allow them to signal to each other as this communication would be faster than light. The mathematical translation of this principle states that the setting of one party can never influence the marginal distribution of the other's outcomes.

Definition 1.2.1 *Let Alice and Bob be the two parties in a process. Let Alice's intervention be conditioned by setting x and resulting in outcome a. Let Bob's be by setting y and outcome b. Write the joint probability distribution of their outcomes as $p(a, b|x, y)$.*
The distribution is ***no-signaling*** *if the marginals of each party are independent*

of the setting of the other party. I.e., if the following holds:

$$\forall a, x, y, y' : \quad \sum_b p(a, b|x, y) = \sum_b p(a, b|x, y') \; ; \tag{1.6a}$$

$$\forall b, x, x', y : \quad \sum_a p(a, b|x, y) = \sum_a p(a, b|x', y) \; . \tag{1.6b}$$

If only Equation (1.6a) *holds, then the distribution is said* ***no-signaling from Bob to Alice****. If such distribution is not no-signaling, then it allows for* ***(one-way) signaling from Alice to Bob*** *and the following holds:*

$$\forall a, x, y, y' : \quad \sum_b p(a, b|x, y) = \sum_b p(a, b|x, y') \; ; \tag{1.7a}$$

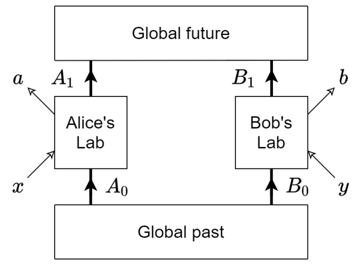
$$\exists b, x \neq x', y : \quad \sum_a p(a, b|x, y) \neq \sum_a p(a, b|x', y) \; . \tag{1.7b}$$

Similarly, if only Equation (1.6b) *holds, the distribution is* ***no-signaling from Alice to Bob****. It allows for* ***(one-way) signaling from Bob to Alice*** *if the following holds:*

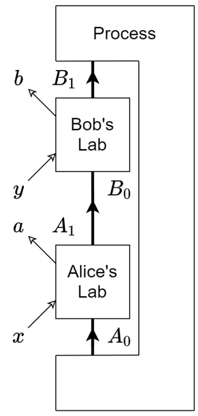
$$\exists a, x, y \neq y' : \quad \sum_b p(a, b|x, y) \neq \sum_b p(a, b|x, y') \; ; \tag{1.8a}$$

$$\forall b, x, x', y : \quad \sum_a p(a, b|x, y) = \sum_a p(a, b|x', y) \; . \tag{1.8b}$$

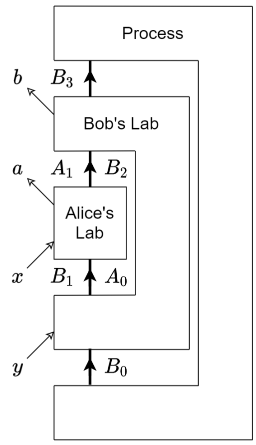
Finally, if neither of Equations 1.6 hold, the distribution allows for ***two-way signaling****.*



(a) No-signaling



(b) Signaling from Alice to Bob



(c) Signaling in both directions

Figure 1.3.: Graphical representation of bipartite processes allowing for different signaling scenarios. Graphically, no-signaling is pictured as an outcome produced below a setting. For instance, in Figure 1.3b, Alice's outcome a is below Bob's setting y, so a cannot depend on y and thus she cannot signal to him

A no-signaling distribution is the standard statement that no local measurement scheme can be used to gain knowledge of the other party's actions deterministically. The first condition (1.6a), states that Alice's outcome distribution cannot be used to determine which setting y Bob has used and thus that the measurement result of Alice cannot be used to guess Bob's own choice of measurement. Hence, it entails that Bob cannot signal to Alice by choosing his intervention according to the y he wishes to transmit.

Therefore, no-signaling is a condition forbidding the possibility of transmitting information from one party to another. In contrast, locality and statistical independence are two strengthenings of conditions (1.6) that rely on other heuristics added on top of the inability to correlate one party's settings with another's outcomes[7]. Similarly, when neither of these conditions are satisfied, another heuristic can be introduced so that some constraint can still be imposed at the level of the joint distribution. This extra *a priori* on the distribution of outcomes is what *causal* correlations are about. It is a constraint that relies on the expectation that a signaling distribution allows for signaling *in one direction at a time*. In other words, a joint distribution can be expected to be either signaling from Alice to Bob or from Bob to Alice. Any in-between situation would then be a mixture of the two directions conditioned by a classical hidden variable.

7: These notions are briefly reviewed in Appendix A.2 for completeness.

For the technical definition, first notice that signaling distributions can be characterized as a factorization of the distribution.

Lemma 1.2.1 *A bipartite distribution which is no-signaling from Bob to Alice as in Equations* (1.7) *admits the following decomposition:*

$$p(a,b|x,y) = p(a|x)p(b|x,y,a)\ , \tag{1.9}$$

for all values a, b, x, y.

Proof. Almost direct from the definitions: the joint distribution is first rewritten in terms of the marginal on a, $p(a|x,y) := \sum_b p(a,b|x,y)$, and the conditional on b given a, $p(b|x,y,a) = \frac{p(a,b|x,y)}{\sum_b p(a,b|x,y)}$, so that

$$p(a,b|x,y) = p(a|x,y)p(b|x,y,a)\ . \tag{1.10}$$

Then Equation (1.6b) is used to simplify the marginal: if $\forall y,y'\colon p(a|x,y) = p(a|x,y')$ then

$$p(a|x,y) = p(a|x)\ . \tag{1.11}$$

□

Pretty much like a Local Hidden Variable Model is a decomposition of a no-signaling distribution into independent distributions conditioned by a variable λ, causal correlations decompose a general distribution into one-way signaling distributions conditioned by a variable λ. As there are only two different signaling directions between two parties, lambda is a dichotomic variable such that

$$p(a,b|x,y) = p(\lambda=0)p(a,b|x,y,\lambda=0) + p(\lambda=1)p(a,b|x,y,\lambda=1)\ , \tag{1.12}$$

where $p(a,b|x,y,\lambda=0)$ is one-way signaling from Alice to Bob, and $p(a,b|x,y,\lambda=1)$ is from Bob to Alice. Naturally, $p(\lambda)$ can be replaced by some $q \in [0,1]$ so that $p(\lambda=0) = q$ and $p(\lambda=1) = 1-q$, and Lemma 1.2.1 can be used as well, doing so leads to the original formulation by Oreshkov, Costa, and Brukner [5].

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

Definition 1.2.2 (Bipartite Causal Correlations) *Let Alice and Bob be two parties in a process. Let Alice's intervention be conditioned by settings x and resulting in outcome a. Let Bob's by settings y and outcome b. Write the joint probability distribution of their outcomes as $p(a,b|x,y)$.*
Then the distribution is ***causal*** *if and only if it admits the following decomposition:*

$$\begin{aligned}&\exists\, q \in [0,1]:\\ &p(a,b|x,y) = q\, p(a|x)p(b|x,y,a) + (1-q)\, p(b|y)p(a|x,y,b)\ .\end{aligned} \tag{1.13}$$

Finally, observe that no-signaling and one-way signaling as in Definition 1.2.1 are pairwise constraints that guarantee the absence of signaling in, respectively, two and one directions. At their core, these definitions are about the impossibility of signaling for all choices of interventions. For this reason, the definition can only be generalized by applying it pairwise to any number of parties. If a tripartite distribution is no-signaling between Alice and Bob, meaning that neither Bob nor Alice can signal to the other, and no-signaling between Bob and Charlie, it does not entail that it will be necessarily no-signaling between Alice and Charlie. In other words, no-signaling is not a transitive property. But to define a distribution with fixed causal order (i.e., with a single and fixed signaling direction) for more than two parties then requires a special kind of no-signaling constraints that are transitive. Indeed, in such a distribution, if Alice can signal to Bob and Bob can signal to Charlie, it *may* be possible for Alice to signal to Charlie as all parties would agree on Alice being first, Bob second, and Charlie third so that a unique notion of *direction* in the signaling structure can be established. However, it should never be possible for Charlie to signal to Alice, as this would form a loop and so the notion of a fixed direction in the signaling structure cannot be established. Thus, if a tripartite distribution has a fixed signaling direction and it is known that Charlie cannot signal to Bob and that Bob cannot signal to Alice, it entails that Charlie cannot signal to Alice. This distinction is important as the notion of causal correlations relies on the correlations which have at most a single direction of signaling at once. Hence, not on the pairwise impossibility of signaling from one party to another, but rather on the transitive impossibility of signaling from one party to all the others. This requires the following generalization of Definition 1.2.2 [38, 40].

[38]: Oreshkov et al. (2016), *Causal and causally separable processes*.
[40]: Abbott et al. (2016), *Multipartite causal correlations: Polytopes and inequalities*.

Definition 1.2.3 (Multipartite Causal Correaltions) *Let there be a multipartite process involving n parties labeled $A^{(1)}$, $A^{(2)}$, $A^{(3)}$, $\ldots A^{(n)}$ so that each party $A^{(i)}$'s intervention is conditioned on setting x_i and results in outcome a_i. Let $\vec{a} := (a_1, a_2, \ldots a_n)$ and $\vec{x} := (x_1, x_2, \ldots x_n)$ be the vectors of outcomes and settings, respectively, so that the process is associated with the distributions $p(\vec{a}|\vec{x})$.*
Let σ_j be an element of the permutation group on n symbols S_n indexed by j so that, e.g., $\sigma_1(1) = 1, \sigma_1(2) = 2 \ldots, \sigma_1(n) = n$; $\sigma_2(1) = 2, \sigma_2(2) = 3 \ldots, \sigma_2(n) = 1$ etc.
*Then, the distribution $p(\vec{a}|\vec{x})$ is **causal** if, for all choices of strategies, there exists $q_j \in [0, 1]$ such that the distribution factorizes into a mixture of causal distributions:*

$$p(\vec{a}|\vec{x}) = \sum_{j=1}^{|S_n|=n!} q_j p(a_{\sigma_j(1)}|x_{\sigma_j(1)}) p(\vec{a}_{\backslash a_{\sigma_j(1)}}|\vec{x}_{\backslash x_{\sigma_j(1)}}, a_{\sigma_j(1)}) \,, \tag{1.14}$$

where $\sum_j q_j = 1$; where $\vec{a}_{\backslash a_{\sigma_j(1)}}$ indicates the vector of $n-1$ components obtained by removing $a_{\sigma_j(1)}$ from $\vec{a}$; and where $p(\vec{a}_{\backslash a_{\sigma_j(1)}}|\vec{x}_{\backslash x_{\sigma_j(1)}}, a_{\sigma_j(1)})$ is a causal distribution for $n-1$ parties.

The above definition thus recursively reduces the number of parties in the conditional probability distributions until it obtains a decomposition featuring only bipartite ones, in which case Definition 1.2.2 applies. In accordance with the definition, a process is called causal if it consists of causal distributions only.

1.3. Local Quantum Theory

The process formalism developed in the previous section is now applied to the concrete case of a single party whose intervention obeys the rules of quantum theory. The main purpose of the following subsections is to review the quantum theory needed to attain the mathematical description of a single quantum process. More notation will be set up along the way.

A simplification made in this thesis is that every system is assumed to be finite-dimensional. This simplification is not necessary, though. The process formalism could accommodate continuous random variables as possible observations, like, for example, the position or momentum of a system, so that the state space of systems could be infinite-dimensional (see Reference [68] for instance); but this results in mathematical complications that are beyond the scope of this thesis.

[68]: Giacomini et al. (2016), *Indefinite causal structures for continuous-variable systems*.

Postulate 1: Local Quantum Theory.

The local operations of each party are described by quantum theory [5].

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

With this assumption, the probability distributions of local interventions can now be obtained by applying the rules of quantum theory. The results of an experiment in a closed lab will correspond to some quantum circuit: the strategy will condition how many systems are prepared in which state, how they evolve, and how they are measured; the observation is the outcome set of this procedure, and the distribution $p(a|x)$ of outcomes a given settings x is obtained through the *Born rule* (see Reference [69, 70] for instance).

[69]: Peres (1993), *Quantum Theory: Concepts and Methods*.
[70]: Ziman (2008), *Process positive-operator-valued measure: A mathematical framework for the description of process tomography experiments*.

1.3.1. Notation

Postulate 1 is the statement that the systems are general quantum systems, which are represented by density operators [71] defined over a finite-dimensional Hilbert space.

[71]: von Neumann (1932), *Mathematical Foundations of Quantum Mechanics*.

In the following, Hilbert spaces are denoted by the calligraphic letter $\mathcal{H}$ and labeled with superscript capital letters like $\mathcal{H}^A$, $\mathcal{H}^B$, $\mathcal{H}^C, \ldots$. These superscripts indicate a specific party: Alice, Bob, Charlie,...etc. and the letters indicate the system prepared by the party. When needed, these are complemented with a number, for example, when a party is preparing a composite system, and each part must be individually referred. For example, A_0, A_1, A_2 indicate three subsystems that belong to Alice. Note that subsystems may sometimes be treated as independent, *local* parties when a fine-grained description is needed. Likewise, several subsystems potentially associated with different parties can be treated as a composite system associated with a single, *global* party when a coarse-grained description is needed.

Dirac bra-ket notation is used for vectors on a Hilbert space; the inner product of a Hilbert space is indicated with brackets followed by the label of the space in subscript like $\langle \cdot\,, \cdot \rangle_A, \langle \cdot\,, \cdot \rangle_B, \ldots$. The dimension of a Hilbert space will be indicated with the letter d with a matching subscript like $d_A, d_B, \ldots$.

Arbitrary linear maps and operators will be denoted using capital Latin letters like N, M, V, W[8]. The set of linear maps from space $\mathcal{H}^A$ to $\mathcal{H}^B$ is noted as $\mathcal{L}\left(\mathcal{H}^A, \mathcal{H}^B\right)$. Space $\mathcal{H}^A$ will often be referred to as the *input space* and $\mathcal{H}^B$ as the *output space*. An operator[9] is a linear map from a space

8: In order not to confuse them with labels, the letters N, M, V, W will be the ones mainly used to indicate linear maps. Labels, on the other hand, will be assigned following the alphabetical order $A, B, C, \ldots$, and the number of labels needed will never be large enough to reach M.

9: For completeness, the main classes of operators mentioned in this thesis, like the positive ones for instance, are briefly reviewed in Appendix A.1.1 alongside the notion of a trace.

to itself, in which case the shorthand notation $\mathcal{L}\left(\mathcal{H}^A\right) := \mathcal{L}\left(\mathcal{H}^A, \mathcal{H}^A\right)$ is used for the set of operators on $\mathcal{H}^A$. The Hilbert-Schmidt inner product in a space of linear operators[10] is indicated with parenthesis instead of brackets like $(\cdot\,,\,\cdot)_A$, $(\cdot\,,\,\cdot)_B$, ...; it is defined through the trace as

$$(V\,,\,N)_A := \mathrm{Tr}_A\left[V^\dagger \cdot N\right]\ , \tag{1.15}$$

where $V, N \in \mathcal{L}\left(\mathcal{H}^A\right)$; the subscript in the trace is used to refer over which space it is taken, and $\dagger$ indicates the adjoint in $\mathcal{H}^A$. When needed, subscripts or superscripts labels will be put on operators to remind in which space they are defined.

10: Briefly reviewed in Appendix A.1.3.

Finally, the tensor factors appearing in an expression defined on several subsystems will be sorted in lexical and numerical orders whenever possible by convention. For example, a tensor product of V_{A_0}, N_{A_1} and U_B will be sorted as $V_{A_0} \otimes N_{A_1} \otimes U_B$ instead of another order like $U_B \otimes V_{A_0} \otimes N_{A_1}$. Thus, the isomorphism $\mathcal{H}^A \otimes \mathcal{H}^B \cong \mathcal{H}^B \otimes \mathcal{H}^A$ will be used whenever necessary. In other words, it is the labels rather than the position that will be relevant in the expressions built using a tensor product.

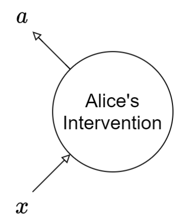


Figure 1.4.: Graphical depiction of a local experiment. A local experiment is an intervention ignoring its input and output system; it can be thought of as the trivial map between the 1-dimensional systems A_0 and A_1, associated with a probability $p(a|x)$. This operation relies only on the local setting and outcome, and it has been obtained by some procedure that can be described entirely within Alice's lab.

1.3.2. Quantum Systems as Density Operators

To present local quantum theory as the intervention of a single party Alice, it is assumed in this section that she ignores her environment: she discards the input system and outputs a random quantum system. The intervention picture of Figure 1.1 is reduced to the *local quantum operation* picture of Figure 1.4. Everything happening in her local lab is abstracted in the probability distribution $p(a|x)$ of her outcome given her setting.

[69]: Peres (1993), *Quantum Theory: Concepts and Methods*.
[72]: Kraus (1983), *States, Effects, and Operations: Fundamental Notions of Quantum Theory*.

What happens within the lab can be reduced to an overall preparation and measurement scenario involving a single system (see References [69, 72] for instance). This situation is depicted in Figure 1.5: First, the preparation of a system according to the settings x, represented by a quantum state. Second, the (destructive) measurement of it yielding the outcome a, represented as a quantum effect. The state of a quantum system is encoded as a trace-1 positive operator called *density operator* (or matrix). Quantum states are noted with the Greek lowercase letters ρ, η, σ instead of capital Latin letters so as to set apart these special operators. Similarly, the effects of quantum measurements are a collection of positive operators forming a Positive Operator-Valued Measure (POVM; see [52]). The Latin uppercase letters E and F are reserved to set them apart.

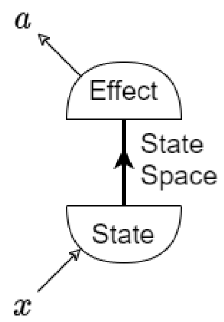


Figure 1.5.: Breakdown of a local experiment as a prepare-and-measure scenario. The quantum systems are represented by thick wires, measurements by bottom-facing half-circles, and state preparation by top-facing half-circles.

In the present case, Alice's intervention consists of preparing the system A_0 in a first time. The state of the system, conditioned by her setting x, is represented as a linear operator $\rho^{A_0}_{|x} : \mathcal{H}^{A_0} \to \mathcal{H}^{A_0}$. This operator is a quantum state, which is Positive Semi-Definite (PSD)[11] and of trace 1 [12].

In a second time, she measures her system through a procedure modeled by a POVM[13] $\{E_a\}_{a\in\Omega_{a|x}}$, the probability distribution of the outcome a given the setting x is then given by the Born rule:

$$p(a|x) = \left(E_a\,,\,\rho_{|x}\right)_{A_0}\ , \tag{1.16}$$

11: Often abridged into 'positive' in this thesis.
12: See Appendix A.1.1 for a refresher on these notions.
13: Her overall measurement procedure also depends on the setting x. However, the effects themselves, each associated with a specific outcome a, are independent of x since quantum theory is *measurement non-contextual*.

where $(\cdot\,,\,\cdot)_{A_0} := \mathrm{Tr}_{A_0}\left[\cdot^\dagger\cdot\right]$ is the Hilbert-Schmidt inner product in $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$. The diversity of prepare-and-measure scenarios that Alice has access to forms her *state space*, i.e. the space of density operators in $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$. The (local) Hilbert space where Alice's state and state space are defined is depicted as a thick black wire in Figure 1.5. (See Appendix A.3 for a short explanation of diagrammatic methods used in this thesis.)

[72]: Kraus (1983), *States, Effects, and Operations: Fundamental Notions of Quantum Theory*.
[73]: Holevo (2011), *Probabilistic and Statistical Aspect of Quantum Theory*.

The state space is convex so as to represent randomization procedures (see, e.g., References [72, 73]): if Alice randomly chooses between preparing a system in state ρ_{A_0} with probability q or η_{A_0} with probability $1-q$, this is represented by another state σ_{A_0} given by

$$\sigma_{A_0} = q\rho_{A_0} + (1-q)\eta_{A_0}\,. \tag{1.17}$$

Using a setting x, this can written as $q := p(x=0)$, $\rho_{A_0} := \rho^{A_0}_{|x=0}$, $(1-q) := p(x=1)$, $\eta_{A_0} := \rho^{A_0}_{x=1}$, so that $\Omega_x = \{0,1\}$. The operator σ_{A_0} is then the *averaged state* ρ^{A_0} so that

$$\rho^{A_0} = p(x=0)\rho^{A_0}_{|x=0} + p(x=1)\rho^{A_0}_{|x=1}\,. \tag{1.18}$$

[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.

This convex set has a distinguished element, its geometrical center, obtained by averaging over all possible randomizations (see, e.g., Section 8.2 in Reference [74]). It is called the *maximally mixed state*, and it is represented by the identity operator, noted $\mathbb{1}_{A_0}$, divided by the dimension of $\mathcal{H}^{A_0}$ in order to obtain a trace of 1. In other words, this state represents non-pre-selected distributions; Alice can make no prediction on what the outcome of any non-trivial measurement of such a state will be. On the other hand, Alice's strategy[14] can lead to a deterministic intervention: a setting x so that she knows beforehand what outcome it will yield. A typical instance of deterministic strategies is when the party prepares and measures in the same basis. Recall that any two states that Alice can deterministically tell apart, i.e., for which there exists a single measurement that will associate each state with a single outcome like $p(a|x=\text{“state 1 was prepared”}) = \delta_{a,x}$, are represented by orthogonal states. That is, states ρ_{A_0}, η_{A_0} with zero overlap in the Hilbert-Schmidt inner product,

14: Remark that the word strategy will be often used to mean 'course of action' or 'choice of intervention'. This is done so as to fit with the terminology introduced in References [12, 75].
[12]: Gutoski et al. (2007), *Toward a General Theory of Quantum Games*.
[75]: Gutoski (2010), *Quantum Strategies and Local Operations*.

$$(\rho_{A_0}\,,\,\eta_{A_0})_{A_0} = \mathrm{Tr}_{A_0}\left[\rho^\dagger_{A_0}\eta_{A_0}\right] = 0\,. \tag{1.19}$$

Such states are called *perfectly discriminable*[15]. The systems a party can prepare are consequently represented on a Hilbert space of a dimension as big as the maximum number of discriminable states in which this system can be prepared.

15: Or *distinguishables* (from each other).

Because of that, the maximal number of perfectly discriminable states for a single measurement procedure is equal to the dimension d_{A_0} of $\mathcal{H}^{A_0}$. Moreover, the states in such sets are projectors[16], so each can be measured in a way that yields a definite measurement outcome. Such states ρ_{A_0} are therefore represented by rank-1 projectors and called *pure states*. They correspond to states of the system that have a measurement procedure that yields a definite outcome and such that no extra information can be gained from a measurement with more possible effects. They consequently represent the system in a state of maximal knowledge, as opposed to the maximally mixed state.

16: Indeed, it can be shown that there are at most d_{A_0} pairwise orthogonal states $\{\rho^{(i)}\}_{i=0}^{d_{A_0}-1}$ in $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$ and that these correspond to a set of commuting rank-1 projectors $\{\rho^{(i)} = |\varphi_i\rangle\langle\varphi_i|\}_{i=0}^{d_{A_0}-1}$ resolving the identity, $\mathbb{1}_{A_0} = \sum_{i=0}^{d_{A_0}-1}|\varphi_i\rangle\langle\varphi_i|$.

This brief review of density operators was conducted so to remind the reader about the structure of the state space because similar structures will play an important role in the following. Mathematically, the unnormalized state space is the convex cone formed by positive operators on a Hilbert space; the state space is then the hyperplane obtained by fixing the trace of each positive operator to be one. This hyperplane is perpendicular to the center of the cone, spanned by $\mathbb{1}$. Whereas the center of the cone corresponds to the state of no knowledge (called the maximally mixed state); the boundary of the cone, its extremal states, correspond to states of maximal knowledge (called pure states).

1.3.3. Evolution as Maps Between Density Operators

The state and effect picture presented in the previous section represents a static situation: the system has a fixed state from its preparation to its measurement. This picture must be complemented by a dynamical ingredient to represent the evolution of a given state according to an external cause. In the process-theoretic picture presented here, this external cause is typically assumed to be the intervention of a party on the system. As her intervention is making the state of the system change, it must be represented as an evolution from the set of density operators to a set of density operators. This evolution is assumed to be as general as possible: the dimension of the output space can be different than the input, and the dynamics can be open in general, meaning that the pure states do not have to be necessarily mapped to other pure states. The most general evolution of a quantum state is provided by the *quantum channel*, which is a Completely Positive (CP) Trace-Preserving (TP) linear map $\mathcal{M}$ from the input space $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$ to the output space $\mathcal{L}\left(\mathcal{H}^{A_1}\right)$. These linearity and CPTP conditions are necessary to ensure that the set of density operators in the input space is correctly mapped to the one in the output space.

More generally, the following properties of linear maps will be used in the rest of the thesis:

Definition 1.3.1 (Features of linear maps) *Let $\mathcal{M} : \mathcal{L}\left(\mathcal{H}^{A}\right) \to \mathcal{L}\left(\mathcal{H}^{B}\right)$ be a linear map between two spaces of linear operators over not necessarily isomorphic Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$. Let $\mathcal{H}^C$ be an arbitrary Hilbert space and let $\mathcal{I}^C : \mathcal{L}\left(\mathcal{H}^{C}\right) \to \mathcal{L}\left(\mathcal{H}^{C}\right)$ be the identity map from $\mathcal{L}\left(\mathcal{H}^{C}\right)$ to itself. Then,*

1. *$\mathcal{M}$ is **Hermitian-Preserving** (HP) if and only if it maps self-adjoint operators to self-adjoint operators,*

$$\forall \rho_A \in \mathcal{L}\left(\mathcal{H}^{A}\right) : \ \rho_A = \rho_A^{\dagger}, \quad \mathcal{M}\left(\rho_A\right) = \mathcal{M}\left(\rho_A\right)^{\dagger} \, . \tag{1.20}$$

2. *$\mathcal{M}$ is **Positive** (P) if and only if it maps positive operators to positive operators,*

$$\forall \rho_A \in \mathcal{L}\left(\mathcal{H}^{A}\right) : \ \rho_A \geq 0, \quad \mathcal{M}\left(\rho_A\right) \geq 0 \, . \tag{1.21}$$

3. *$\mathcal{M}$ is **Completely Positive** (CP) if and only if it maps positive operators to positive operators even when the Hilbert spaces are extended*

through the tensor product by an arbitrary space $\mathcal{H}^C$,

$$\forall \eta_{AC} \in \mathcal{L}\left(\mathcal{H}^A\right) \otimes \mathcal{L}\left(\mathcal{H}^C\right) : \eta_{AC} \geq 0, \\ \left(\mathcal{M}^A \otimes \mathcal{I}^C\right) \{\eta_{AC}\} \geq 0 \,. \tag{1.22}$$

4. *$\mathcal{M}$ is **Unital** if and only if it maps the identity to the identity,*

$$\mathcal{M}\left(\mathbb{1}_A\right) = \mathbb{1}_B \,. \tag{1.23}$$

5. *$\mathcal{M}$ is **Trace-Preserving** (TP) if and only if it preserves the trace,*

$$\forall \rho_A \in \mathcal{L}\left(\mathcal{H}^A\right), \quad Tr_B\left[\mathcal{M}\left(\rho_A\right)\right] = Tr_A\left[\rho_A\right] \;; \tag{1.24}$$

*It is **Trace-non-Increasing** (TnI) if and only if it contracts the trace,*

$$\forall \rho_A \in \mathcal{L}\left(\mathcal{H}^A\right), \quad Tr_B\left[\mathcal{M}\left(\rho_A\right)\right] \leq Tr_A\left[\rho_A\right] \;; \tag{1.25}$$

Therefore, the most general linear map from quantum states to quantum states is a CPTP map as defined above (for a review, see e.g. Chapter 8 in Reference [52]).

[52]: Nielsen et al. (2009), *Quantum Computation and Quantum Information*.

Definition 1.3.2 (Quantum Channel) . *A Completely-Positive (CP) Trace-Preserving (TP) map $\mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ is called a quantum channel*[17].

17: Or 'channel' in short.

The important part of this definition is the CP condition: from Equation (1.22), it can be interpreted as allowing a quantum channel to be applied to a subsystem defined on a larger space without interfering with the validity of the global quantum state. Hence, this condition is important to allow for the extension of systems by ancillary degrees of freedom and to talk about systems shared by several local parties.

1.3.4. Interventions as Quantum Instruments

Knowing the structure of state spaces in which the systems are represented and how the evolution of a system from one state to another is represented, the next step is to study the structure of maps representing the interventions as in Figure 1.1. The most general kind of quantum intervention between an input and an output quantum systems are the non-destructive measurement [76], for which the most general form allowed by quantum theory is represented by a *quantum instrument* [14].

[76]: Ozawa (1984), *Quantum measuring processes of continuous observables*.

[14]: Davies et al. (1970), *An operational approach to quantum probability*.

Definition 1.3.3 (Quantum Instrument) *A collection $\{\mathcal{M}_i\}_{i=1}^n \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ of completely positive (CP) Trace non-Increasing (TnI) maps that **resolves**, i.e. that sums up to, a CPTP map $\mathcal{M}$,*

$$\sum_{i=1}^{n} \mathcal{M}_i = \mathcal{M} \,, \tag{1.26}$$

*is called a **quantum instrument***[18].

18: Or 'instrument' in short.

These act on a system labeled A_0 in a state[19] ρ^{A_0} and output a system A_1 in a state $\sigma^{A_1}_{|a}$ in the following manner:

19: The reference to the label of the Hilbert space has been put in superscript for notational convenience.

$$\sigma^{A_1}_{|a} = \frac{1}{p(a)} \mathcal{M}_a \left(\rho^{A_0}\right) \ . \tag{1.27}$$

Where the probability $p(a)$ of seeing outcome a is obtained through a modification of the *Born rule*:

$$p(a) = \mathrm{Tr}\left[\mathcal{M}_a\left(\rho^{A_0}\right)\right] \ . \tag{1.28}$$

According to Definition 1.1.1, this rule can be understood as a shortcut notation for a process like

$$p(a|x, \{\mathcal{M}_{a|x}\}, \mathcal{W}) = \mathrm{Tr}\left[\mathcal{M}_{a|x}\left(\rho^{A_0}\right)\right] \ , \tag{1.29}$$

where $\mathcal{W}$ is the environment that supplies Alice with state ρ^{A_0}. In the above, the setting x was ignored by Alice as well. As a consequence, referring to x in the description of the map is superfluous and thus swallowed into its definition. This results in the quantum instrument/intervention $\{\mathcal{M}_{a|x} := \mathcal{M}_a\}$. Each instrument element $\mathcal{M}_a$ results in probability $p(a)$ for a given state. Summing over all possible outcomes leads to a deterministic operation of the party since $\sum_{a\in\Omega_{a|x}} p(a) = 1$. Accordingly, summing over the elements of the instrument gives the averaged state resulting from averaging the intervention over all possible outcomes as $\mathcal{M} = \sum_a \mathcal{M}_a$. This can be pictured as if the party has forgotten the measurement outcome. Indeed, the weighted sum of all possible output states reads (the reference to the spaces has been dropped for conciseness):

$$\sum_a p(a)\sigma_{|a} = \sum_a \frac{p(a)}{p(a)} \mathcal{M}_a\left(\rho\right) = \left(\sum_a \mathcal{M}_a\right)\left(\rho\right) = \mathcal{M}\left(\rho\right) \ , \tag{1.30}$$

where $\mathcal{M}$ corresponds to the CPTP map resolved by the elements of the instrument, so that $\mathcal{M}\left(\rho\right)$ is a valid state (but defined in a different space). Hence, when averaging over all possible outcomes and settings, the intervention of a party results in a deterministic modification of the system, mathematically expressed in the form of a *transformation* from a valid state in $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$ to one in $\mathcal{L}\left(\mathcal{H}^{A_1}\right)$. Such maps representing deterministic operations are the usual quantum channels of Definition 1.3.2.

In general, however, the parties' interventions depend also on their setting. To represent this, each setting is mapped to a different instrument. Alice's intervention is then defined as the *quantum operation*[20]. The quantum operation is the name of the collection of CPTnI maps $\{\mathcal{M}_{a|x}\}$ associated with Alice's choice of course of action during the intervention.

20: The name is chosen to fit the terminology of References [10, 11] while setting a technical term to refer to the definition, but 'intervention' or 'strategy' will be used as well in colloquial explanations.

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

Definition 1.3.4 (Quantum operation) *In a quantum process, a party's intervention is represented as a* ***quantum operation***[21]*. These consist of a collection of quantum instruments indexed by the setting x. In each instrument, each element is in turn indexed by the outcomes a.*
Mathematically, let $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$ be the Hilbert space where the input state is represented, and let $\mathcal{L}\left(\mathcal{H}^{A_1}\right)$ be the one where the output state is. A ***(probabilistic) quantum operation*** *is the representation of an intervention*

21: Or 'operation' in short.

as a collection of CPTnI maps

$$\{\mathcal{M}_{a|x}\} \subset \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right) \ , \tag{1.31}$$

such that each subset $\{\mathcal{M}_{a|x}\}$ *indexed by the same setting* x *forms a quantum instrument.*
The related ***deterministic quantum operation*** $\mathcal{M}_{|x}$ *(noted using the same letter but without the outcome in the subscript) is the action of the operation when averaged over its outcomes,*

$$\mathcal{M}_{|x} := \sum_a \mathcal{M}_{a|x} \ , \tag{1.32}$$

which is a quantum channel.
And the related ***averaged quantum operation*** $\mathcal{M}$ *(noted using the same letter but without subscripts) is the action of the operation when averaged over all of its settings and outcomes, i.e. it is the quantum channel defined by*

$$\mathcal{M} := \sum_{a,x} p(x)\ \mathcal{M}_{a|x} \ , \tag{1.33}$$

where $p(x)$ *is the distribution of the setting.*

1.3.5. Environment as Single-Partite Process Functional

Having fixed local interventions as *quantum operations*, the next part involves letting the party interact with her environment to determine how the environment is represented. The situation is shifted from what is depicted in Figure 1.4 towards Figure 1.6.

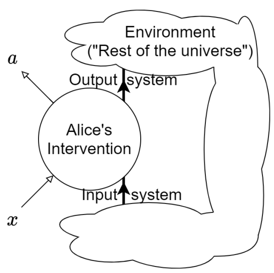


Figure 1.6.: Principle of a single partite process: Alice's local intervention results in an interaction with the environment through the exchange of systems.

In the first picture, it was assumed that Alice was not interacting with her environment. This is operationally equivalent to Alice discarding her input and replacing it with a maximally mixed state, applying her operation on it, and then discarding the output state again and outputting another maximally mixed state to the environment. That way, no information can be received or sent to the environment. This situation leads to the probability distribution being computed from the generalized Born rule as

$$p(a|x) = \mathrm{Tr}_{A_1}\left[\mathcal{M}_{a|x}\left(\frac{\mathbb{1}_{A_0}}{d_{A_0}}\right)\right] \ . \tag{1.34}$$

With respect to that, it is as if Alice's environment were represented as the trivial process, $p(a|x) = p(a|x, \{\mathcal{M}_{a|x}\}, \mathcal{W}_{trivial})$. This process is always seen as white noise by the party, no matter her choice of interventions:

$$\mathcal{W}_{trivial}(\bullet) = \mathrm{Tr}_{A_1}\left[\bullet\left(\frac{\mathbb{1}_{A_0}}{d_{A_0}}\right)\right] \ . \tag{1.35}$$

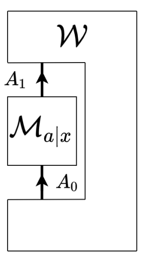


Figure 1.7.: Graphical representation of a process with a single party, Equation (1.37).

Instead, when Alice interacts with her environment, the process is no longer replaced by the trivial process; her distribution changes from $p(a|x, \{\mathcal{M}_{a|x}\}, \mathcal{W}_{trivial})$ to $p(a|x, \{\mathcal{M}_{a|x}\}, \mathcal{W})$ with $\mathcal{W}$ no longer trivial,

$$\mathcal{W} : \mathcal{L}\left(\mathcal{H}^{A_0}, \mathcal{H}^{A_1}\right) \to [0, 1] \ . \tag{1.36}$$

In accordance with Definition 1.1.1, the environment is now represented as a specific functional $\mathcal{W}$ from the space of linear maps to the space of probabilities called the *process functional* (or, loosely, the *process* since it encodes everything that is not the interventions of parties). The probability distribution becomes:

$$p(a|x) := p(a|x, \{\mathcal{M}_{a|x}\}, \mathcal{W}) = \mathcal{W}\left(\mathcal{M}_{a|x}\right) \ . \tag{1.37}$$

However, the single-partite picture obtained at Equation (1.37) does not exactly specify *what is* the set of valid $\mathcal{W}$, i.e., what is the set of the allowed process functionals representing Alice's possible environments.

At that point, physical heuristics can be relaxed into an admissibility criterion: any transformation that can be applied to other transformations in a manner that does not spoil the probabilistic interpretation is a valid candidate for representing an environment [4]. Accordingly, as purely abstract objects, the set of allowed processes is defined out of logical consistency [5] (see e.g., [77]): a process is valid if and only if it leads to valid probability distributions for all choices of interventions [5]. This assumption is called the *no-restriction hypothesis* [78, 79].

[4]: Chiribella et al. (2013), *Quantum computations without definite causal structure*.

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

[77]: Baumeler et al. (2016), *The space of logically consistent classical processes without causal order*.

[78]: D'Ariano et al. (2017), *Quantum Theory from First Principles: An Informational Approach*.

[79]: Plávala (2021), *General probabilistic theories: An introduction*.

Postulate 2: No-Restriction Hypothesis.

All maps that satisfy all mathematical requirements for representing a transformation within the theory will be actual transformations of the theory.

For completeness, it should be pointed out that quantum theory can be derived without assuming such a broad principle; see References [50, 80] and [78]. Whether higher-order generalizations of quantum theory necessarily require it is a question left open for future work.

[50]: Chiribella et al. (2010), *Probabilistic theories with purification*.

[80]: Chiribella et al. (2011), *Informational derivation of quantum theory*.

Remark that this Postulate 2 has been stated in terms of *transformations*. This is because all mathematical objects representing a process can be seen as mappings from one space to another. Indeed, states are maps from the trivial system –the number 1– to the state space; effects are the dual maps from the state space to a probability; and operations are mappings from state space to state space. Seeing the singleton $\{1\}$ as a state space, the objects are all transformations between state spaces[22]. In particular, the process functional $\mathcal{W}$, representing the environment, is a mapping from the intervention of Alice to a probability. Consequently, the process is a transformation of an object transforming a system; it is a *higher-order transformation*. This leads to the first observation motivating this thesis: **The process formalism is about higher-order transformations** [4, 8, 10, 11, 33, 37]. In that sense, the process formalism is a higher-order generalization of quantum theory.

22: The theory of categories provides a rigorous framework for formulating this point precisely. By phrasing it as 'higher-order processes are morphisms in the category of state spaces over finite-dimensional Hilbert spaces' for instance; see in particular Reference [81].

[81]: Coecke et al. (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*.

[4]: Chiribella et al. (2013), *Quantum computations without definite causal structure*.

[8]: Chiribella et al. (2008), *Transforming quantum operations: Quantum supermaps*.

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.

[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

[37]: Milz et al. (2023), *Transformations between arbitrary (quantum) objects and the emergence of indefinite causality*.

In Postulate 2, the 'mathematical requirements' still have to be made explicit in order to define the *admissible* process functionals $\mathcal{W}$. The admissible $\mathcal{W}$ must yield valid probability distributions in Equation (1.37) irrespectively of the choice of intervention made by the party, since any intervention is allowed on any state by the no-restriction hypothesis. Moreover, it can be shown that this admissibility requirement constrains the set of process functionals to the set of linear and (completely) positive functionals on $\mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$ which preserves the normalization of

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

probabilities [5]. Linearity and preservation of the normalization are indeed necessary to preserve the convex structure of the theory, which in turn allows Alice to mix any two of her operations. On the other hand, Complete Positivity (CP) is necessary to allow for arbitrary extensions of the process, for example, by letting Alice keep an ancillary memory or by adding a second party.

Explicitly, preservation of the normalization means that $\mathcal{W}$ maps every quantum operation of the party Alice to a valid probability. Whence, according to the definition of an operation, Definition 1.3.4, this implies that the process functional $\mathcal{W}$ must send elements of quantum instruments (CPTnI maps) to the interval $[0, 1]$ and moreover that it is normalized on quantum channels[23] (CPTP maps). This leads to the following definition (adapted from [5]).

23: Since

$$\begin{aligned} 1 &= \sum_a p(a|x) \\ &= \sum_a \mathcal{W}\left(\mathcal{M}_{a|x}\right) \\ &= \mathcal{W}\left(\sum_a \mathcal{M}_{a|x}\right) . \end{aligned} \tag{1.38}$$

The first line follows by the definition of a distribution, the second by Definition 1.1.2, and the third by linearity. Equation (1.39) is then obtained through Definition 1.3.4.

Definition 1.3.5 (Admissible Single-Partite Quantum Process) *A functional $\mathcal{W} : \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right) \to \mathbb{C}$ is a **single-partite process functional** as in Definition 1.1.2 which is **admissible** for quantum theory if 1) it is linear; 2) it maps all elements (CPTnI maps) of all quantum operations as in Definition 1.3.4 defined on space $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ to the interval $[0, 1]$, and it keeps the deterministic operations (CPTP maps) normalized, i.e.,*

$$\forall \mathcal{M} \text{ CPTP}, \quad \mathcal{W}(\mathcal{M}) = 1 . \tag{1.39}$$

The generalization of this definition to the multipartite case is straightforward: a process is a multilinear functional from the operations of several parties to a probability.

1.4. Defining Higher-Order Transformations

The definition of a process functional was obtained by requiring compatibility with mixing strategies, keeping an ancilla, or adding an extra party. These are indeed possible transformations in quantum theory, which is valid locally by Postulate 1. Because of that, the intervention as well as the process normalized on it must be compatible with the probabilistic structure of the theory, which requires linearity and complete positivity. Pushing Postulate 2 further, these are essentially the same constraints that should be imposed on *all higher-order transformations*, not just the functionals on operations. This generalization is captured by the idea of *admissibility* of higher-order transformations [82], first defined by Chiribella [9], Perinotti [10], and Bisio [11].

[82]: Wilson et al. (2023), *Quantum Supermaps are Characterized by Locality*.

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.

[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

As mentioned above, the quantum process functional is an example of a transformation of an object which is a transformation itself: it sends a channel, which is the transformation between quantum states, to the number one. This 'second-order transformation' is a special case of the quantum supermap [8], which is the transformation between quantum channels, obtained when considering the number one as the only channel that can be defined over a one-dimensional Hilbert space.

Consider the following construction of a supermap as the guiding example for the general definition of an *admissible higher-order transformation*. It is defined in full analogy as with how the quantum channel is defined. Let $\{\mathcal{M}^A\}$ be the set of all quantum channels defined in $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ and let $\{\mathcal{N}^B\}$ be the one in $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{B_0}\right), \mathcal{L}\left(\mathcal{H}^{B_1}\right)\right)$. Then, a *supermap* (or *superoperator* when $\mathcal{H}^{A_0} \cong \mathcal{H}^{B_0}$ and $\mathcal{H}^{A_1} \cong \mathcal{H}^{B_1}$) is a map $\mathcal{S}$ that sends the set of channels in A onto the one in B:

$$\begin{aligned} \mathcal{S} : \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right) &\to \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{B_0}\right), \mathcal{L}\left(\mathcal{H}^{B_1}\right)\right), \\ \forall \mathcal{M}^A \text{ CPTP, } \exists \mathcal{N}^B \text{ CPTP} : &\quad \mathcal{S}(\mathcal{M}^{\mathcal{A}}) = \mathcal{N}^B \, . \end{aligned} \tag{1.40}$$

Next, the supermap must be compatible with the probabilistic structure: the acted-upon channel is assumed to be under the control of a party who can in general perform a quantum operation as in Definition 1.3.4. Compatibility with the probabilistic structure is the requirement that not only all channels in A[24] have an image in the set of channels in B, but also all possible choices of operations in A have one in the set of operations in B. This means that all elements of all operations in A should be mapped to elements of operations in B. This entails two conditions. First, all quantum instruments must be mapped to quantum instruments in B. This is the requirement that all elements of a resolution are maps to elements of a resolution, i.e. that every CPTnI map in A (indexed by some outcome a) must be mapped to a CPTnI in B (now indexed by outcome b). This condition is compatibility under probabilistic operations, but the second condition is compatibility under randomizations. A party can act deterministically but in a way randomized between several deterministic operations according to a setting x. In such case, its averaged quantum operation is given by $\mathcal{M}^A = p(x=0)\mathcal{M}^A_{|x=0} + p(x=1)\mathcal{M}^A_{|x=1} + \dots$ and this behavior must be mapped homogeneously to B so that

24: A and B will be used here as a quick way to respectively mention the spaces $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ and $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{B_0}\right), \mathcal{L}\left(\mathcal{H}^{B_1}\right)\right)$.

$$\mathcal{S}\Big(\sum_x p(x)\mathcal{N}^A_{|x}\Big) = \sum_x p(x)\mathcal{S}\Big(\mathcal{N}^A_{|x}\Big) \, . \tag{1.41}$$

From this latter condition, it can be shown that $\mathcal{S}$ is linear (see e.g. References [5, 9, 10]).

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.
[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.
[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.

The argument so far is similar to how to axiomatically define a channel as a mapping from a set of states to a set of states. However, sending valid states to valid states, i.e. positive trace one operators to positive trace one operators, is not sufficient to define a channel. This is because the local application of the channel on one part of a bipartite state may result in a non-valid state. In the same way, sending valid channels to valid channels is also not sufficient to define a supermap: applying it to one part of a bipartite channel may not result in a valid bipartite channel. In other words, the linear map $\mathcal{S}$ must not only send CP maps to CP maps, but it must also do it in a 'completely CP-preserving' manner. In symbols, $\mathcal{S}$ is completely CP-preserving if for all possible $\mathcal{H}^C$, so that it can extend A with a space of linear maps $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{C_0}\right), \mathcal{L}\left(\mathcal{H}^{C_1}\right)\right)$ ($\mathcal{H}^C \cong \mathcal{H}^{C_0} \cong \mathcal{H}^{C_1}$) in order to define a bipartite channel $\mathcal{M}^{AC}$, $\mathcal{S}$ sends this bipartite channel to a bipartite channel $\mathcal{N}^{AC}$ in the following manner:

$$\begin{gathered} \forall \mathcal{M}^{AC} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{C_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1} \otimes \mathcal{H}^{C_1}\right)\right) : \ \mathcal{M}^{AC} \text{ CP}, \\ \exists \mathcal{N}^{BC} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{B_0} \otimes \mathcal{H}^{C_0}\right), \mathcal{L}\left(\mathcal{H}^{B_1} \otimes \mathcal{H}^{C_1}\right)\right), \ \mathcal{N}^{BC} \text{ CP} : \\ \left(\mathcal{S} \otimes \mathcal{I}^C\right) \{\mathcal{M}^{AC}\} = \mathcal{N}^{BC} \, , \end{gathered} \tag{1.42}$$

where $\mathcal{I}^C$ is the identity mapping in $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{C_0}\right), \mathcal{L}\left(\mathcal{H}^{C_1}\right)\right)$.

This linearity and complete CP-preservation essentially define a quantum supermap, but to complete the picture of a supermap being a transformation between channels in the same regard as a channel being a transformation between states, the supermap should be interpretable as 'under the control of a party'. In analogy with Definition 1.3.4, the party should be able to implement a 'superoperation' consisting of probabilistic resolutions and randomization of their choice(s) of supermap(s).

This is due to a special case of supermaps consisting of plugging a pair of channels on both sides of the input channel, like $\mathcal{S}(\mathcal{M}^A) = \mathcal{N}^{A_1 \to B_1} \circ \mathcal{M}^A \circ \mathcal{N}^{B_0 \to A_0}$. Such a pair of channels can be assumed under the control of the party realizing the supermap, say Bob, while the input channel $\mathcal{M}^A$ is under the control of a different party, Alice. Bob can apply a quantum operation on both sides of the input channel like $\mathcal{S}^B(\mathcal{M}^A) = \mathcal{N}^{A_1 \to B_1}_{b_1|y_1} \circ \mathcal{M}^A \circ \mathcal{N}^{B_0 \to A_0}_{b_0|y_0}$, and grouping the labels together, $b := (b_0, b_1), y := (y_0, y_1)$ the overall operation can be defined as

$$\mathcal{S}^B_{b|y}(\mathcal{M}^A) = \mathcal{N}^{A_1 \to B_1}_{b_1|y_1} \circ \mathcal{M}^A \circ \mathcal{N}^{B_0 \to A_0}_{b_0|y_0} , \tag{1.43}$$

From this special case, the concept of an operation can be generalized to all completely CP-preserving trace non-increasing resolutions as well as randomizations of quantum supermaps.

Hence, the quantum supermap is a higher-order transformation that was obtained by a formal analogy of the definition of the quantum channel case. The set of all such quantum supermaps forms the set of *admissible* transformations of quantum channels. Generalizing the procedure is what is meant by 'defining a higher-order quantum transformation'[25].

25: Remark that the generalization presented in this thesis, which relies on formal analogies and operational heuristics, can be made rigorous using the framework of category theory. In particular, while this thesis only considers higher-order quantum processes, the formulation of which is reliant on Hilbert spaces and CP maps, the categorical treatment can consider different kinds of objects and morphisms instead, like sets and relations for example. See Reference [33] for more information.

[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

The first step towards the general definition is to allow transformations between any kind of input and output. The formalism should be able to define a set of transformations from channels to supermaps in the same way that it can define states to states. In order to do so, the trivial system must also be considered, so that anything can be considered as a transformation from the trivial transformation to itself. For example, the quantum states are a set of admissible transformations as they transform the trivial system (the number 1) to the quantum state they are representing. In Equation (1.29), the input system is represented by a state $\rho_{A_0} \in \mathcal{L}\left(\mathcal{H}^{A_0}\right)$, and a transformation $\mathcal{M}_a$ acts on it. However, this state is itself interpretable on the same footing as the channels and supermap, that is, as an admissible linear map. In that case, it is a linear map from the trivial system 1 to itself, $\rho_{A_0} \in \mathcal{L}\left(\mathbb{C}, \mathcal{L}\left(\mathcal{H}^{A_0}\right)\right) : \rho_{A_0}(1) = \rho_{A_0}$. It is pretty straightforward to see that it will map any trivial system to any quantum state in a manner that respects extensions of the input system, since Equation (1.22) applied to $\mathcal{H}^{A_0} = \mathbb{C}$ reads

Notice that in this example, the 'trivial system' could have been itself lifted as a special subset of the quantum systems, whose state space is restricted to an element proportional to the maximally mixed state $\{\mathbb{1}\}$. In general, it will be possible to map any set of admissible transformations to a set that resembles a constrained state space. This kind of set will be called a *state structure* and is the backbone of the characterization presented in this thesis. The exact definition is delayed until Chapter 3.

$$\begin{aligned} &\forall \eta_C \in \mathbb{C}^{A_0} \otimes \mathcal{L}\left(\mathcal{H}^C\right) \cong \mathcal{L}\left(\mathcal{H}^C\right) : \eta_C \geq 0, \\ &(\rho_{A_0} \otimes \mathcal{I}^C)\{1 \times \eta_C\} = \rho_{A_0}(1) \otimes \mathcal{I}^C(\eta_C) = \rho_{A_0} \otimes \eta_C \geq 0 \, . \end{aligned} \tag{1.44}$$

It is also possible to see that it will trivially preserve the probabilistic structure as both a resolution $1 = \sum_a p(a)$ and a randomization $\{p(x) \times 1 | \sum_x p(x)\}$ amounts to considering scalar multiplication by probability weights. The homogeneity condition reads $\rho_{A_0}(\sum_x p(x)1) = \sum_x p(x)\rho_{A_0}(1) = \rho_{A_0}$, whereas the trace non-increasing condition reads $\mathrm{Tr}\left[\rho_{A_0}(p(a))\right] \leq \mathrm{Tr}\left[p(a)\right]$. Finally, the notion of an operation can be lifted to the state: it amounts to preparing ensembles conditioned by a random variable; the operation is deterministic when the value of the random variable is known, and a randomization amounts to considering an ensemble of states $\{\rho_{|x}\}$ indexed by a setting x (known a prori); it is probabilistic when it is not, and a resolution amounts to considering the ensemble as a collection $\{\rho_a := p(x)\rho_{|x}\}$ resolving the average state $\rho_{A_0} = \sum_a \rho_a = \sum_x p(x)\rho_{|x}$ and indexed by an outcome a (not known a priori).

From this example, one can conclude that the notion of an order is not an absolute thing. With respect to quantum states, the quantum channel is a first-order transformation, but it is also a second-order transformation with respect to the trivial system (whose state is transformed into a quantum state which is in turn transformed by the action of the channel). However, it can always be seen as a first-order transformation from the trivial system to itself. Therefore, what is important in the theory is not how a set of higher-order transformations is defined, but rather with respect to which other set it is defined. An interesting consequence is that sometimes certain sets will have several seemingly unrelated valid ways to be defined, for example, the quantum 2-network, which will be presented in Subsection 2.3.1, is a specific kind of bipartite CPTP map, but at the same time, it happens to be the quantum supermap. The study of the relations between the sets from how they are defined will be conducted in Chapter 5, with the general reason behind this equivalence being explored as the concluding example of this chapter.

The next step in the generalization is to properly define what should be the complete preservation of complete positivity when dealing with higher-order transformations between two different kinds of transformations. The idea is to recursively generalize the definition by noticing that complete positivity, Equation (1.22), depends on the notion of positivity of the input and output spaces, and iteratively, that complete-complete-positivity-preservation, Equation (1.42), depends on the notion of CP of its input and output spaces.

Definition 1.4.1 (Generalized Complete Positivity) *The space of positive operators on a Hilbert space $\mathcal{L}(\mathcal{H})$ is completely positive in the generalized sense and noted $\mathrm{CP}_{\mathcal{L}(\mathcal{H})}$.*
Let $\mathcal{K}$ and $\mathcal{J}$ be two arbitrary spaces of linear maps between Hilbert spaces, for which the input and output spaces can themselves be arbitrary spaces of linear maps between Hilbert spaces. Let the respective notions of generalized complete positivity for these spaces noted $\mathrm{CP}_{\mathcal{K}}$ and $\mathrm{CP}_{\mathcal{J}}$ and let $\mathcal{K}'$ be a copy of $\mathcal{K}$. Let $\mathcal{S} \in \mathcal{L}(\mathcal{K}, \mathcal{J})$ be a linear map between these spaces, then $\mathcal{M}$ is completely positive in the generalized sense, and noted $\mathrm{CP}_{\mathcal{L}(\mathcal{K},\mathcal{J})}$ if and only if

$$\begin{aligned} &\forall \mathcal{M} \in \mathcal{K} \otimes \mathcal{K}' : \ \mathcal{M}\ \mathrm{CP}_{\mathcal{K}\otimes\mathcal{K}'}, \\ &\exists \mathcal{N} \in \mathcal{J} \otimes \mathcal{K}',\ \mathcal{N}\ \mathrm{CP}_{\mathcal{J}\otimes\mathcal{K}'} : \\ &\left(\mathcal{S} \otimes \mathcal{I}^{K'}\right)\{\mathcal{M}\} = \mathcal{N}\,, \end{aligned} \tag{1.45}$$

where $\mathcal{I}^{K'}$ is the identity mapping on $\mathcal{K}'$.

Colloquially, generalized complete positivity is the 'preservation of CP-ness' from the input to the output when the map is acting on a subsystem. In the following, whenever a map is called CP, this should be understood in the general sense except when explicitly said otherwise.

This definition leads to the second observation: **Higher-order quantum transformations are about defining CP maps between CP maps, i.e. nested CP maps** [9, 13, 83, 84][26]. Indeed, observe how the previous section concerned the shifting from the usual representation of local experiments obeying the rules of quantum theory to a representation in terms of quantum operations, as in Definition 1.3.4. In mathematical terms, this shifting amounts to requiring a representation in which every object is represented by a completely positive (CP) map on some space[27]: in Equation (1.29), the input system is represented by a state ρ_{A_0}, which a CP map from $\mathbb{C}$, where the trivial system 1 is defined, to $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$, the intervention is a CP map $\mathcal{M}_{a|x} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$, and the output system is the unnormalized state $\mathcal{M}_{a|x}(\rho_{A_0})$, which is a CP map from $\mathbb{C}$ to $\mathcal{L}\left(\mathcal{H}^{A_1}\right)$. Similarly, in Equation (1.37), $\mathcal{M}_{a|x}$ is the same CP map and the process functional $\mathcal{W}$ is a CP map from $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ to $\mathbb{C}$. The set of admissible single-partite is thus a set of CP maps in Hilbert space $\mathcal{L}\left(\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right), \mathbb{C}\right)$ acting on a set of CP maps in $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ (the operations) which in turn is acting on a set of CP maps in $\mathcal{L}\left(\mathbb{C}, \mathcal{L}\left(\mathcal{H}^{A_0}\right)\right)$ (the states).

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.
[13]: Chiribella et al. (2008), *Quantum Circuit Architecture*.
[83]: Jenčová (2012), *Generalized channels: Channels for convex subsets of the state space*.
[84]: Wilson et al. (2022), *A Mathematical Framework for Transformations of Physical Processes*.

26: See also References [4, 33, 37].

27: This point presented in more detail in Appendix A.3 together with some more information about how to read diagrams.

With respect to that, defining quantum processes is defining nested CP maps preserving the probabilistic structure from one set of maps to another. Formally,

Definition 1.4.2 (Admissibility of Higher-Order Transformation) *A* ***Higher-Order Transformation*** *is a map from any set of admissible higher-order transformations to any set of admissible higher-order transformations. A higher-order transformation is* ***admissible*** *if 1) it is linear; 2) it preserves the probabilistic structure meaning that it maps probabilistic resolutions to probabilistic resolutions; 3) it is CP in the generalized sense.*

This definition involves notions that will be made precise only in Chapter 3. So far, it should only be seen as a heuristic extension of the process formalism. The cyclicity of the definition (the need for higher-order transformations to define higher-order transformations) will be lifted once the notion of a state structure is introduced in Definition 3.2.2, and from there, the notion of resolution will be defined in Definition 3.2.4. Both of these notions rely on the Choi-Jamiołkowski (CJ) isomorphism [85, 86], which will be introduced in Definition 2.2.1, Chapter 2. With this isomorphism, generalized complete positivity will be replaced by the positivity of the elements in a state structure[28]. This will be shown by doing over the reasoning of this section but using the introduced concepts, thus converging to the proper definition, Definition 3.4.3

[85]: Jamiołkowski (1972), *Linear transformations which preserve trace and positive semidefiniteness of operators*.
[86]: Choi (1975), *Positive Linear Maps on Complex Matrices*.

28: Whether this condition is stronger or equal to generalized complete positivity is still to be clarified in a future work. I am grateful to the reviewers for pointing out this issue as well as issues with the previous definition of an admissible transformation in the earlier version of this manuscript.

A last postulate, in line with the no-restriction hypothesis, was slipped in the above definition: that all admissible maps will be allowed.

Postulate 3: Admissibility.

Any admissible higher-order transformation that can be defined is assumed susceptible to represent the intervention of a party in the theory of higher-order processes.

This postulate states that anything that mathematically looks like an intervention will be a valid intervention, putting aside its exact operational interpretation as well as its physical implementation. Postulates 1, 2, and 3 are taken as the core ideas of the theory of higher-order quantum processes this thesis is about. The main concern of the present manuscript will be the characterization of all admissible sets of higher-order transformations. Before proceeding, the next chapter will present some heuristic reasons for this last postulate as well as some of the theoretical frameworks it can recover.

Multipartite Processes and Indefinite Causal Order

2.

The law of causality, I believe, like much that passes muster among philosophers, is a relic of a bygone age, surviving, like the monarchy, only because it is erroneously supposed to do no harm.

Russell (1912), *On the Notion of Cause* [87]

In the previous chapter, the process formalism was reviewed in a way that led to the definition of the single-partite process as a pair of objects, the quantum operation and the process functional, representing, respectively, the intervention of a party and her environment. This class of processes first involves one party whose interventions obey the rules of quantum theory. Then, under the no-restriction hypothesis, the set of all process functionals in this class –called the *admissible processes*– are defined as every mapping from the interventions to a valid probability that obeys the conditions of Definition 1.3.5. This admissibility requirement led to identifying processes with a constrained set of CP maps called process functionals.

When considering multipartite processes, such a broad definition of *admissible processes on local quantum interventions* is actually enough to recover most of the previous instances of (linear) higher-order quantum process theories that appeared in the literature like the supermaps [8], the quantum comb formalism [9], or the process matrix formalism [5] as was first formalized in [10]. This is done simply by adding more parties to the process and then requiring specific constraints on the signaling structure, as shown in this chapter.

[8]: Chiribella et al. (2008), *Transforming quantum operations: Quantum supermaps*.
[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.
[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.
[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.

The idea conveyed in this chapter is that the process functional can itself be seen as an intervention on the interventions of the parties – a higher-order intervention. In that regard, the notion of *admissibility* is a consistency requirement used to define operations on operations so that the process they represent in the end always results in valid probability distributions. In the next chapter, this notion of admissibility will be generalized so that interventions, as well as every way to define interventions on interventions, can all be seen as some kind of admissible mappings themselves.

2.1. Multiround and Higher-order Interventions

Notice that the word *process* is used in two different manners in the above: first in the information-theoretic sense, where *the process* refers to a collection of probabilities as in Definition 1.1.1, and by extension, the process functional used to compute them. Then, in the information-processing sense, where *a process* refers to applying a *transformation* on an object. For instance, interventions are quantum processes, and process functionals are higher-order quantum processes since they are transformations (sending to a probability) of transformations (the interventions). Compared to the previous chapter, where the notion of *causal* was distinguished from *no-signaling*, here the ambiguity is the point: processes are processes[1]. This is a consequence of the assumption of tomography: stating that the process (in the probabilities sense) can be represented by a process functional (which is a process in the transformation sense) means that the collection of distributions associated with all possible interventions (the process) are enough to characterize the map $\mathcal{W}$ uniquely (the process functional).

1: This is a short but confusing way of phrasing the observation made in the previous chapter that "the process formalism is about higher-order transformations".

Under that logic, the process functional $\mathcal{W}$ itself can be seen as the intervention of a party acting deterministically on every party's input and output systems and outputting a trivial system (nothing). Since it is a deterministic intervention on interventions, it can be abstracted as a higher-order intervention. However the process functional is far from being the only way to define a higher-order intervention. Actually, even certain quantum operations in specific processes can themselves be interpreted as higher-order interventions.

2: See also the discussion in Appendix B.1 for a heuristic account without an explicit reference to process functionals.

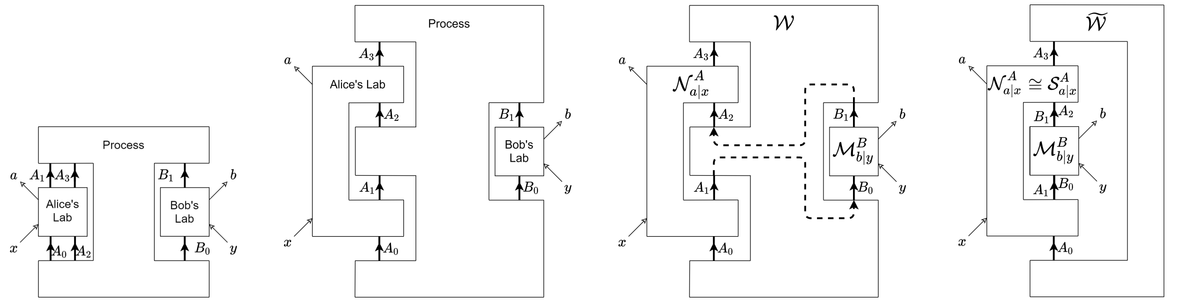
To see how the process formalism can feature higher-order interventions as possible interventions, multi-round interventions must be introduced first[2]. Consider Figure 2.1a: because the systems received and sent by the parties can generally be multipartite, Alice's input and output systems can be split into two subsystems each. In that case, during her intervention she receives two subsystems A_0 and A_2 and sends back two subsystems A_1 and A_3. In addition, the formalism does not impose that parties have to send and receive all parts of their systems simultaneously, and in particular, it does not preclude a party's intervention from being split into several rounds. Consider Figure 2.1b for instance, in this case, Alice is assumed to act in two rounds: she first receives A_0 and sends A_1 *then* she receives A_2 and sends A_3. In such special cases, Alice's second round is assumed to be in the causal future of her first; the two rounds happened in her *local* lab, and the order in which she acted on the subsystems is locally fixed. Such particular cases of local interventions are generally called *multi-round* [1, 34]. Because multi-round interventions are allowed, the process can in particular be like the one represented in Figure 2.1c, such that Bob's intervention always happens in between Alice's two rounds, as in Figure 2.1d. By doing so, Alice's operation, seen as a single overall operation has been defined as a higher-order operation [8, 70].

[1]: Hoffreumon et al. (2021), *The Multi-round Process Matrix*.
[34]: Hoffreumon (2019), *Processes with indefinite causal structure in quantum theory: The Multi-Round Process Matrix*.
[8]: Chiribella et al. (2008), *Transforming quantum operations: Quantum supermaps*.
[70]: Ziman (2008), *Process positive-operator-valued measure: A mathematical framework for the description of process tomography experiments*.

In equations, the process in the first two situations consisted of Alice's quantum operation $\mathcal{N}_{a|x} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_2}\right), \mathcal{L}\left(\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_3}\right)\right)$, of Bob's $\mathcal{M}_{b|y} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{B_0}\right), \mathcal{L}\left(\mathcal{H}^{B_1}\right)\right)$, and of a process functional $\mathcal{W}$ acting on these two operators to yield the probability:

$$p(a, b|x, y) = \mathcal{W}\left(\mathcal{N}_{a|x}, \mathcal{M}_{b|y}\right) \ . \tag{2.1}$$

Going to the last situation, in which Alice's operation happens to be higher-order, implies that there exists a map $\mathcal{S}_{a|x}$ and a *reduced* single-partite process $\widetilde{\mathcal{W}}$ such that the operation of Alice can be identified with



(a) The systems in Alice's intervention can be bipartite and she does not have to send and receive them at the same time...

(b) ...hence her operation can be split into two 'nodes', in that case her intervention is *multirounds*.

(c) The process can be such that Bob's operation ends up exactly between Alice's first and second nodes...

(d) ...in that case, her intervention is the environment seen by Bob; it is of *higher-order* compared to Bob's.

Figure 2.1.: Multiround interventions allow for higher-order processes. The last two figures are the diagrammatic representations of the terms appearing on both sides of Equation (2.2).

the *supermap* $\mathcal{S}_{a|x}$ [8] so that:

$$\begin{aligned} \mathcal{N}_{a|x} \mapsto \mathcal{S}_{a|x} : \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{B_0}\right), \mathcal{L}\left(\mathcal{H}^{B_1}\right)\right) \to \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_3}\right)\right) \ , \\ \mathcal{W} \mapsto \widetilde{\mathcal{W}} : \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_3}\right)\right) \to [0,1] \ : \\ \mathcal{W}\left(\mathcal{N}_{a|x}, \mathcal{M}_{b|y}\right) = \widetilde{\mathcal{W}}\left[\mathcal{S}_{a|x}\left(\mathcal{M}_{b|y}\right)\right] \ . \end{aligned} \tag{2.2}$$

In such a description, Bob would describe Alice's intervention in the same way he would describe his environment. In Alice's perspective, however, Bob is treated as a black box within her local lab. Bob can be absorbed as a part of Alice's description over which she has no deterministic control. By doing so, the 'superoperation' of Alice, $\mathcal{S}_{a|x}$, becomes an operation with respect to Alice's environment $\widetilde{W}$, defined as

$$\begin{aligned} \widetilde{\mathcal{N}}_{a,b|x,y} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_3}\right)\right) \ : \\ \widetilde{\mathcal{N}}_{a,b|x,y} := \mathcal{S}_{a|x}\left(\mathcal{N}_{b|y}\right) \ . \end{aligned} \tag{2.3}$$

Such operations in which a party has been removed from the description after they have performed their operation is called a *reduced operation*. From the environment perspective, Alice is the only party in the description. All that the environment 'sees' is the reduced operation of Alice, with 'outcome' $\tilde{a} := (a, b)$ and 'setting' $\tilde{x} := (x, y)$. In its description, the situation is reduced into the following single-partite process as in (1.37)

$$p(\tilde{a}|\tilde{x}) = p(a, b|x, y) = \widetilde{\mathcal{W}}\left(\widetilde{\mathcal{N}}_{a,b|x,y}\right) \ . \tag{2.4}$$

Conversely to that example, by reducing a bipartite scenario into a single-partite one, the environment in the immediate surroundings of a party can always be assumed under the control of an extra party. As a consequence of admissibility, there is indeed always a way to fine-grain the description from a single-partite process to a bipartite process where an intermediate higher-order intervention has been 'slipped' in between the party and the process [88].

[88]: Apadula et al. (2022), *No-signalling constrains quantum computation with indefinite causal structure.*

What this means concretely is that, on the one hand, *some* bipartite processes $\mathcal{W}$ such as Equation (2.1) happen to lead to a scenario where one of the parties is 'totally around' another one, like Equation (2.2), so that the situation can be reduced to a single partite scenario, like Equation (2.4). On the other hand, *every* single-partite process description can always be fine-grained into a bipartite, higher-order scenario, either by splitting the environment description or the party's local lab description. That is, it is always possible to go from (2.4) to (2.2). This is the case because one can always choose the intervention of the intermediate party to do nothing, $\mathcal{S}_{a|x} = \mathcal{I}$, where $\mathcal{I}$ is the identity map, which is an admissible mapping. But, less trivially, the intervention can only have no consequence on average: $\mathcal{S}_{a|x} : \sum_a \mathcal{S}_{a|x} = \mathcal{I}$. This is also an admissible mapping so long that the distributions it induces are well-defined and[3]

3: Which amounts to do the identification $A_0 = A_1 = B_0$ and $B_1 = A_2 = A_3$ when the outcome a has been ignored.

$$\sum_a p(a,b|x,y) = \sum_a \widetilde{\mathcal{W}}\left[\mathcal{S}_{a|x}\left(\mathcal{M}_{b|y}\right)\right] = \widetilde{\mathcal{W}}\left(\mathcal{M}_{b|y}\right) = p(b|y)\,. \tag{2.5}$$

Hence, the lower-order party cannot infer from his outcome distribution whether or not the higher-order party was in the environment at all.

Put another way, the difference between the environment and the parties is a matter of assumptions, and nothing prevents relaxing these assumptions by promoting the environment as a party and defining an environment for a new party, effectively defining a *higher-order process*[4].

4: Recursively defining and characterizing a hierarchy of higher-order processes in that manner will be considered as the concluding example of this thesis in Section 6.1.

In that regard, the process functional as well as the higher-order and multi-round interventions are all instances of quantum operations. The only difference is that these are defined over different Hilbert spaces. But this observation is sufficient to define any extension of quantum theory based on higher-order processes in full generality: higher-order quantum operations are defined as quantum operations on quantum operations simply by enforcing their *admissibility* [10, 11][5]. That is, to define them as CP maps that preserve the normalization of probabilities. The obtained *hierarchy* of higher-order processes is a hierarchy of CP maps defined on CP maps recursively.

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

5: This point can also be phrased as a categorical construction, see References [33, 84].

[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.
[84]: Wilson et al. (2022), *A Mathematical Framework for Transformations of Physical Processes*.

This thesis develops the tools to fully characterize this hierarchy in the Choi-Jamiołkowksi picture. In addition, it develops tools that allow the decomposition of a given higher-order theory in term of its signaling structure, which in turn allow answering a question like "Which processes admit a decomposition like Equation (2.2)?".

2.2. Representation of Processes: Channel-State Duality

However, characterizing processes or using them to compute probabilities may prove tricky when dealing with nested CP maps only. For this reason, the methods developed in this work rely heavily on representing linear maps as operators. This is often referred to as a channel-state duality: the property that a channel between spaces $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$ and $\mathcal{L}\left(\mathcal{H}^{A_1}\right)$ can be represented as a state in space $\mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$. This duality has a two-fold advantage. On the one hand, any process can be reduced to a state-and-effect pair. On the other hand, Choi-Kraus theorem applies: the channel-state duality sends higher-order completely positive maps into subspaces of the cone of positive operators. This implies that the structure of these maps is encoded into the structure of their associated subspace. Therefore, simple linear characterization techniques for spaces, like projectors, can be used. This key property underlies the projective characterization presented in the next part.

2.2.1. Choi-Jamiołkowski (CJ) Isomorphism

The exact implementation of channel-state duality that will be in use is the following definition of the Choi-Jamiołkowski (CJ) isomorphism[6] [85, 86, 89]. Remark that Definition 2.2.1 differs slightly from the one used in the quantum information literature (e.g. in [74, 90]): an extra transposition has been added in the definition. This is a convenience of notation introduced in Reference [5] in order to write fewer partial transpositions in the CJ picture. A few comments about the interpretation of this transpose as an antilinear identification of a space with its dual is provided for completeness in Appendix B.2.

6: Also called *CJ correspondence*.

[85]: Jamiołkowski (1972), *Linear transformations which preserve trace and positive semidefiniteness of operators*.
[86]: Choi (1975), *Positive Linear Maps on Complex Matrices*.
[89]: Pillis (1967), *Linear transformations which preserve hermitian and positive semidefinite operators*.
[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.
[90]: Watrous (2018), *The Theory of Quantum Information*.
[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

Definition 2.2.1 (Choi-Jamiołkowski isomorphism) *Let $\mathcal{M}$ be a linear map from $\mathcal{L}\left(\mathcal{H}^A\right)$ to $\mathcal{L}\left(\mathcal{H}^B\right)$. Let $\{|i\rangle\langle j|\}_{i,j=0}^{d_A-1,d_A-1}$ be the standard basis of $\mathcal{L}\left(\mathcal{H}^A\right)$ and let $\mathcal{I}$ be the identity map on $\mathcal{L}\left(\mathcal{H}^A\right)$. Define $M_{AB} \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ as follows:*

$$M_{AB} := \left[\sum_{i=0}^{d_A-1}\sum_{j=0}^{d_A-1} (\mathcal{I}\otimes\mathcal{M})\{|i\rangle\langle j| \otimes |i\rangle\langle j|\}\right]^T , \tag{2.6}$$

*where T indicates transposition w.r.t. the standard basis. This operator is the **Choi operator** of the map $\mathcal{M}$. The ensuing correspondence between linear maps $\mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right)$ and operators[7] $M \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ is called the **Choi-Jamiołkowski (CJ) isomorphism**.*
*To recover the action of the map $\mathcal{M}$ on an arbitrary operator $V_A \in \mathcal{L}\left(\mathcal{H}^A\right)$, the **reverse direction** of the CJ correspondence is used:*

$$\mathcal{M}(V_A) = \left(Tr_A\left[M_{AB}\cdot(V_A\otimes\mathbb{1}_B)\right]\right)^T , \tag{2.7}$$

where $\mathbb{1}_B$ the identity operator in $\mathcal{L}\left(\mathcal{H}^B\right)$.

7: When clear from the context or not necessary, the subscripts on Choi operators will be dropped to lessen clutter.

Note that the correspondence will also be used in the text as a "de Pillis" linear mapping [89] in order to refer to it more easily. This mapping, noted $\mathfrak{C}$, is given as:

$$\begin{aligned} \mathfrak{C} &: \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^A\right)\right) \to \mathcal{L}\left(\mathcal{H}^A\otimes\mathcal{H}^B\right) , \\ \mathfrak{C}(\bullet) &:= \left[\sum_{i=0}^{d_A-1}\sum_{j=0}^{d_A-1} (\mathcal{I}\otimes\bullet)\{|i\rangle\langle j| \otimes |i\rangle\langle j|\}\right]^T . \end{aligned} \tag{2.8}$$

The reason for choosing the Choi-Jamiołkowski representation rather than any alternative representation like Kraus or Stinespring (see e.g., Chapter 2 in [90]) comes from the following 'enjoyable' properties of the correspondence.

[90]: Watrous (2018), *The Theory of Quantum Information*.

Proposition 2.2.1 (Properties of the Choi-Jamiołkowski isomorphism) *Let $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ be a Hilbert space of operators with its Hilbert-Schmidt inner product noted as $\left(\cdot\,,\,\cdot\right)_{AB} := Tr_{AB}\left[\cdot^{\dagger}\cdot\right]$ with '$\dagger$' indicating the adjoint. Let $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right)$ be a Hilbert space of linear maps with its Hilbert-Schmidt inner product noted as $[\cdot,\cdot] := \sum_{\mu}\left(\cdot(e_\mu)\,,\,\cdot(e_\mu)\right)_A$, where $\{e_\mu\}_{\mu=0}^{d_A^2-1}$ is a shorthand notation for the standard basis of $\mathcal{L}\left(\mathcal{H}^A\right)$, $\{e_\mu\}_{\mu=(i=0,j=0)}^{d_A-1,d_A-1} := \{|i\rangle\langle j|\}_{i,j=0}^{d_A-1,d_A-1}$. For any two arbitrary elements of this space, $\mathcal{M}$, $\mathcal{W}$, such that $\mathfrak{C}\left(\mathcal{M}\right) = M$ and $\mathfrak{C}\left(\mathcal{W}\right) = W$, the following properties hold:*

1. *The Choi-Jamiołkowski correspondence is a bijection.*
2. *The Choi-Jamiołkowski correspondence is an isometry:*

$$\left(M\,,\,W\right)_{AB} = [\mathcal{M},\mathcal{W}]\ . \tag{2.9}$$

3. *$\mathcal{M}$ is Hermitian-Preserving (HP) $\Leftrightarrow$ M is Hermitian*[8]*.*
4. *$\mathcal{M}$ is Positive (P) $\Leftrightarrow$ M is Positive On Pure Tensors*[9] *(POPT).*
5. *$\mathcal{M}$ is Completely Positive (CP) $\Leftrightarrow$ M is Positive SemiDefinite (PSD).*
6. *$\mathcal{M}$ is Unital $\Leftrightarrow$*

$$Tr_A\left[M\right] = \mathbb{1}_B\ . \tag{2.10}$$

7. *$\mathcal{M}$ is Trace-Preserving (TP) $\Leftrightarrow$*

$$Tr_B\left[M\right] = \mathbb{1}_A\ . \tag{2.11}$$

8. *The adjoint of a map is mapped to the complex conjugate of its Choi operator,*

$$\mathfrak{C}\left(\mathcal{M}^*\right) = \overline{M}\,, \tag{2.12}$$

where * indicates the adjoint in $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right)$ and $\overline{\cdot}$ the (entry-wise) complex conjugation.

8: Or self-adjoint.

9: See Definition A.1.2.

The proof of all these properties is standard, except for 4 and 8 but these can be shown from the definitions. For completeness, the proof of property 8 is given in Appendix B.2 alongside the graphical interpretation of the CJ correspondence and some remarks on its 'hidden antilinearity' that were explored in a work outside of this thesis [3].

The property 5, called Choi or Choi-Kraus theorem, is arguably the most compelling feature of the representation as mentioned above. Indeed, CP maps represent higher-order quantum transformation, but complete positivity is not an easy property to prove, and the geometry of the space of CP maps is convoluted (see Reference [74] for instance). The Choi-Jamiołkowski representation allows the mapping of this set into the cone of positive operators, making the characterization of its properties easier.

In particular, mapping it to the cone of positive operators allows the utilization of semi-definite programming (SDP) methods [41] to characterize its interesting elements. See References [40, 43–45, 91–93] for such examples of SDP methods in the context of higher-order quantum processes.

[3]: Carette et al. (2023), *Complete Graphical Language for Hermiticity-Preserving Superoperators*.

[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.

[41]: Vandenberghe et al. (1996), *Semidefinite Programming*.

[40]: Abbott et al. (2016), *Multipartite causal correlations: Polytopes and inequalities*.

[43]: Feix et al. (2016), *Causally nonseparable processes admitting a causal model*.

[44]: Chiribella et al. (2016), *Optimal quantum networks and one-shot entropies*.

[45]: Bavaresco et al. (2019), *Semi-device-independent certification of indefinite causal order*.

[91]: Quintino et al. (2019), *Probabilistic exact universal quantum circuits for transforming unitary operations*.

[92]: Bavaresco et al. (2021), *Strict Hierarchy between Parallel, Sequential, and Indefinite-Causal-Order Strategies for Channel Discrimination*.

[93]: Bavaresco et al. (2022), *Unitary channel discrimination beyond group structures: Advantages of sequential and indefinite-causal-order strategies*.

2.2.2. The Link Product

Thus far, the isomorphism may not look practical because its reverse direction, Equation 2.7, is quite an involved expression. Yet, graphically, it just consists of 'linking' the wire of the input state with the correct wire representing the channel. This hints at a general composition operation. One for which the tensor product is a special case (no wire getting connected), and every other action is a form of sequential composition with the inner product as the other limiting case (all wires getting connected). This recovers the idea of the previous section that every operation is, to some extent, just a composition of some CP maps. Under the CJ correspondence, all these maps are but vectors on a tensor product of Hilbert spaces, so their composition is a form of pairing on this space. This pairing is formalized as the *link product* [9].

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.

Definition 2.2.2 (Link Product) *Let* $M_A \in \mathcal{L}\left(\bigotimes_{i=0}^{n_A-1} \mathcal{H}^{A_i}\right)$ *and* $N_B \in \mathcal{L}\left(\bigotimes_{j=0}^{n_B-1} \mathcal{H}^{B_j}\right)$ *be two composite operators, acting respectively on* m_A *and* n_B *subsystems.*
Let $C := A \cap B$ *labeling the set of* n_C *subsystems they have in common, i.e.* $C := \{C_k | \forall (i,j) : A_i \in A, B_j \in B, A_i = B_j := C_k\}$ *so that* $\mathcal{H}^{C_k} \cong \mathcal{H}^{A_i} \cong \mathcal{H}^{B_j}$. *Let* $A \setminus C$ *be the complement of set* C *in set* A, *i.e.* $A \setminus C := \{A_i | \forall C_k \in C : A_i \neq C_k\}$ *and* $B \setminus C$ *be the one in* B.
*The **link product** of operator* M_A *with* N_B, *noted with* $*$, *is the operator* $M_A * N_B \in \mathcal{L}\left(\mathcal{H}^{A\setminus C} \otimes \mathcal{H}^{B\setminus C}\right)$ *defined as*

$$M_A * N_B := Tr_C \left[\left(M_A^{T_C} \otimes \mathbb{1}_{B\setminus C} \right) \cdot \left(\mathbb{1}_{A\setminus C} \otimes N_B \right) \right] . \tag{2.13}$$

Proposition 2.2.2 (Properties of the link product) *Let* $\mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right)$, *let* $\mathcal{N} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^B\right), \mathcal{L}\left(\mathcal{H}^C\right)\right)$, *and let their composition be* $\mathcal{N} \circ \mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^C\right)\right)$. *Then, the following holds:*

1. *The composition of two linear maps is represented by the link product of their Choi operators:*

$$\mathfrak{C}\left(\mathcal{N} \circ \mathcal{M}\right) = \mathfrak{C}\left(\mathcal{N}\right) * \mathfrak{C}\left(\mathcal{M}\right) \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^C\right) . \tag{2.14}$$

2. *The link product is associative.*
3. *The link product is commutative*[10].
4. *The link product of Hermitian operators is Hermitian.*
5. *The link product of Positive Semi-Definite (PSD) operators is PSD.*

10: Note that this 'commutativity' of the link product is up to a reorganization of the tensor factors (or 'up to a SWAP gate') as in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^C\right) \cong \mathcal{L}\left(\mathcal{H}^C \otimes \mathcal{H}^A\right)$. This reorganization will always be implicit since the factors are assumed to be sorted by their labels by convention.

The interest in using the link product stems from its versatility: it encompasses the composition, the tensor product, and the inner product as a single operation. In Definition 2.2.2, the link product of two operators defined on different spaces indeed reduces to a tensor product; let $M_A \in \mathcal{L}\left(\mathcal{H}^A\right)$ and $N_B \in \mathcal{L}\left(\mathcal{H}^B\right)$, then

$$M_A * N_B \overset{(2.13)}{=} M_A \otimes N_B \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right) . \tag{2.15}$$

Similarly, the link product of two operators defined over the same Hilbert space reduces to a trace, which can be expressed as a Hilbert-Schmidt inner product; let $N_A, V_A \in \mathcal{L}\left(\mathcal{H}^A\right)$, then

$$V_A * N_A \overset{(2.13)}{=} \mathrm{Tr}_A\left[V_A^T \cdot N_A\right] \overset{(1.15)}{=} \left(N_A^\dagger\,,\, V_A^T\right)_A \in \mathbb{C}\,. \tag{2.16}$$

With the link product, the following shorthand notation for expressions like Equation (2.7) can be used:

$$\mathcal{M}\left(V_A\right) = \left(M_{AB} * V_A^T\right)^{T_B} = \left(M_{AB}\right)^T * V_A\,. \tag{2.17}$$

The link product is a very effective tool for writing compact CJ expressions dealing with traces on multiple spaces at once; for example,

$$\mathrm{Tr}_B\left[\mathcal{M}\left(V_A\right)\right] = \mathbb{1}_B * M_{AB} * V_A^T\,, \tag{2.18}$$

which uses $\mathrm{Tr}_B\left[\cdot\right] = \mathrm{Tr}_B\left[\mathbb{1}_B \cdot\right]$ as well as $\mathbb{1}_B^T = \mathbb{1}_B$. Moreover, since the link product is commutative, the single-partite terms can be regrouped in the above expression:

$$\mathrm{Tr}_B\left[\mathcal{M}\left(V_A\right)\right] = M_{AB} * \left(V_A^T \otimes \mathbb{1}_B\right)\,. \tag{2.19}$$

This kind of manipulation involving the link product will often be used to write shortened formulae in the following.

2.3. Indefinite Causal Order

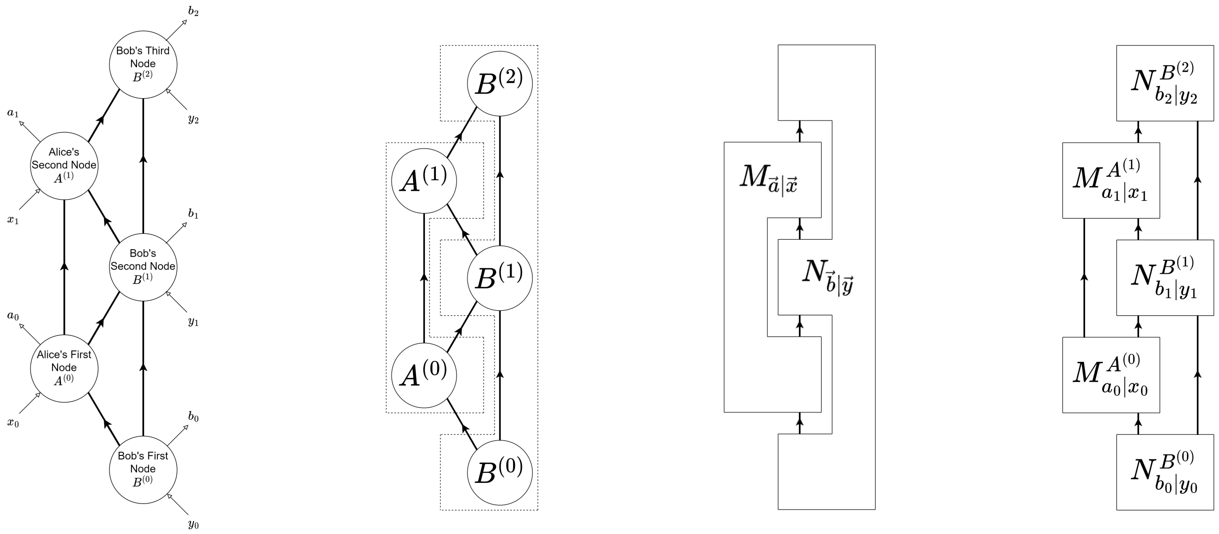
Before moving to the result part, it remains to see why signaling and causality are relevant in process formalism.

2.3.1. Quantum Networks and Combs

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks.*

The first formalization of a class of higher-order processes that appeared in the literature is the *quantum network formalism* [9]. It has been developed for the representation of interconnected multi-round interventions. A multi-round intervention of a party Alice is associated with a network in such a way that each of her interventions is a *node* in the network. Several networks can then be combined to represent back-and-forth exchanges of messages between two or more parties; see Figure 2.2a. In between their message exchanges, the parties are allowed to keep a memory of their previous interventions stocked into an ancillary system. A network with n nodes associated with the successive interventions of party Alice is concisely referred to as Alice's n-network. For example, the figure represents the composition of some 2-network of Alice together with some 3-network of Bob.

Seen differently, the network formalism consists of abstracting fragments of quantum circuits that are associated with some party. The exchange of systems amounts to connecting the fragments to complete the circuit and realize a protocol. As mentioned in the previous example, such multi-round interventions actually correspond to higher-order processes. In the figure, Alice's 2-network can be seen as a higher-order map acting



(a) Abstract depiction of Alice and Bob's interconnected networks as two sets of nodes (interventions) exchanging systems.

(b) Grouping of the nodes into combs.

(c) The network as the composition of the two combs associated with the actions of the two parties.

(d) Decomposition of the network into node-wise quantum operations.

Figure 2.2.: Main aspects of the quantum network formalism.

on Bob's second node, whereas Bob's 3-network can be seen as a higher-order map on Alice's 2-network. This recursive aspect is what defines the network formalism.

Definition 2.3.1 (Quantum Networks) *A **deterministic quantum 1-network** is a quantum channel. A **probabilistic quantum 1-network** is a quantum operation.*
*A **deterministic quantum n-network** is an admissible transformation from a deterministic $(n-1)$-network to a deterministic 1-network. A **probabilistic quantum n-network** is a resolution of the deterministic quantum n-network.*

Here, the concept of a resolution generalizes the usage made in POVM and quantum instrument formalisms: a resolution of a CPTP supermap is any collection of CP TnI supermaps that sum up to the CPTP supermap. A quantum network with n nodes is, therefore, a supermap that takes a network with $(n-1)$ nodes and outputs a network of 1 node. Remark that in the example of the figure, Bob's 3-network is particular as it has a trivial first input (no wires coming into Bob's first node) and a trivial last output (none coming out of his third node). His 3-network is said to output the trivial node, i.e. a 1-dimensional one, and therefore results in a probability distribution rather than a quantum operation. In that sense, Bob's 3-network can be seen as the process functional upon which Alice intervenes with her 2-network. This particular kind of process/network is called a *quantum 2-tester* [94] (because it 'tests' a fragment of a circuit with two nodes and outputs a probability).

[94]: Chiribella et al. (2008), *Memory Effects in Quantum Channel Discrimination.*

To deal with these supermaps numerically, for example to optimize a communication protocol or circuit, the networks are represented by their CJ operators. In that case, these are called *quantum combs* (because of their shape as diagrams, see Figure 2.2c).

Definition 2.3.2 (Quantum Combs) *A **(quantum) comb** is the Choi-Jamiołkowski representation of a quantum network.*

The epithets linked to a network carry to the comb. For example, the CJ representation of a probabilistic quantum 3-network is called a probabilistic quantum 3-comb.

As operators on a composite Hilbert space, combs have an easy characterization [1] (adapted from [9, Theorem 5]).

[1]: Hoffreumon et al. (2021), *The Multi-round Process Matrix.*

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks.*

Theorem 2.3.1 (Characterization of deterministic n-combs) *Let $M \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1} \otimes \ldots \otimes \mathcal{H}^{A_{2n-1}}\right)$ be an operator. Let j be an integer and let the following operators be defined from M: $\forall j : 1 \leq j \leq n$, $M^{(j)} := \frac{1}{d_{A_{2j}} d_{A_{2j+2}} \ldots d_{A_{2n-2}}} Tr_{A_{2j} A_{2j+1} \ldots A_{2n-1}} [M]$, such that $M^{(n)} = M$.*
Then, M is a deterministic quantum n-comb if and only if the following holds:

$$M \geq 0\,; \tag{2.20a}$$

$$Tr_{A_0 A_1}\left[M^{(1)}\right] = d_{A_0}\,; \tag{2.20b}$$

$$\forall i \in 1, \ldots n\ :$$

$$Tr_{A_{2i-1}}\left[M^{(i)}\right] = Tr_{A_{2i-2}A_{2i-1}}\left[M^{(i)}\right] \otimes \frac{\mathbb{1}_{A_{2i-2}}}{A_{2i-2}}\,. \tag{2.20c}$$

This characterization implies that the quantum networks, which are abstract supermaps, are all realizable as a causally ordered succession of channels [9, Theorem 6]. This is indeed the content of equations (2.20): each $M^{(i)}$, $i < n$, appearing in the recursive characterization rule represents a network in which the last node was detached as an independent quantum channel (or instrument in the probabilistic case). See Figure 2.2d: Equation (2.20) implies that Alice's 2-comb $M_{\vec{a}|\vec{x}}$, which is a supermap, decomposes as a succession of two maps $M^{A^{(1)}}_{a_1|x_1}$ and $M^{A^{(0)}}_{a_0|x_0}$. The two of which are elements of quantum instruments defined on each of Alice's nodes linked by an ancillary system (here named A'):

$$M_{\vec{a}|\vec{x}} = M^{A_2 A_3 A'}_{a_1|x_1} * M^{A_0 A_1 A'}_{a_0|x_0}\,, \tag{2.21}$$

where

$$M^{A^{(1)}}_{a_1|x_1} := M^{A_2 A_3 A'}_{a_1|x_1}\,, \tag{2.22}$$

and

$$M^{A^{(0)}}_{a_0|x_0} := M^{A_0 A_1 A'}_{a_0|x_0}\,. \tag{2.23}$$

Whence, the result is that $M^{A^{(1)}}_{a_1|x_1}$ and $M^{A^{(0)}}_{a_0|x_0}$ are also quantum operations. However, compared to seeing Alice's intervention as a supermap from Bob's second node to an operation as in Equation (2.3),

$$M_{\vec{a}|\vec{x}} * N^{B^{(1)}}_{b_1|y_1} = M^{A_0 A_3 B_1' B_2'}_{\vec{a}, b_1|\vec{x}, y_1}\,. \tag{2.24}$$

The decomposition sees Alice's intervention as a special kind of bipartite operation as in Equation (2.1). This means that the combs can be decomposed into subsequent uses of two quantum instruments, which have a known physical realization [76], and therefore the combs have a physical realization.

[76]: Ozawa (1984), *Quantum measuring processes of continuous observables.*

The point of this result is that the quantum networks are supermaps, but these supermaps decompose into a causally ordered succession of operations at each node. As such, this class of higher-order quantum processes is not a higher-order generalization of quantum theory: every object they involve can be decomposed and represented within quantum theory as a composition of some quantum instruments and ancillary systems.

2.3.2. The Quantum Switch and Causal Non-Separability

Fragments of quantum circuits are quantum networks; quantum networks are quantum combs; and quantum combs are (representations of) supermaps. But are all supermaps networks? As it turns out, there is a counterexample to the potential universality of quantum combs: *the quantum switch* [4].

[4]: Chiribella et al. (2013), *Quantum computations without definite causal structure*.

The idea behind the quantum switch is to coherently control the order of the operations applied on a system. Two parties, Alice and Bob, get as input a target system $|\psi\rangle_t$ and a control system $|\psi\rangle_c$, both in a pure state. Then, depending on the value of the control, Alice either applies her local operation first on the target system or it is Bob who does. When the control system is in a pure state, the circuit formalism holds on: the operations of Alice and Bob are black boxes applied in a particular order or another depending on this control bit. It also holds when the control is in a probabilistic mixture as a natural consequence of the convexity of the space of density operators. But when the control bit is in a superposed state, like $|\psi\rangle_c = |+\rangle \equiv \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, it breaks down. The signaling structure of the circuit appears to be in an entangled state. In the paper [4], this is formulated as a no-go theorem:

> "The [SWITCH supermap] cannot be computed deterministically by a circuit in which the two unknown oracles [i.e. the operations of Alice and Bob] are called a single time in a fixed causal order."

As mentioned in the previous chapter, this *indefinite causal order* can furthermore be formalized as the theory-dependent notion of *causally separable processes* [5]. The switch can be seen as a four-partite process shared by four parties. In addition to Alice and Bob, Charlie is added in the global past, and David is added in the global future. Charlie's role is to prepare the control and the target systems before passing them onto the process so that Alice and Bob can act on them, whereas David's is to measure the target and control systems destructively after Alice and Bob have acted on them. As can be shown, for certain interventions of Charlie, the reduced tripartite process shared by Alice, Bob, and David cannot be split into a convex mixture of terms that have a fixed signaling direction for all choices of interventions[11]. In such case, the quantum switch is *causally non-separable*, the different causal orders it shows, i.e. Alice's operation happening in a superposition of before and after Bob's, are effectively in *more than a classical mixture*. It is an *indefinite causal order* (ICO).

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

11: This is an outline of the argument. See References [35, 95] for the complete derivation.

[35]: Araújo et al. (2015), *Witnessing causal nonseparability*.

[95]: Branciard (2016), *Witnesses of causal nonseparability: an introduction and a few case studies*.

However, in contrast with no-signaling and causality, causal separability is a theory-dependent property. It is inferred at the level of the mathematical description of the process, rather than at the correlations it allows.

2.3.3. The Process Matrix and Violation of Causal Inequalities

The quantum switch example reveals that not all processes are quantum networks since not all processes can be understood as a causally ordered succession of parties' operations. Generally, the environments, and thus the higher-order interventions, are different admissible processes functional than just quantum testers. The CJ representation of this broader class of process functionals is called the process matrix.

This representation of processes using the CJ correspondence has still not been addressed. To begin with, the single-partite process as in Equation (1.37) can be represented as a *single-partite process matrix* using the CJ correspondence on the quantum operation,

$$\mathcal{M}_{a|x} \overset{(2.6)}{\mapsto} M_{a|x} \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right) \ , \tag{2.25}$$

as well as on the process functional[12]

$$\mathcal{W} \overset{(2.6)}{\mapsto} W \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right) \ . \tag{2.26}$$

12: Actually it is used twice on the functional: once to go from $\mathcal{L}\left(\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right), \mathbb{C}\right)$ to $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right) \otimes \mathbb{C}$ which is isomorphic to $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ (this amounts to applying a Riesz mapping) and then a second time.

This turns the computation of the probabilities in the single-partite process of Equation (1.37) into a trace rule

$$p(a|x) = \mathcal{W}\left(\mathcal{M}_{a|x}\right) = \mathrm{Tr}\left[W \cdot M_{a|x}\right] \ , \tag{2.27}$$

Actually, since both W and $M_{A|x}$ are positive by Choi theorem, they are in particular self-adjoint, i.e. $W^\dagger = W$, hence this rule can be interpreted as the inner product[13] in $\mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$:

$$p(a|x) = \mathrm{Tr}\left[W^\dagger \cdot M_{a|x}\right] \overset{(1.15)}{=} \left(W \ , \ M_{a|x}\right)_{A_0 A_1} \ . \tag{2.28}$$

13: Since $\mathcal{W}$ is a positive functional, W is none other than the unique operator obtained through Riesz representation theorem (see Reference [96] for instance).

[96]: Roman (2008), *Advanced Linear Algebra*.

This recovers a state-and-effect kind of relation: two positive operators united through the inner product to yield a probability distribution. For that reason, this rule is called a *generalized Born rule* [5, 60, 70, 94]. In the CJ picture, the matrix W, representing the process variable, is called a *single-partite process matrix*. Therefore, the higher-order theory (single-partite process matrix) has been represented in the same form as the base theory (quantum prepare-and-measure scenarios): A state and effect pair linked by the inner product (the generalized Born rule).

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.
[60]: Shrapnel et al. (2018), *Updating the Born rule*.
[70]: Ziman (2008), *Process positive-operator-valued measure: A mathematical framework for the description of process tomography experiments*.
[94]: Chiribella et al. (2008), *Memory Effects in Quantum Channel Discrimination*.

This picture is also valid for multipartite processes. For instance, in Equation (2.1), the probability distribution can also be cast as a generalized Born rule:

$$p(a,b|x,y) = \mathrm{Tr}\left[W \cdot (M^A_{a|x} \otimes M^B_{b|y})\right] \ , \tag{2.29}$$

where $\{M^A_{a|x}\}$ is a quantum operation associated with Alice's intervention and $\{M^B_{b|y}\}$ one with Bob's. Defining general processes featuring multiple parties is the systematic method for considering every situation in order to uncover those that may feature indefinite causal order. In that regard, it bypasses the question of how concretely the process is implemented and moves on to the question of which correlations it allows.

Definition 2.3.3 (Process Matrix) *Let* $\{\mathcal{M}^A_{a|x}\}$, $\{\mathcal{M}^B_{b|y}\}$, *... and let* $\mathcal{W}^{AB\cdots}$ *be defined as in Definition 1.1.2. Let* $\{M^A_{a|x}\}$, $\{M^B_{b|y}\}$, *... be the Choi-Jamiołkowski (CJ) representation of the operations. The operator* $W^{AB\cdots} \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1} \otimes \mathcal{H}^{B_0} \otimes \mathcal{H}^{B_1} \otimes \ldots\right)$ *representing the process functional* $\mathcal{W}^{AB\ldots}$ *such that*

$$\mathcal{W}^{AB\ldots}\left(\mathcal{M}^A_{a|x}, \mathcal{M}^B_{b|y}, \ldots\right) = \mathrm{Tr}\left[W^{AB\cdots} \cdot \left(M^A_{a|x} \otimes M^B_{b|y} \otimes \ldots\right)\right], \tag{2.30}$$

is called the ***process matrix*** *[5].*

Theorem 2.3.2 (Characterization of the Process Matrix) *An operator* $W^{AB\cdots} \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1} \otimes \mathcal{H}^{B_0} \otimes \mathcal{H}^{B_1} \otimes \ldots\right)$ *is a* ***process matrix*** *if and only if it obeys the following conditions [5]:*

$$W \geq 0\,; \tag{2.31a}$$

$$\mathrm{Tr}\left[W \cdot \left(M^A \otimes N^B \otimes \ldots\right)\right] = 1\,, \tag{2.31b}$$

For all $M^A \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$ *that is the CJ representation of a quantum channel applied by party Alice, and for all* $N^B \in \mathcal{L}\left(\mathcal{H}^{B_0} \otimes \mathcal{H}^{A_1}\right)$ *that is one applied by Bob, and etc. for all remaining parties.*

As it turns out, the quantum switch is not the most exotic process that can be expressed as a process matrix with no counterpart in circuit formalism. As shown in Reference [5], some causally non-separable processes result in distributions of outcomes that are non-causal in the sense of Definition 1.2.3. This means that for some processes, indefinite causal order can be certified by a theory-independent bound on correlations.

To show this, the authors considered a general bipartite process matrix and proposed a communication game in which the winning probability is related to causality. This game is similar to the "guess your neighbor's input" (GYNI) game [97], but where an external referee randomly designates a party that has to guess the other party's input in order for both to win. The parties are allowed to coordinate their strategies beforehand and to share any environment for as long as they each intervene only once on it. Alice and Bob each receive an evenly distributed input bit x and y, and the referee's decision is represented by another bit z that the two have access to. In such a process, it is assumed that beyond the random bits, the parties act deterministically under an agreed-upon strategy so that Alice's settings[14] are x, z, and Bob's are y, z. The game is won if Bob guesses Alice's setting correctly when $z = 0$ or if Alice guesses Bob's when $z = 1$. Encoding the guess of each player on their outcomes, the game-winning probability is then equal to

$$p_{succ.} := \sum_{a,b} 1/2p(a = y, b|x, y, z = 0) + 1/2p(a, b = x|x, y, z = 1)\,, \tag{2.32}$$

and the parties' goal is to maximize this probability.

Assuming classical resources, an optimal strategy does not depend on z: Alice can always send her setting to Bob. By doing so, she fixes him in her signaling future; the joint distribution becomes one-way signaling, $p(a, b|x, y, z) = p(a|x, z)p(b|x, y, z, a)$ as in Equation (1.7). Hence, she cannot obtain the future setting of Bob y any differently than by guessing.

[97]: Almeida et al. (2010), *Guess Your Neighbor's Input: A Multipartite Nonlocal Game with No Quantum Advantage*.

14: The use of 'setting' is slightly altered here to phrase the process as a variant of GYNI game. The bit z is not a setting in the sense of local interventions, since it is not tied to a single lab. It is rather a piece of information obtained as the outcome of a previous round of communication in which both parties independently received the setting z of the referee, representing his decision. Nevertheless, as Bob never uses z in the winning strategy, it could have been said that z is part of Alice's settings, whereas Bob's settings consist of y only.

By consequence, she wins perfectly half of the time and guesses the other half, whence they win with a $p_{succ.} = 1/2 \times 1 + 1/2 \times 1/2 = 3/4$ chance. A similar optimal strategy with the same probability of winning is obtained by Bob sending his setting to Alice; in that case, the situation is reversed but the probability of winning is still the same, $p_{succ.} = 3/4$ as well. Actually, any optimal classical strategy is a mixture of these two [97]. Hence, a strategy involving a classical process matrix will conclude that the winning probability is $3/4$.

There is a good reason for that; it can be shown that this optimal probability is the maximal amount obtainable by assuming that the distribution is causal as in Definition 1.2.2 [5]. Therefore, this bound of $3/4$ is a *causal inequality*: Any process obtaining a value exceeding it will prove that it is non-causal.

While the bipartite processes that assume classical theory locally are causal, bipartite process matrices, which are those assuming local quantum theory locally, are not. Oreshkov, Costa, and Brukner introduced a particular example of a bipartite process matrix defined over 2-dimensional input and output systems to show it. This operator is commonly referred to as *OCB process matrix* and is usually expressed in the *Pauli basis*.

Definition 2.3.4 (Pauli basis) *Let*

$$\mathbb{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \; X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.33}$$

The set $\{\mathbb{1}, X, Y, Z\}$ forms a basis for the space of operators on a two-dimensional Hilbert space $\mathcal{L}(\mathcal{H}) \cong \mathbb{C}^{2\times 2}$, called the ***Pauli basis****.*

Note that this basis has the particularity of being constituted of unitary and self-adjoint operators. Because of this convenient property, this basis is extensively used for the examples presented throughout this thesis.

The OCB process matrix represents a process functional on Alice and Bob's operations, respectively represented by $M^A_{a|x} \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$ and $M^B_{b|y} \in \mathcal{L}\left(\mathcal{H}^{B_0} \otimes \mathcal{H}^{B_1}\right)$; it is the operator[15]

15: In the following, the references to the subsystems will be omitted to lessen clutter. Recall that the systems are always sorted alphabetically and then numerically. In this specific case, the omitted subscripts are $\cdot_{A_0} \otimes \cdot_{A_1} \otimes \cdot_{B_0} \otimes \cdot_{B_1}$ for instance.

$$W_{OCB} := \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} + \frac{1}{\sqrt{2}}\left(X\otimes Z\otimes Z\otimes\mathbb{1} + Z\otimes\mathbb{1}\otimes\mathbb{1}\otimes Z\right)\right) , \tag{2.34}$$

such that $W_{OCB} \in \mathcal{L}\left(\mathcal{H}^{A_0}\otimes\mathcal{H}^{A_1}\otimes\mathcal{H}^{B_0}\otimes\mathcal{H}^{B_1}\right)$. The claim is that the parties can use this process matrix to win the GYNI game with a probability greater than $3/4$. Here is the winning strategy: in his local lab, Bob always follows the same course of action, which consists of measuring his input system B_0 in the eigenbasis of Z and recording his measurement results as his outcome set $b = \{0, 1\}$ [16], then preparing B_1 in the same basis but according to his setting. The CJ representation of his quantum operation has the form

16: Recall that in the process formalism, the measured value like '$(+z, -z)$') are substituted some ordered label $a = (0, 1)$, i.e. the outcome.

$$M_{b|y} = \frac{1}{4}\left(\mathbb{1} + (-1)^b Z\right) \otimes \left(\mathbb{1} + (-1)^y Z\right) . \tag{2.35}$$

On the other hand, Alice's choice of operation depends on her setting z. When $z = 1$, she measures in the eigenbasis of Z and outputs a random system; when $z = 0$, she measures in the eigenbasis of X instead and encodes both her outcome a and her other setting x into a Z state of A_1. The CJ representation of her operation has the form

$$M_{a|x,z} = \delta_{z,0}\left(\frac{1}{4}\left(\mathbb{1}+(-1)^a X\right)\otimes\left(\mathbb{1}+(-1)^{a+x}Z\right)\right)+\delta_{z,1}\left(\frac{1}{4}\left(\mathbb{1}+(-1)^a Z\right)\otimes\mathbb{1}\right). \tag{2.36}$$

Where $\delta_{z,0}$ is the Kronecker symbol. The probability distribution is computed in the CJ picture using the generalized Born rule:

$$p(a,b|x,y,z) = W_{OCB} * (M_{a|x,z}\otimes M_{b|y}) = \mathrm{Tr}\left[W_{OCB}\cdot(M_{a|x,z}\otimes M_{b|y}\right], \tag{2.37}$$

which yields the following distribution:

$$p(a,b|x,y,z) = \delta_{z,0}\frac{1}{2}\left(1+\frac{(-1)^{b+x}}{\sqrt{2}}\right)+\delta_{z,1}\frac{1}{2}(1+\frac{(-1)^{a+y}}{\sqrt{2}}). \tag{2.38}$$

This gives a success probability of $p_{succ.} = \sum_{a,b} 1/2p(a,b=x|x,y,z=0)+1/2p(a=y,b|x,y,z=1) = 1/2(1+1/\sqrt{2}) \approx 0.85$.

Since the game is won with a better probability than any causal theory, the OCB process matrix is said to violate a causal inequality and represents a non-causal process. In addition, the bound is at the level of the probabilities alone. Hence, the OCB process matrix presents indefinite causal order in a theory-independent manner, a stronger property than causal non-separability.

The difference between the quantum combs, the quantum switch, and the process matrix reviewed in this section is everything that is needed from the theory of indefinite causal order so to motivate the formalism developed in this thesis. Before concluding, here are some remarks intended to give some directions for further enquiries to the interested reader.

First, notice the formal analogy between the theory of quantum entanglement (see [74, 98] for instance) and causal non-separability. A fair amount of the important developments of the theory were actually obtained by transposing concepts encountered in the theory of entangled states and non-locality into the theory of non-separable and non-causal process matrices. While it is necessary for a state to be entangled in order to be non-local which is then proven by beating a Bell inequality, it is necessary for a process matrix to be causally non-separable in order to be non-causal which is then proven by beating a causal inequality [5]. Conversely, a non-local state is entangled, and the same way, a non-causal process is causally non-separable. The analogy can be pushed forward: like there exist *local* entangled states that cannot beat a Bell inequality (see [56, 98]), there exist process matrices that are causally non-separable but yet that cannot beat a causal inequality. This is the case of the quantum switch for instance [35, 38].

Thus far, it may look like a process matrix has to be based on local quantum theory to be non-causal; but this is only the case for bipartite processes. For tripartite processes, there exist processes which interventions are locally described by classical probability theory[17] that can violate a causal inequality [99]. On the same account, a causal process is not necessarily classical. See Reference [100] for more details on this disambiguation.

[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.

[98]: Horodecki et al. (2009), *Quantum entanglement*.

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

[56]: Brunner et al. (2014), *Bell nonlocality*.

[98]: Horodecki et al. (2009), *Quantum entanglement*.

[35]: Araújo et al. (2015), *Witnessing causal nonseparability*.

[38]: Oreshkov et al. (2016), *Causal and causally separable processes*.

17: More precisely on the dephased classical limit of quantum theory. That is, in the limit where the Choi operators representing the interventions can all be diagonalized in the same basis. Such limit effectively reduce the quantum channels acting on quantum states description into the bistochastic matrices acting on probability vectors one.

[99]: Baumeler et al. (2014), *Maximal incompatibility of locally classical behavior and global causal order in multiparty scenarios*.

[100]: Kunjwal et al. (2023), *Nonclassicality in correlations without causal order*.

2.4. Summary

Before proceeding to the main part, it should be noted that what has been called process formalism so far is actually a generalization of the original framework. Indeed, the possibility of multi-round interventions as well as the higher-order interventions they induce were not part of the original formulations. This was only developed during my master thesis [34] and by two other groups that used different approaches [10, 11, 33].

[34]: Hoffreumon (2019), *Processes with indefinite causal structure in quantum theory: The Multi-Round Process Matrix*.
[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.
[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

Recapitulating the rationale so far, it has been shown that the processes formalism (with higher-order and quantum interventions) involves the same linear maps as higher-order quantum processes. And that these processes are mathematically the theory of nested CP maps.

Looking at the consequences of the formalism, it was shown that among the allowed processes, some feature more than one fixed signaling direction between the parties it relates together. Among those with more than one signaling direction, some have indefinite signaling direction[18]. Moreover, among the processes with an indefinite signaling direction, some can truly be non-causal: some of the outcome distributions associated with these processes cannot be decomposed as in Definition 1.2.2. In other words, higher-order processes can violate causal inequalities.

18: Keep in mind that this notion has a rigorous definition using the theory-dependent notion of causal non-separability. This notion will not be reviewed in this thesis.

In order to precisely understand these non-causal effects, one can then wonder *when* do the higher-order processes feature these exotic signaling structures. The purpose of the present thesis is to fully characterize all processes that can be built under this formalism and to systematically track how the signaling structure in a process can be decomposed on a party-wise basis.

As a whole, this thesis defines what higher-order processes are and the main part of the technical results is about devising a method to characterize them: *the projector algebra*. The generalization of the notion of causal separability for higher-order processes, as well as the characterization of the causal polytopes are too remote to be attained within this couple hundred pages. Despite this shortcoming, the hope is that the methods developed in this thesis will serve as a starting point to study these natural follow-up questions.

The Projective Characterization of Higher-Order Processes

3. State Structures

Young man, in mathematics you don't understand things. You just get used to them.

John von Neumann*

The point of this chapter is to show that the state space of all families of admissible higher-order processes has the same abstract structure. Because of that, the characterization will be conducted on this structure first, and then applied to concrete cases.

This chapter starts by introducing this ubiquitous mathematical structure appearing everywhere in the characterization. The goal is to do it gradually using minimal assumptions while keeping track of them so as to pinpoint the principles underlying them. Some elements of discussion about their validity will also be provided in the appendices.

This structure is called a **state structure**. The name was chosen because the simplest example of such a structure is the set of finite-dimensional quantum states in density matrix form. One may think of a state structure operationally as 'the set of allowed deterministic interventions a party can locally perform'. In the case of quantum states for example, the deterministic intervention is a preparation, i.e. to choose a *specific* state of the system, say ρ; this intervention is then phrased as 'the party decided to prepare state ρ in her lab and to pass it to the environment'.

Therefore, one of the core messages of this thesis is that, in the Choi-Jamiołkowski picture, every conceivable intervention is represented as an element of a state structure. The nature of the state structure then reflects the kind of intervention: the preparation of a system in a given state, a given measurement procedure of the system, a transformation of the system, a higher-order transformation of the system, etc. are all represented by an element of different state structures. Characterizing the state structure associated with a party then amounts to characterizing the kind of intervention the party can perform on the system and vice-versa.

This property of inferring a new state structure out of its relations with another one applies to all conceivable interventions. This is the next core message of this thesis: while all higher-order processes are state structures, state structures are moreover defined through their relations with each other. Concretely, after introducing the state structure, the concept of a measurement of a state structure, as well as the parallel composition of, the sequential composition of, and the transformation between two state structures will be introduced. Again, the assumptions for why a set of operations *represents* each of these four concepts will be carefully tracked. Each of these concepts will then be shown to be representable by a state structure as well.

* Reply, according to Dr. Felix T. Smith of Stanford Research Institute, to a physicist friend who had said "I'm afraid I don't understand the method of characteristics"; as quoted in Gary Zukav (1979), *The Dancing Wu Li Masters: An Overview of the New Physics*, Bantam Books, p. 208, footnote.

Another related core message of the thesis is that the algebra of these projectors reflects (in the sense of *is homomorphic to*) the relations between these state structures. Since these state structures are derived from their relations with one or two 'base' state structure(s), and since each state structure is associated with a projector, the relations with the base(s) are encoded as specific operations to apply on the base projector(s). This procedure yields a projector characterizing the newly defined state structure. For example, the set of transformations between two state structures will be shown to be a state structure as well, and its characterization will be reduced to a specific way of forming a new projector out of the projectors characterizing the input and output state structures.

3.1. Motivating Example: From POVM to Instrument Formalism

The idea behind state structures is then to abstract the common mathematical skeleton out of the theory of processes. An occurrence of it lies in the similarities between POVM formalism and quantum instrument formalisms when expressed in the CJ picture.

Consider the quantum communication protocol where Alice transfers a message to Bob by sending him a quantum system. A simple picture of it is that Alice will encode her setting x on a system $\rho_{|x}$ and then send it to Bob's lab in which he will measure it. If the two parties have agreed on a basis for encoding the message and the system has a dimension big enough for the message to be reliably encoded, Bob's outcome distribution will tend to a perfect transmission where each setting is identified with a single outcome, $p(b|x) = \delta_{b,x}$.

The situation can be represented both with destructive or non-destructive measurements, and shifting from one to the other is a prototypical example of how to construct a higher-order transformation. As will be shown, this example is actually related to the construction of the single-partite process as in Subsection 1.3.5.

In the destructive case, Alice's encoding consists of her operation being the preparation of a state among a collection of possible states $\{\rho_{|x}\}_{x\in\Omega_x}$ (preferably orthogonal states to maximize discriminability). The state transits through a perfect channel (represented by an identity map; in this scenario, the channel plays the role of the environment shared by Alice and Bob) and reaches Bob's lab in which he measures POVM $\{E_b\}$ (preferably a projective measurement in the same orthogonal basis as Alice). The probability of observing the outcome b is given by the Born rule:

$$p(b|x) := p(b|x, \{\rho_{|x}\}, \{E_b\}) = \left(E_b\,,\,\rho_{|x}\right) \equiv \mathrm{Tr}\left[E_b^\dagger\,\rho_{|x}\right]\,. \tag{3.1}$$

His effects can be seen as *probabilistic functionals* from the state space to a probability, i.e. $\mathcal{E}_b = \left(E_b\,,\,\cdot\right)_{B_0} = \mathrm{Tr}\left[E_b^\dagger\,\cdot\right] : \mathcal{L}\left(\mathcal{H}^{B_0}\right) \to [0,1]$. These functionals sum up to a *deterministic functional* $\mathrm{Tr}\left[\mathbb{1}\cdot\right] : \mathcal{L}\left(\mathcal{H}\right) \to 1$ that send each state to a probability of 1.

When one considers non-destructive measurement, the representation of Bob's operation is instead given by a quantum operation. Assuming for simplicity that this collection is deterministic, i.e. that it depends on a single-valued setting, it can be represented as the quantum instrument $\{\mathcal{M}_b\} : \mathcal{L}\left(\mathcal{H}^{B_0}\right) \to \mathcal{L}\left(\mathcal{H}^{B_1}\right)$ *resolving* the quantum channel $\mathcal{M}$, i.e., $\sum_b \mathcal{M}_b = \mathcal{M}$. The probability of observing outcome b as well as the output state $\frac{1}{p(b|x)}\mathcal{M}_b\left(\rho_{|x}\right) \in \mathcal{L}\left(\mathcal{H}^{B_1}\right)$ is given by

$$p(b|x) = \mathrm{Tr}\left[\mathcal{M}_b\left(\rho_{|x}\right)\right] . \tag{3.2}$$

But who discards the system so to end the protocol? From the perspective of Bob, this does not matter as this system exited his lab and causality[1] forbids any subsequent intervention on it to change what Bob has recorded. For all intents and purposes, Bob's output is as if discarded by a third local party, Charlie, whose intervention has no influence on Bob's. Referring to the labels of the systems explicitly, Alice's output is identified with Bob's input, $A_1 = B_0$, and Charlie's input is identified with Bob's output, $B_1 = C_0$, so that the probability is the inner product

1: In the sense of no-signaling from the future to the past.

$$p(b|x) = \left(\mathbb{1}^{C_0=B_1}, \mathcal{M}_b^{B_0\to B_1}\left(\rho_{|x}^{A_1=B_0}\right)\right)_{C_0} . \tag{3.3}$$

(A reference to the spaces where the operators and the map are defined have been put in superscripts for clarity.)

The similarity with the destructive case becomes striking in the CJ picture. From Bob's perspective, the probability reads $p(b|x) = \mathbb{1}^{B_1} * M_b^{B_0B_1} * (\rho_{|x}^{B_0})^T$. Using the link product, this can be rephrased as another inner product as in Equation (2.16),

$$p(b|x) = \left(M_b , \rho_{|x} \otimes \mathbb{1}^T\right)_{B_0B_1} . \tag{3.4}$$

This is back to the situation of an inner product without in-between mapping. Here, Alice's preparation and Charlie's discarding can be bundled into a 'space-time state' $W_{|x} = \rho_{|x}\otimes\mathbb{1} \in \mathcal{L}\left(\mathcal{H}^{A_1=B_0} \otimes \mathcal{H}^{C_0=B_1}\right)$ that constitutes Bob's environment. Bob's operation is the corresponding 'space-time effect' so that the inner product is the generalized Born rule [5, 60, 94]:

$$p(b|x) = \left(M_b , W_{|x}\right) . \tag{3.5}$$

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.
[60]: Shrapnel et al. (2018), *Updating the Born rule*.
[94]: Chiribella et al. (2008), *Memory Effects in Quantum Channel Discrimination*.

The 'state' $W_{|x}$ is once more destructively measured by an 'effect' (or probabilistic functional). $W_{|x}$ is indeed a positive and trace-normalized operator while $\{M_i\}$ is a collection of positive operators resolving a trace-normalized positive operator M.

The destructive and non-destructive representations of the protocol look formally similar in the CJ picture. However, the 'effects' $\mathcal{M}_b$ of the quantum instrument formalism are mappings between the states of the POVM formalism; they are thus *higher-order effects* compared to E_b. Consequently, switching the representation from POVMs to quantum instruments can be seen as the construction of a higher-order process[2].

2: This example will reappear but treated using the tools developed in this thesis as part of the concluding example. See Subsection 6.1.1.

Notice the common threads that will be the starting point of the generalization: the two formalisms involve 'states' ($\rho_{|x}$ and $W_{|x}$, respectively) that are measured by a 'deterministic functional' or 'unit effect' ($\mathbb{1}$ and M), resolved into 'probabilistic functionals' or 'effects' (E_b and M_b). What changes between the two situations is that the set of deterministic functionals $\{\mathbb{1}\}$ has a single element in the POVM case, whereas the set $\{M\}$ belongs to a subspace of $\mathcal{L}\left(\mathcal{H}^{B_0} \otimes \mathcal{H}^{B_1}\right)$. Remark, however, that the sets of states and unit effects have a similar structure: those of positive and trace-normalized subspaces. These are precisely the structures that will be abstracted under the name 'state structures', which form the backbone of the characterization methods. Moreover, observe that the *state structures* of both cases are related: W is the tensor product of a quantum state ρ with a quantum effect, $\mathbb{1}$; the set of valid W's is thus a composite state structure obtained by *the tensor product of the state structures* of ρ and $\mathbb{1}$. Likewise, the set of valid M's consists of all mappings between the state structure of ρ in space B_0 to a similar state structure but on space B_1 obtained under the link product as $<$ State in $B_1 >= M* <$ State in $B_0 >$; it is itself a composite state structure, *the transformation between two state structures*.

This example motivates the common features of higher-order objects and their projective characterization. As explained in the introductory chapters, the quantum comb formalism and the process matrix formalism are also characterized by a pair of 'states' and 'unit effects': the environment and the deterministic interventions. These sets correspond to state structures in the CJ picture, as they are sets of positive and normalized operators with support on a specific subspace. The projective characterization that will be presented in this chapter then consists of finding the projector to that subspace and phrasing how the state structures relate to each other as operations on the projectors. Three such ways of being related that will be formalized in the following already appeared in the example: *(deterministic) functional on*, *tensor of*, and *transformation between* state structure(s). As will be shown, relations are actually imposed by certain no-signaling constraints on correlations, as in Definition 1.2.1.

Finally, note that $W_{|x}$ is effectively a higher-order state; it can be seen as a mapping from an effect in B_0 to an effect in B_1. However, it is not a local closed box, i.e., it is not under the control of a single party. The higher-order picture can be completed by assuming that Bob's post-measurement state was returned to Alice. In this case, he pictures her as his environment: Alice's lab encloses Bob's. The spaces can be identified such that Bob's input corresponds to Alice's (first node) output $B_0 = A_1$ and his output to her (second node) input $B_1 = A_2$. What changes in this new scenario is that Alice can now learn things about Bob. For example, she could have entangled the state she sent to Bob with an ancilla of hers so that when she receives Bob's output, she can jointly measure it with her ancilla to learn about Bob's operation. This approach is yielding the PPOVM formalism [70, 94], generalized into the single-round process matrix formalism [5, 101]. Both are the CJ representations of the single-partite and single-round quantum process functional, as presented in the introduction, Definition 1.3.5 in Subsection 1.3.5, and then in Subsection 2.3.3.

[70]: Ziman (2008), *Process positive-operator-valued measure: A mathematical framework for the description of process tomography experiments*.
[94]: Chiribella et al. (2008), *Memory Effects in Quantum Channel Discrimination*.
[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.
[101]: Morimae (2014), *The process matrix framework for a single-party system*.

3.2. Abstracting the Admissible Quantum Operations

In this section, the abstract kind of structure to interpret $W_{|x}$ and M_b in Equation (3.5) as a 'higher-order' state and effect pair of a 'higher-order quantum theory' are defined and characterized. Introducing mappings between these higher-order objects as 'higher-higher-order objects' will follow in Section 3.4.

3.2.1. Defining State Structures

State structures are denoted with a script letter $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$, chosen to be the same as the label of the party they are associated to (and thus, in accordance with the notation, they will come with subscripts when associated to subsystems). For example, the set of deterministic quantum operations a party A can do on Hilbert space $\mathcal{L}\left(\mathcal{H}^A\right)$ is represented by 'generalized states'[3] in a state structure $\mathscr{A}$. These states represent quantum operations, but the exact interpretation of which, like the order, the number of input and output subsystems, etc., is left open until precised. For example, these operations can be the preparations of a quantum state in which case $\mathscr{A}$ is the set of density operators; but it can also be the choice of a quantum channel to apply between subsystems A_0 and A_1, in which case $\mathscr{A}$ is a different set of operators: the (CJ representation of the) set of quantum channels in $\mathcal{L}\left(\mathcal{H}^A\right) = \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$. When several abstract state structures need to be defined on the same space for comparison, these will be distinguished by a prime like $\mathscr{A}, \mathscr{A}', \mathscr{A}''$.

3: Thereafter just called 'states'; the terminology 'quantum states' will be reserved to a regular quantum state preparation, resulting in density operators, i.e. an element of the positive trace-1 set of operators on the space.

As concluded from the examples so far, admissible deterministic quantum operations of a certain kind are represented under the CJ isomorphism as a constrained set of positive operators on some composite space. The abstract structure of these sets is captured under the name state structure that is yet to be defined in this section. Since state structures aim to be abstract enough to infer general properties and relations between higher-order transformations, the question is: What are the key aspects of the sets obtained through CJ representation to be abstracted in it?

A geometrical perspective first: the set of density matrices is a set of positive operators on a Hilbert space with a trace norm of 1. These constraints underlie the probabilistic interpretation of density operators under the Born rule as they shape the spectrum of the operator into a probability vector. The exact form of the operator is then just a 'basis-dependent packaging' around this probability vector. On the one hand, the feature of positive operators is that they form a convex cone in the space of operators, the center[4] of which is made by all elements proportional to the identity operator. On the other hand, the feature of the trace-normalized operators is that they form a plane perpendicular to the identity operator, as the normalization can be seen as an inner product constraint: $\mathrm{Tr}\,[\rho] = (\mathbb{1}\,,\,\rho) = c \in \mathbb{C}$. While the space of operators is a complex linear space, each of these two features actually restricts this linear space to a space with a more complex geometry: the PSD cone is a real convex space, whereas the trace-normalized plane is a complex affine space. Both are more delicate to characterize than a linear space,

4: The center of a set of operator is defined using the distance induced by the Hilbert-Schmidt inner product.

especially the cone of positive operators as positivity is a non-linear constraint.

However, one can focus on the smallest linear subspace contained in the span of both sets. In the case of density operators, as all positive operators are self-adjoint, this set spans the space of self-adjoint operators, a real linear subspace of the space of operators. The trace condition, on the other hand, does not bring another linear constraint besides itself so the sought subspace cannot be further restricted.

If one moves to the next level, mappings between quantum states, that is, quantum channels represented by the set of CPTP maps, a similar reasoning can be conducted in the CJ picture. By Proposition 2.2.1 (or Theorem 2.3.1), the CJ representation $M \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$ of a CPTP map $\mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ must obey the following two conditions:

$$M \geq 0\,; \tag{3.6a}$$
$$\mathrm{Tr}_{A_1}\left[M\right] = \mathbb{1}_{A_0}\,. \tag{3.6b}$$

The first line (3.6a), stating that the CP condition becomes a PSD condition under CJ correspondence, again constraints the operators to the cone of positive operators in $\mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$. This set is again spanning the (real sub-)space of self-adjoint operators.

On the other hand, the TP condition (3.6b) is no longer just a trace condition. As it was first noticed by Araújo and colleagues[5] [35], it actually can be split between a trace normalization,

5: Actually, not in the context of quantum channels, but rather in the context of process matrices.

[35]: Araújo et al. (2015), *Witnessing causal nonseparability*.

$$\mathrm{Tr}\left[M\right] = d_{A_0} \tag{3.7}$$

and a linear constraint

$$\mathrm{Tr}_{A_1}\left[M\right] - \mathrm{Tr}\left[M\right]\frac{\mathbb{1}_{A_0}}{d_{A_0}} = 0\,. \tag{3.8}$$

This linear constraint on M can be expressed in terms of a linear supermap $\mathcal{P}_A : \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right) \to \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$ which is actually an orthogonal superoperator projector. It is defined as:

$$\mathcal{P}_A\left\{M\right\} := M - \mathrm{Tr}_{A_1}\left[M\right] \otimes \frac{\mathbb{1}}{d_{A_1}} + \mathrm{Tr}\left[M\right]\frac{\mathbb{1}_{A_0} \otimes \mathbb{1}_{A_1}}{d_{A_0} d_{A_1}}\,, \tag{3.9}$$

so that condition (3.8) is expressed as a complex linear subspace defined by this projector[6],

6: For completeness, the proof of necessity and sufficiency of Equation (3.10) can be found in the Appendix A.2 of Reference [1].

[1]: Hoffreumon et al. (2021), *The Multi-round Process Matrix*.

$$\mathrm{Tr}_{A_1}\left[M\right] - \mathrm{Tr}\left[M\right]\frac{\mathbb{1}_{A_0}}{d_{A_0}} = 0 \iff \mathcal{P}_A\left\{M\right\} = M\,. \tag{3.10}$$

Hence, the set of channels in CJ representation has support on the linear subspace supporting both the positive cone and the subspace defined by condition (3.6b). Compared to density operators in $\mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$, their span is now smaller than the full space of self-adjoint operators: Equation (3.10) further restricts it to a subspace.

Generalizing from this example, on the one hand, it will be shown that the admissibility condition at each level of the hierarchy guarantees that the

CJ representations of all sets of quantum operations are sets of positive operators. These therefore have support in the space of self-adjoint operators. On the other hand, the TP-preserving condition is expressed as a pair of conditions, one of which being a linear constraint, expressed as a projector, which restricts the support of the set of quantum operations to a complex linear subspace of the space of operators. Combining these two ideas, the linear subspace spanned by a state structure is assumed to be a subspace of the self-adjoint operators.

The convexity of the sets of quantum operations, in conjunction with the second part of the trace-preserving condition (3.7), hints that these sets must always contain an element proportional to the identity operator. The physical interpretation of this fact is randomization: without access to the setting, any probabilistic operation can be thought of as a mixture of several deterministic operations that some party can implement. In particular, the party can always choose to do a uniform mixture of every deterministic procedure so their operation is indistinguishable from white noise. The 'maximally randomized operation' obtained through this procedure is indeed represented by the center of the convex set of positive operators which is proportional to the identity. Alternatively, this is the operation that forbids the transmission of any information between the party's lab and the environment. At all levels of the hierarchy, it corresponds to tracing out the input state, whose CJ representation is the identity operator, followed by repreparing the maximally mixed post-measurement 'state' as output, whose CJ representation is proportional to the identity operator.

Recapitulating, the linear space supporting sets of CJ operators have the structure of a subspace of Hermitian operators containing the identity. This kind of set belongs to a class that has been studied in operator theory under the name *operator system* [102, 103][7] (see References [106, 107] for an up-to-date introduction).

[102]: Kadison (1957), *Unitary Invariants for Representations of Operator Algebras*.
[103]: Choi et al. (1977), *Injectivity and operator spaces*.

7: Definition 3.2.1 is actually the one used by Choi and Effros. Kadison's definition is actually more restricted: it is a real Jordan algebra of self-adjoint operators meaning that it is closed under real-linear combinations as well as the Jordan product $\rho \circ \sigma := \frac{\rho\cdot\sigma+\sigma\cdot\rho}{2}$. In finite dimension, this is a special case of a C^*-algebra called a JC-algebra [104, 105]. In an earlier version of the thesis, the implicit assumption that all elements in a state structure were self-adjoint was made because of the latter definition; Definition 3.2.2 was indeed inspired by the C^*-algebraic formulation of quantum theory. I am grateful to the reviewers for bringing this to my attention.

Definition 3.2.1 (Operator system [103]) *For a given space of operators, an* ***operator system*** *is a subspace that contains the identity and that is closed under the adjoint.*

A real subspace of self-adjoint operators is indeed a special case of an operator system. Now, the actual sets containing the CJ representation of some kind of higher-order transformations are built on top of an even more special kind of operator system: those that are closed with respect to the complex conjugation. This is due to the fact that the adjoint of a linear map is mapped to the conjugate of its Choi operator, see property 8 of Proposition 2.2.1. Hence, a Choi operator and its conjugate essentially represent the same map, and since the interesting operators are the self-adjoint ones, the set of such operators must be closed under the transpose as well. For these reasons, in this thesis when mentioning an 'operator system', it will always be meant 'the real subspace spanned by a set of self-adjoint operators containing the identity and which is closed under the transposition with respect to the basis used to define the CJ isomorphism'.

[104]: Topping (1957), *Jordan Algebras of Self-Adjoint Operators*.
[105]: Effros et al. (1967), *Jordan Algebras of Self-Adjoint Operators*.
[106]: Sinclair et al. (2008), *Finite von Neumann Algebras and Masas*.
[107]: Hiai et al. (2014), *Introduction to Matrix Analysis and Applications*.

Similar to the quantum states and quantum channels cases, the admissibility of a set of maps to represent a set of higher-order operations will be shown to entail the positivity and trace-normalization of the operators.

This in turn restricts the operator system to the cone of positive operators and a hyperplane orthogonal to the identity. These two conditions define the abstract structure that will play a central role in the characterization of higher-order quantum transformations, named *state structure*.

Definition 3.2.2 (State structure) *A* ***state structure*** $\mathscr{A} \subset \mathcal{L}\left(\mathcal{H}^A\right)$ *is a set of positive operators that share the same non-zero trace, which is closed under the transposition, and that spans an operator system.*

The linear span of a state structure will be called *the operator system on which it is defined*. For simplicity, the same script letter will be used to refer to a state structure and the operator system it spans, as it should be clear from the context which one is under consideration. Note that at least two operator systems can be defined on every Hilbert space: itself and its real subspace of self-adjoint operators. In accordance with the above discussion, this is the latter that will implied when talking about the *largest operator system* defined on a space as all operator systems are assumed made of self-adjoint operators. Since a state structure can be defined for any Hilbert space, a special case of which is the one defined on a 1-dimensional system: the *state structure of the trivial system*, which by definition is always normalized to be the number 1.

Definition 3.2.3 (The state structure of the trivial system) *A system associated with a one-dimensional state space is called* ***trivial****; it always has the same* ***trivial state*** 1*. The* ***state structure of the trivial system***[8] *is the state structure in the operator system* $\mathbb{R}$ *that consists of the number* 1*.*

8: The name 'trivial state structure' will also be used as a shortcut for 'state structure of the trivial system'.

Likewise, note that two state structures defined on the same underlying operator system only differ by their normalization: they are isomorphic up to a rescaling of their trace.

The interest of working with state structures is that they abstract the operations into abstract sets that can be compared and combined. The knowledge of the state structure associated with the local operations and two parties, combined with a constraint on their signaling relation, defines a new composite state structure, which represents all the joint operations these parties can do. For example, knowing that two parties can prepare quantum states, inferring that globally they prepare states from the set of bipartite states is an example of combining two state structures into a new one: the set of density operators defined in the space associated with party A and the one of B have been combined into the set of density operators defined in the composite space.

3.2.2. Probabilistic Content: Defining Resolutions

State structure represents sets of deterministic quantum operations. What about the probabilistic content of the theory? The CJ isomorphism has the advantage of being a module homomorphism, meaning it maps the vector space of quantum operations onto a vector space in the CJ representation. Because of that, the convex combinations of two operations are mapped to the same convex combinations of the two operators representing them. The representation of probabilistic operations is accordingly mapped to a resolution of the representation of deterministic operations.

In the motivating example, a POVM is the deterministic 'destructive measurement' operation represented by the unit quantum effect $\{\mathbb{1}\}$ resolved into probabilistic 'measure outcome a' represented by a collection of quantum effects $\{E_a\}$ so that the (probabilistic) effects sum up to the (deterministic) effect,

$$\sum_a E_a = \mathbb{1} \ . \tag{3.11}$$

A coarse-graining of the outcomes amounts to redefining their set as one in which one or more outcomes are mapped to the same new outcome. For example, a coarse-graining may consist of combining two outcomes a_i and a_j into a joint one $\tilde{a}_k$ so that $p(\tilde{a}_k) = p(a_i \cup a_j)$. In terms of the quantum implementation, this coarse-graining is obtained by summing the associated effects, $\tilde{E}_{\tilde{a}_k} = E_{a_i} + E_{a_j}$ [72, 108]. This coarse-graining defines a new valid POVM $\{\tilde{E}_{\tilde{a}}\}$ under the identification

[72]: Kraus (1983), *States, Effects, and Operations: Fundamental Notions of Quantum Theory*.
[108]: Ludwig (1983), *Foundations of Quantum Mechanics I*.

$$\begin{aligned} \{E_a\} \mapsto \{\tilde{E}_{\tilde{a}}\} \ &: \\ \forall l, l \neq i,j,k \ : \ E_{a_l} &= \tilde{E}_{\tilde{a}_l} \ ; \\ E_{a_i} + E_{a_j} &= \tilde{E}_{\tilde{a}_k} \ . \end{aligned} \tag{3.12}$$

In the same manner, a quantum instrument $\{\mathcal{M}_a\}$ can also be coarse- and fine-grained using the addition operation like $\widetilde{\mathcal{M}}_{\tilde{a}} = \mathcal{M}_{a_i} + \mathcal{M}_{a_j}$ [14]. Since the Choi-Jamiołkowski representation is linear, this coarse-graining is also represented through the addition in the CJ picture, i.e.

[14]: Davies et al. (1970), *An operational approach to quantum probability*.

$$\widetilde{\mathcal{M}}_{\tilde{a}} = \mathcal{M}_{a_i} + \mathcal{M}_{a_j} \overset{(2.6)}{\mapsto} M_{\tilde{a}} = M_{a_i} + M_{a_j} \ . \tag{3.13}$$

Since the elements in the quantum instrument are CP, their CJ representation is a collection of positive operators $\{M_a\}$, *the effects at the level of instruments*, resolving a unit effect M. This is a similar structure to the elements of a POVM; but the difference is that there are now more than one unit effects M to sum up to, with each corresponding to the representation of a channel. Accordingly, the setting plays a role more important for instruments than for POVMs because two quantum instruments do not necessarily sum up to the same quantum channel. Consequently, different settings x, x' can now lead to different deterministic operations (unit effects): $\sum_a M_{a|x} \neq \sum_{a'} M'_{a'|x'}$. Compare it to the only effect in the POVM case $\sum_a E_{a|x} = \sum_{a'} E'_{a'|x'} = \mathbb{1}$.

Generalizing from the example, the probabilistic content of the theory is phrased under the concept of a *resolution* by formal analogy[9] with the POVMs as well as the quantum operations of Definition 1.3.4.

9: Remark that nothing guarantees a priori that *every* positive operator represents an experimentally feasible probabilistic operation. Or even that any element of a resolution can be obtained from a combination of elements of a lower-order operation. Therefore, this definition may hide a mathematical hypothesis underlying the theory presented in this thesis; see Appendix C.3.1 for some comments concerning this hypothesis.

Definition 3.2.4 (Resolution of a state structure and resolution of an element of a state structure) *Let $\mathscr{A}$ be a state structure in $\mathcal{L}\left(\mathcal{H}^A\right)$. A set of operators* ***resolving*** *an element of $\mathscr{A}$ is a collection of positive operators summing up to this element. That is, a set of operators $\{E_i\}_{i\in\Omega_i}$ is a* ***resolution of an element*** *of $\mathscr{A}$ if*

$$\forall i \in \Omega_i : \quad E_i \geq 0 \, ; \tag{3.14a}$$

$$\sum_{i\in\Omega_i} E_i \in \mathscr{A} \ . \tag{3.14b}$$

The resolutions can be used to represent anything probabilistic in the state structure. For example, if the state structure is a set of quantum state preparations, the preparation of a state conditioned by a setting is a resolution[10] $\{\rho_x := p(x)\rho_{|x}\}$ of the mean state $\rho = \sum_x p(x)\rho_{|x}$. The mean state preparation is the deterministic operation (unit effect) resolved by the probabilistic operations (effects) consisting of preparing states from the set $\{\rho_{|x}\}$.

10: Notice the multiplication by a probability as it is a conditional distribution; the notation is coherent with a Bayesian interpretation of the states as 'state of knowledge' of the system [109].

[109]: Leifer et al. (2013), *Towards a formulation of quantum theory as a causally neutral theory of Bayesian inference*.

3.2.3. The Projective Characterization of Single-Partite State Structures

As mentioned in the previous sections, the great interest in working in the CJ picture is that all sets have the same abstract structure. This structure in turn has a remarkably simple characterization [35].

[35]: Araújo et al. (2015), *Witnessing causal nonseparability*.

Proposition 3.2.1 (Characterization of a State Structure) *For every state structure $\mathscr{A}$ as in Definition 3.2.2, there exists a unique superoperator projector*[11] $\mathcal{P}_A : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^A)$ *characterizing its span. That is, for every operator* $V_A \in \mathcal{L}(\mathcal{H}^A)$*,*

$$V_A \in \mathscr{A} \iff$$
$$V_A \geq 0\,, \tag{3.15a}$$
$$Tr[V_A] = c_A\,, \tag{3.15b}$$
$$\mathcal{P}_A\{V_A\} = V_A\,, \tag{3.15c}$$

where c_A is a positive real number.

11: Recall that a superoperator is a linear map from a space of operators to itself and that a projector is an idempotent linear map. In the case of a projector characterizing the operator system spanned by a state structure, it belongs to a special kind of projector that will be defined in the following, Definition 3.2.7. See Appendix C.1.3 for a longer introduction of these projectors.

The first two conditions, positivity and normalization, are common to all state structures. They follow from the admissibility of linear maps to represent a higher-order operation (which will be made formal in the following). The positivity is related to generalized complete positivity, thus required for defining arbitrary compositions of state structures. While important, this condition does not tell much about the current state structure at hand because all of them obey this condition. The normalization is trace-preservation, required for the normalization of probabilities. It is fixed for a whole hierarchy of state structures as soon as it has been fixed for the state structure of the lowest order, called the 'base state structure'[12].

12: The base state structure defines what is taken as the 'first order' operation under consideration. Usually taken to be the preparation of quantum states, hence the state structure of density matrices.

Therefore, the relevant bit in the characterization of the higher-order operations is the third line: the projector defining the operator system spanned by the state structure. Two projectors that are defined for all spaces of operators acting on a Hilbert space always result in state structure as in Definition 3.2.2. The first is the identity.

Definition 3.2.5 (Identity Map) *The superoperator* $\mathcal{I} : \mathcal{L}(\mathcal{H}^A) \to \mathcal{L}(\mathcal{H}^A)$ *defined by*

$$\forall V_A,\ \mathcal{I}_A(V_A) = V_A\,, \tag{3.16}$$

is called the ***identity map****. Its CJ representation*[13] *is the maximally entangled operator*

$$\mathfrak{C}(\mathcal{I}) = \sum_{i,j} |i\rangle\langle j|_{A_0} \otimes |i\rangle\langle j|_{A_1}\ , \tag{3.17}$$

13: As in Equation (2.8).

where the input and output spaces have been labeled to disambiguate them: $\mathcal{H}^{A_0} \cong \mathcal{H}^{A_1} \cong \mathcal{H}^{A}$.

The second is the depolarizing superoperator, because an operator system always contains a maximally mixed element.

Definition 3.2.6 (Depolarizing Superoperator) *The superoperator* $\mathcal{D} : \mathcal{L}\left(\mathcal{H}^{A}\right) \to \mathcal{L}\left(\mathcal{H}^{A}\right)$ *defined by*

$$\mathcal{D}_A(V_A) := \frac{\mathbb{1}_A}{d_A} Tr\left[V_A\right] , \tag{3.18}$$

is called the ***depolarizing superoperator****. Its CJ representation is proportional to the identity operator*

$$\mathfrak{C}\left(\mathcal{D}\right) = \sum_{i,j} |i\rangle \langle i|_{A_0} \otimes \frac{|j\rangle \langle j|_{A_1}}{d_{A_1}} = \mathbb{1}_{A_0} \otimes \frac{\mathbb{1}_{A_1}}{d_{A_1}} , \tag{3.19}$$

where the input and output spaces have been labeled to disambiguate them: $\mathcal{H}^{A_0} \cong \mathcal{H}^{A_1} \cong \mathcal{H}^{A}$.

These two projectors are examples of *projectors on an operator system*. The identity is associated with the operator system of all self-adjoint operators since it projects the space on itself. Thus, this is the biggest operator system that can be defined on a space of operators. Whereas the depolarizing superoperator projects on the span of the identity, which is the smallest one[14].

The projectors characterizing state structure are defined in between these two cases. The only requirement to projector on a state structure is that, like the identity and depolarizing superoperators, they must preserve the identity and be closed under adjoints.

Definition 3.2.7 *A* ***projector on an operator system*** $\mathcal{P}_A : \mathcal{L}\left(\mathcal{H}^{A}\right) \to \mathcal{L}\left(\mathcal{H}^{A}\right)$ *is a linear orthogonal superoperator projector to a subspace closed under the adjoint that contains the identity.*
Mathematically[15]*, it is a linear superoperator* $\mathcal{P}_A \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A}\right), \mathcal{L}\left(\mathcal{H}^{A}\right)\right)$ *that obeys the following conditions:*

$$\mathcal{P}_A \circ \mathcal{P}_A = \mathcal{P}_A ; \tag{3.20a}$$
$$\mathcal{P}_A^* = \mathcal{P}_A ; \tag{3.20b}$$
$$\mathcal{P}_A \circ \dagger \circ \mathcal{P}_A = \dagger \circ \mathcal{P}_A ; \tag{3.20c}$$
$$\mathcal{P}_A \circ \mathcal{D}_A = \mathcal{D}_A , \tag{3.20d}$$

where '' indicates the adjoint in* $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A}\right), \mathcal{L}\left(\mathcal{H}^{A}\right)\right)$ *and where '†' means taking the adjoint in* $\mathcal{L}\left(\mathcal{H}^{A}\right)$.

Remark that the first two conditions in the definition are just the definition of an orthogonal projector.

The handiness of using projectors is that most of the properties of the state structure they characterize are encoded in them. Furthermore, the relations and combinations of local state structures with global state structures –for example, the state structure obtained by the parallel

14: Notice that these two projectors are moreover always valid operations on any state structure. The identity is the 'do nothing and pass the system on', whereas the depolarizing is the 'replace by white noise' operation. The fact that they are valid higher-order operations, no matter the level of the hierarchy, will play a role in interpreting the characterization method as a lattice, leading to the logic-like structure of the characterization presented in Chapter 5.

15: The exact mathematical phrasing of the conditions (3.20) is derived around Equations (C.20), (C.25), (C.28), and (C.31) presented in Appendix C.1 alongside a reminder on some of the properties of projectors.
Some examples of projectors on operator systems in the Pauli basis are presented in Appendix C.1.4.

composition of two local ones, or the state structure transforming a state structure into another one (representing a higher-order transformation)– can be expressed as composition rules on the projectors. The signaling constraints between the operations of several parties are encoded into their global state structure which is characterized by a global projector. The idea of projective characterization is breaking the global projector into local projectors combined through operations reflecting the signaling constraints. These rules are what is derived in the following sections.

3.3. The Projective Characterization of States and Effects

The first relation between state structures considered in the characterization is those forming states and effects pairs. Consider the two projectors introduced in the previous section, $\mathcal{I}$ and $\mathcal{D}$. The first projector is associated with the state structure of quantum states in density matrix form, $\rho \in \mathcal{L}(\mathcal{H})$, characterized by

$$\rho \geq 0 \,, \tag{3.21a}$$
$$\mathrm{Tr}\,[\rho] = 1 \,, \tag{3.21b}$$
$$\mathcal{I}\{\rho\} = \rho \,. \tag{3.21c}$$

The second projector is associated with the state structure of the quantum unit effect, characterized by

$$\mathbb{1} \geq 0 \,, \tag{3.22a}$$
$$\mathrm{Tr}\,[\mathbb{1}] = d \,, \tag{3.22b}$$
$$\mathcal{D}\{\mathbb{1}\} = \mathbb{1} \,. \tag{3.22c}$$

This latter state structure made up of a single element $\{\mathbb{1}\}$, is what the elements of a POVM, the effects, are summing up to, i.e., *are resolving*. These two structures are not independent of each other. On the contrary, they define each other through the inner product; a valid effect always sends a valid state to a probability as

$$\mathrm{Tr}\left[E_a^\dagger \cdot \rho\right] = p(a) \,. \tag{3.23}$$

In that regard, the quantum states are linear functional from the resolutions of the unit quantum effect to a probability. This is actually a well-known result, called the Gleason-Busch theorem [110, 111]: imposing a measure[16] on the set of observables uniquely fixes the set of states.

[110]: Gleason (1957), *Measures on the Closed Subspaces of a Hilbert Space*.
[111]: Busch (2003), *Quantum States and Generalized Observables: A Simple Proof of Gleason's Theorem*.

16: In the sense of a measurable space.

3.3.1. Effects as Functionals

This theorem can be generalized to state structures. Following the terminology introduced in a series of papers generalizing the original theorem on projective measurements to POVMs [112, 113] and beyond [60, 114, 115], the possible resolutions of a state structure can be seen as a σ-algebra whose intersections and unions are represented by, respectively, multiplication and addition of operators. As a consequence, a measure can be defined on it. Concretely, a function mapping each effect to a

[112]: Caves et al. (2004), *Gleason-Type Derivations of the Quantum Probability Rule for Generalized Measurements*.
[113]: Barnett et al. (2014), *Quantum probability rule: a generalization of the theorems of Gleason and Busch*.
[60]: Shrapnel et al. (2018), *Updating the Born rule*.
[114]: Flatt et al. (2017), *Sequential measurements: Busch-Gleason theorem and its applications*.
[115]: Wright et al. (2019), *A Gleason-type theorem for qubits based on mixtures of projective measurements*.

non-negative real number can be defined. All such measures are called *frame functions* [110].

[110]: Gleason (1957), *Measures on the Closed Subspaces of a Hilbert Space*.

Definition 3.3.1 (Frame Function on a State Structure) *Let $\mathscr{A}$ be a state structure in $\mathcal{L}\left(\mathcal{H}^A\right)$. A* ***frame function*** *on this state structure is a (real-valued) functional $f : \mathcal{L}\left(\mathcal{H}^A\right) \to \mathbb{R}$ so that, for all $N \in \mathscr{A}$, for all resolutions $\{E_a\}_{a=1}^{|\Omega_a|}$ of N, and for all arbitrary sequences $\{E_i\} := \{E_j, E_k, \ldots\}_{\{j,k,\ldots\}\subset\Omega_a}$ in the resolutions, the following holds:*

$$f(N) = 1\,; \tag{3.24a}$$
$$f(E_i) \in [0, 1]\,; \tag{3.24b}$$
$$f(E_j + E_k + \ldots) = f(E_j) + f(E_k) + \ldots\,. \tag{3.24c}$$

This definition relies on two hypotheses: that the frame functions are non-contextual[17] and that they associate a probability to all effects in a homogeneous manner, not just to each set of mutually orthogonal projectors[18]. These hypotheses are enough to enforce linearity for all frame functions, whence the frame functions are elements of the dual space $\mathcal{L}\left(\mathcal{H}^A, \mathbb{C}\right) = \mathcal{L}\left(\mathcal{H}^A\right)^*$. As a consequence of the Riesz theorem, these are representable as dual vectors on the same Hilbert space.

17: In the Gleason sense, i.e. similarly to Spekkens' measurement non-contextuality, and not in the Kochen-Speckers one, see [116] for disambiguation.

[116]: Budroni et al. (2022), *Kochen-Specker contextuality*.

18: Both hypotheses have been criticized for the case of POVMs in the literature. Moreover, as with defining resolutions for state structures, defining deterministic functional to be normalized on arbitrary resolutions of state structure instead of POVMs actually may induce an extra hidden mathematical hypothesis in the model. As is the case for the definition of a resolution, the frame function is a definition obtained by formal analogy and that results in the model this thesis is about. The interpretation of the hypotheses their definitions require is left open for future work; for some preliminary elements, see the discussion in Appendix C.3.2.

Lemma 3.3.1 *The set of all frame functions $\{f\} \subset \mathcal{L}\left(\mathcal{H}^A\right)^*$ can be represented as a set $\{V\}$ contained in $\mathcal{L}\left(\mathcal{H}^A\right)$ which forms a complex subspace. Their action is given by the inner product so that*

$$\begin{gathered} \mathcal{L}\left(\mathcal{H}^A\right)^* \ni f \mapsto V \in \mathcal{L}\left(\mathcal{H}^A\right)\ : \\ \forall X \in \mathcal{L}\left(\mathcal{H}^A\right), \quad f(X) = (V\,,\,X) = \mathit{Tr}\left[V^\dagger \cdot X\right]\,. \end{gathered} \tag{3.25}$$

Proof. Any N as in Definition 3.3.1 can be taken as proportional to the identity element. Since a frame function must also obey the conditions for $N = \frac{c_A}{d_A}\mathbb{1}$, they can be seen as a frame function on $\{\mathbb{1}\}$ and so the Gleason-Busch theorem applies [111], which prove that these maps are linear functionals. By Riesz representation theorem, linear functionals can be identified with a unique element $V \in \mathcal{L}\left(\mathcal{H}^A\right)$ so that Equation (3.25) holds.

[111]: Busch (2003), *Quantum States and Generalized Observables: A Simple Proof of Gleason's Theorem*.

It remains to show that the operator representations of all linear functionals, the set $\{V\}$, is a subspace of $\mathcal{L}\left(\mathcal{H}^A\right)$. For a state structure $\mathscr{A}$, let an orthogonal family of self-adjoint operators $N_i \in \{N_1, N_2, \ldots, N_{d_\mathscr{A}}\}$ be such that it spans this set. Then, condition (3.24a) defines a set of linearly independent constraints on all V,

$$(N_i\,,\,V) = 1\,, \tag{3.26}$$

which defines an affine subspace of dimension $d_A - d_\mathscr{A}$ (where $d_\mathscr{A}$ is the dimension of the affine span of $\mathscr{A}$) . □

Interpreting the frame functions, they are deterministic operations from effects to probabilities: they are the general notion of a deterministic operation but with either trivial input or output space, like a state preparation or a destructive measurement. Following the literature, these are called *deterministic functionals* [10, 11]. They generalize the notion of a *unit effect* [79, 81] by allowing the existence of more than one in the

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.

[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

[79]: Plávala (2021), *General probabilistic theories: An introduction*.

[81]: Coecke et al. (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*.

19: In the sense that the different unit effects are distinguishable by some discrimination task performed using resolution of state structures.

theory[19]. Note that the two terms will be used loosely to talk about the set of linear functionals sending the elements of a state structure to the number 1 as well as the set of their representation as vectors in $\mathcal{L}\left(\mathcal{H}\right)$. All deterministic functionals on a state structure are thus (representable as) operators V mapping all resolutions to given distributions,

$$(V\ ,\ E_a) = p(a|\{E_a\}, N, V)\ , \tag{3.27}$$

so that summing over the resolution amounts to forgetting the value of a, making it a *deterministic procedure* which gives a probability of 1:

$$\sum_a (V\ ,\ E_a) = 1\ . \tag{3.28}$$

Formally, by applying Lemma 3.3.1 to the set of all frame functions, their definition can be rephrased as a set of operators, called the deterministic functionals.

Definition 3.3.2 (Deterministic Functional) *The representation in $\mathcal{L}\left(\mathcal{H}^A\right)$ of a frame function on state structure $\mathscr{A}$ is called a **deterministic functional** (or **unit effect**). The set of all deterministic functional on $\mathscr{A}$ is noted $\overline{\mathscr{A}}$; it contains all operators which take each element of $\mathscr{A}$ to the number 1 through the inner product,*

$$\forall V \in \overline{\mathscr{A}},\ \forall N \in \mathscr{A}\ :\ \mathit{Tr}\left[V^\dagger \cdot N\right] = 1\ , \tag{3.29}$$

and which take each element E_a of every resolution of N to a positive number between 0 and 1, i.e.,

$$\begin{aligned} \forall N \in \mathscr{A}\ ,\ \forall\{E_a\}\ :\ E_a \geq 0\ ,\ \sum_a E_a = N\ , \\ \mathit{Tr}\left[V^\dagger \cdot E_a\right] \in [0,1]\ . \end{aligned} \tag{3.30}$$

With the defining conditions (3.24) rephrased as inner products, the characterization is now a linear problem.

Theorem 3.3.2 (Characterization of Deterministic Functionals) *Let $\mathscr{A}$ be a state structure. Let $\{E_a\}$ be a resolution of an element of $\mathscr{A}$ as in Definition 3.2.4. Let $\overline{\mathscr{A}}$ be the set of all deterministic functionals as in Definition 3.3.2. Then, $\overline{\mathscr{A}}$ is a state structure characterized by the following conditions:*

$$V \in \overline{\mathscr{A}} \iff$$

$$V \geq 0\ , \tag{3.31a}$$

$$\mathit{Tr}\left[V\right] = \frac{d_A}{c_A} =: c_{\overline{A}}\ , \tag{3.31b}$$

$$\mathcal{P}_{\overline{A}}\{V\} := \{\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A\}\left(V\right) = V\ . \tag{3.31c}$$

This theorem is a generalization of various results obtained in previous works on higher-order quantum operations [1, 10, 11, 34, 35, 92][20]. A full proof is presented in Appendix C.2.1.

[1]: Hoffreumon et al. (2021), *The Multi-round Process Matrix*.
[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.
[34]: Hoffreumon (2019), *Processes with indefinite causal structure in quantum theory: The Multi-Round Process Matrix*.
[35]: Araújo et al. (2015), *Witnessing causal nonseparability*.
[92]: Bavaresco et al. (2021), *Strict Hierarchy between Parallel, Sequential, and Indefinite-Causal-Order Strategies for Channel Discrimination*.

20: Its first projective formulation was derived in my work on the Multi-round Process Matrices (MPM) but applied to the restricted case of process matrices and quantum combs [1, 34]. It appeared in a similar projective form but for restricted cases in Appendix B of [35] as well as Theorem 2 in [92]; Its non-projective formulation appeared at Lemma 2 in [10] as well as Lemma 4 in [11].

The set characterized in Theorem 3.3.2 will be called the state structure *complementary* to $\mathscr{A}$ in the following. This defines some sort of duality with respect to condition (3.29): an element of $\overline{\mathscr{A}}$ *is the need of an element* *of $\mathscr{A}$ in order to obtain the number 1.*

Definition 3.3.3 *The state structure $\overline{\mathscr{A}}$ as defined by Equations* (3.31) *is called the* ***dual*** *or* ***complementary*** *state structure to $\mathscr{A}$ defined by Equations* (3.15).

In addition, the following corollary should be obvious from the theorem:

Corollary 3.3.3 *The double dual of a state structure is equal to itself,*

$$\mathscr{A} = \overline{\overline{\mathscr{A}}} \,. \tag{3.32}$$

The projector characterizing the dual state structure can be seen as being obtained from an operation applied on the projector associated with the original state structure. This operation called the **negation** (of a projector) is defined using the 'bar-over-the-original' notation

$$\overline{\mathcal{P}} := \mathcal{I} - \mathcal{P} + \mathcal{D} \,. \tag{3.33}$$

It should be clear that this operation defines a new projector $\overline{\mathcal{P}}$ out of the original one, i.e. that $(\overline{\mathcal{P}})^2 = \overline{\mathcal{P}}$. In the appendix to Chapter 5, which is concerned with operations on projectors as a means to characterize state structures, it will be further shown that this newly defined projector is a projector on operator system as in Definition 3.2.7. Moreover, as commutation is preserved for all linear combinations of projectors, it is also proven that a projector always commutes with its negation, see Appendix D.2.3. As a consequence, the intersection of the two state structures defined by projectors $\mathcal{P}$ and $\overline{\mathcal{P}}$ can be defined as their composition[21], which yields the depolarizing superoperator,

21: This standard result is reminded in Appendix C.1.3, Proposition C.1.3.

$$\mathcal{P} \circ \overline{\mathcal{P}} = \mathcal{D} \,. \tag{3.34}$$

This equation shows that the intersection of the span of a state structure and its dual is the span of the identity. This has an interpretation in terms of the statistical structure of the operations, as discussed next.

3.3.2. Statistical Interpretation

The situation represented by a pair $\{\mathscr{A}, \overline{\mathscr{A}}\}$ has interpretation in the context of a single-partite process. The process is a deterministic functional picked in $\overline{\mathscr{A}}$; it represents the given environment[22] of party Alice. The probabilistic operation of said party is then a quantum operation resolving an element of $\mathscr{A}$. The environment is thus represented by a state V, and the party will apply an averaged unit effect $N \in \mathscr{A}$ on it. The single-partite process thus generalizes the quantum state and effect scenario into a state and effect pair taken from $\{\mathscr{A}, \overline{\mathscr{A}}\}$.

22: Which is equivalently a special party that always chooses the same deterministic strategy, consisting of preparing a fixed $V \in \overline{\mathscr{A}}$.

However, the party's operation can also depend on her setting. In this case, it is lifted from a resolution (generalizing the quantum instruments) to the ad hoc generalization of a quantum operation, Definition 1.3.4, when applied to resolutions of state structures.

Definition 3.3.4 (Operation (from a state structure)) *Let Alice be a party whose deterministic operations are represented as the elements of $\mathscr{A}$.*
When conditioned by setting x and outcome a, Alice's active part in her intervention is represented as their choice of a ***(probabilistic) operation*** *from her state structure $\mathscr{A}$ to be applied on some locally*[23] *dual state structure $\overline{\mathscr{A}}$.*
An operation consists of a collection of deterministic elements $\{N_{|x=0}, N_{|x=1}, \ldots\} \in \mathscr{A}$, her ***deterministic operations****, indexed by the setting x. Each of her deterministic elements are in turn resolved into her probabilistic elements as in Definition 3.2.4: $\{\{N_{a|x=0}\}, \{N_{a|x=1}\}, \{N_{a|x=2}\} \ldots\}$.*
The related ***averaged operation*** *N is the action of the operation when averaged over all of its settings and outcomes, i.e. it is the quantum channel defined by*

$$N := \sum_{a,x} p(x)\, N_{a|x} \;, \tag{3.35}$$

where $p(x)$ is the distribution of the setting.

23: 'Locally' means that $\overline{\mathscr{A}}$ can in general be only a factor in a bigger state structure.

The actual element $N_{a|x}$ that she will implement is determined by the realization of a, but it is independent of x by intervention non-contextuality. However, the values a *might have* taken are dependent on the choice of the given set $\{N_{a|x=0}\}$ (the context). This implies that Alice's realized deterministic operation, $N_{|x=0}$, is dependent on the realized value of the setting, in that case $x = 0$. The distribution of Alice's settings and outcomes for a given choice of operation is thus given by

$$p(a|x) := p(a|x, \{N_{a|x}\}, V) = \left(N_{a|x}\,,\, V\right) := \mathrm{Tr}\left[N_{a|x}^{\dagger} \cdot V\right] \;. \tag{3.36}$$

Of course, this joint distribution is also conditional on the choice of process V. When it will be needed to make this choice explicit, an 'environment setting' y can be introduced so that two different choices of V are labeled by different choices of y, i.e. such that $y \neq y' \iff V_{|y} \neq V_{|y'}$.

Then, different processes correspond to different probability distributions conditioned by random variable y,

$$p(a|x, y) = \left(N_{a|x}\,,\, V_{|y}\right) \;. \tag{3.37}$$

And when the environment is different, $V_{|y} \neq V_{|y'}$, then the processes are different in the sense that there exists at least one operation that can be used to distinguish between the two,

$$\exists a, x, \qquad p(a|x, y) \neq p(a|x, y') \;. \tag{3.38}$$

The probability of occurrence of a given value a is again always independent of the choice of resolution $\{N_{a|x}\}_{x \text{ fixed}}$. This is the intervention non-contextuality [51]: any outcome a appearing in two different resolutions $a \mapsto a|x$ and $a \mapsto a|x'$, $x \neq x'$, should result in the same probability distribution $p(a|x, y) = p(a|x', y)$. In that case, the elements of each resolution have to be the same,

[51]: Shrapnel et al. (2018), *Causation does not explain contextuality*.

$$\forall y, \qquad p(a|x, y) = p(a|x', y) \iff N_{a|x} = N_{a|x'} \;, \tag{3.39}$$

although the measured unit effects can be different, i.e. nothing can be said about whether $N_{|x} = N_{|x'}$.

With regard to that, each effect $N_{a|x}$ of the operation corresponds to a conditional distribution $p(a|x)$ of outcomes a for a given setting x (and for a given state V). Without the knowledge of x nor a, the operation is averaged over all choices of settings and outcomes as the unit effect $N := \sum_x p(x) \sum_a N_{a|x}$; without the knowledge of the outcome, the operation is the unit effect $N_{|x} := \sum_a N_{a|x}$. In these two cases, the distribution sum up to a probability of 1:

$$\big(N_{|x}\,,\, V\big) = \sum_a \big(N_{a|x}\,,\, V\big) = \sum_a p(a|x) = 1\,; \tag{3.40a}$$

$$\big(N\,,\, V\big) = \sum_x p(x) \sum_a \big(N_{a|x}\,,\, V\big) = \sum_x p(x) = 1\,. \tag{3.40b}$$

The only difference is that in the first case, Alice still knows which resolution she chose to apply, represented by the variable x.

Similar to how the conditional distribution can recover the joint distribution of settings and outcomes through $p(a|x)p(x) = p(a,x)$, the probabilistic operation $\{N_{a,x}\}$ with elements defined by[24]

$$N_{a,x} = p(x) N_{a|x} \tag{3.41}$$

recovers the joint distribution of outcomes and settings

$$\big(N_{a,x}\,,\, V\big) = p(a,x)\,. \tag{3.42}$$

Defining such an $N_{a,x}$ amounts to treating the setting x as another outcome. In other words, losing the a priori knowledge of which realization of x happened.

24: This actually follows from the freedom of choice assumption, see Appendix C.3.3.

3.3.3. Quasi-Orthogonality

The joint probability distribution of Alice outcome a and setting x given an environment picked through random variable y is thus given by

$$p(a,x|y) = \big(p(x)\, N_{a|x}\,,\, V_{|y}\big)\ . \tag{3.43}$$

The joint distribution of all three random variables is accordingly given by

$$p(a,x,y) = p(a,x|y)p(y) = \big(p(x)\, N_{a|x}\,,\, p(y)\, V_{|y}\big)\ . \tag{3.44}$$

While the freedom of choice assumption asserts that the settings are independent[25], $p(x,y) = p(x)p(y)$, the fact that all pairings $(V,N) \in (\mathscr{A}, \overline{\mathscr{A}})$ give the number 1 is the assumption that, on average, the freedom of choice assumption is also verified by all choices of operation.

25: In the context of state and effect pair, it requires the choice of strategy on the state side to have no deterministic influence on the choice of strategy on the effect side and vice-versa.

By 'on average' it is meant that, even when the environment is seen as a probabilistic operation under the control of some local party, when summing over the outcomes a and b, the joint distribution of settings is independent. That is, the following sum

$$\begin{aligned}\sum_{a,b} p(a,b,x,y) = \sum_{a,b} \big(p(x)N_{a|x}\,,\, p(y)V_{b|y}\big) &= \big(p(x)N_{|x}\,,\, p(y)V_{|y}\big) \\ &= p(x)p(y)\,\big(N_{|x}\,,\, V_{|y}\big)\end{aligned} \tag{3.45}$$

should be equal to $p(x)p(y)$ for all choices of $V_{|y} \times N_{|x} \in (\mathscr{A}, \overline{\mathscr{A}})$. Otherwise, the choice of the operation of the party is not independent of the choice of its environment. Seen as a state and effect pair, the choice of state V should be independent of the choice of measurement N, although the outcomes of the measurement, a, can be correlated with the choice of V, y and correspondingly, although that b can be correlated with the choice of N, x. Therefore, the defining condition

$$\forall N \in \mathscr{A}, \forall V \in \overline{\mathscr{A}}, \quad (N\,,V) = 1 \tag{3.46}$$

of a 'state and effect' pair of state structures is the (almost tautological) statement that, in a local lab, any deterministic operation (unit effect) on a given environment occurs with a probability of 1 (hence the name). By doing something authorized by the theory, it is certain that *something* has been observed. The knowledge of *what* was observed is obtained through a probabilistic operation (effect).

Theorem 3.3.2 guarantees condition (3.46) to be true whenever the support of the unit effects is orthogonal to the support of the states everywhere but at the identity. This condition is called *quasi-orthogonality* [117][26]. Quasi-orthogonality in turn implies the following property [107, Theorem 2.37 iii)]:

$$\forall V \in \mathscr{A}, \forall N \in \overline{\mathscr{A}}\,, \quad \mathrm{Tr}\,[N \cdot V] = \frac{1}{d_A}\mathrm{Tr}\,[N]\,\mathrm{Tr}\,[V]\ . \tag{3.47}$$

Combining the trace conditions and Equation (3.47) yields the following.

$$\mathrm{Tr}\,[N \cdot V] = \mathrm{Tr}\left[N \cdot \frac{\mathbb{1}}{c_{\overline{A}}}\right] \mathrm{Tr}\left[\frac{\mathbb{1}}{c_A} \cdot V\right]\ . \tag{3.48}$$

This suggests the interpretation of the above as a concrete instance of the difference between randomizing and acting probabilistically: a state and effect pair are two sets of deterministic operations such that the choice of an operation in one set cannot modify the probability distribution seen by the other set; it is as if the other always chooses to do a maximally mixed operation. However, keep in mind that it is always possible to see an influence probabilistically. Indeed, the party on the effect side in Equation (3.48) can still sometimes learn that some V was chosen rather than some other V' by applying a suitable probabilistic resolution $\{N_a\}$ of N; for certain choices of resolutions the probability of seeing certain outcomes will be different: $p(a|\{N_a\}, V) = \mathrm{Tr}\,[N_a \cdot V] \neq \mathrm{Tr}\left[N_a \cdot \frac{\mathbb{1}}{c_{\overline{A}}}\right] = p(a|\{N_a\}, V' = \frac{\mathbb{1}}{c_{\overline{A}}})$. Hence, by repeating the procedure enough times, V and V' can be discriminated by tomography. However, the same can never be achieved deterministically, even if the experiment is repeated a large number of times and the party on the effect side chooses to apply a randomized choice of deterministic functionals like $\{p(x)N_{|x}\}$. This is because all $N_{|x}$ belong to $\overline{\mathscr{A}}$ and therefore $\mathrm{Tr}\left[N_{|x} \cdot V\right] = \mathrm{Tr}\left[N_{|x} \cdot \frac{\mathbb{1}}{c_{\overline{A}}}\right]$ for all choices of $N_{|x}$ and V. Quasi-orthogonality is then the property that no deterministic functional can be applied to a state in order to deterministically gain information about it; repeated probabilistic procedures are needed to distinguish any state from the maximally mixed one.

[117]: Petz (2007), *Complementarity in quantum systems*.

26: Note that this condition was first considered as the definition of statistical independence in the context of Algebraic Quantum Field Theory (AQFT) [118, III.A].

[118]: Haag et al. (1964), *An Algebraic Approach to Quantum Field Theory*.

[107]: Hiai et al. (2014), *Introduction to Matrix Analysis and Applications*.

3.3.4. Statistical Dependence of State and Effects

While Equation (3.48) can be seen as the operator version of the independence of settings, it does not entail the independence of the party from its environment in the sense that the joint outcome distribution has no factorization like $p(a,b|x,y) = p(a|x)p(b|y)$ as it would require the analog of (3.48) to hold for *all* effects not just the unit ones,

$$\mathrm{Tr}\left[N_{a|x} \cdot V_{b|y}\right] = \frac{1}{d_A}\mathrm{Tr}\left[N_{a|x}\right]\mathrm{Tr}\left[V_{b|y}\right] , \tag{3.49}$$

which is obviously impossible as both operators can be any positive element of $\mathcal{L}\left(\mathcal{H}^A\right)$. Moreover, this influence can be used to signal from one side to the other. When seeing $V_{|y}$ as under the control of some party Bob which plays the role of Alice's environment, the information about this environment, represented by variable y, can be obtained through a suitable choice of operation of the party so that she can distinguish between different y, y' which corresponds to different choices $V_{|y}, V_{|y'}$, $V_{|y} \neq V_{|y'}$.

Suppose now that the environment is behaving like a party, in the sense that Bob can also do probabilistic operations $V_{b|y}$. In that case, Alice can obtain information about the average operation of the party in the same manner: a suitable choice of resolution can help them distinguish probabilistically between two different choices x, x'. This point is contained in the following two statements:

$$\forall\{V_{b|y}\},\{V_{b|y'}\} : \; V_{|y} \neq V_{|y'}, \quad \exists N_{a|x} : \quad \sum_b \mathrm{Tr}\left[N_{a|x} \cdot V_{b|y}\right] \neq \sum_b \mathrm{Tr}\left[N_{a|x} \cdot V_{b|y'}\right] ; \tag{3.50a}$$

$$\forall\{N_{a|x}\},\{N_{a|x'}\} : \; N_{|x} \neq N_{|x'}, \quad \exists V_{b|y} : \quad \sum_a \mathrm{Tr}\left[N_{a|x} \cdot V_{b|y}\right] \neq \sum_a \mathrm{Tr}\left[N_{a|x'} \cdot V_{b|y}\right] . \tag{3.50b}$$

These are indeed signaling distributions as in Definition 1.2.1. Substituted by their probability distribution, these equations are exactly the definition of signaling

$$\exists a, x, y \neq y', \quad \sum_b p(a,b|x,y) \neq \sum_b p(a,b|x,y') ; \tag{1.8a}$$

$$\exists b, x \neq x', y, \quad \sum_a p(a,b|x,y) \neq \sum_a p(a,b|x',y) . \tag{1.7b}$$

That is to say, the local information of the environment can be sent to the party and the local information of the party can be sent to the environment.

Notice that the quantum states and effects have a particular behavior with respect to that fact. As there is only one unit quantum effect $N_{|x} = \mathbb{1}$ for all possible settings and strategies, it does not satisfy condition (3.50b). The different choices of settings all lead to the same deterministic operation. This is a convoluted way of arriving at the fact that quantum theory does not allow for procedures that deterministically lead to postselection. Still, in Chapter 5 this fact will be essential for understanding some peculiar isomorphisms that only happen at the lowest levels of the hierarchy of higher-order quantum processes.

3.4. The Projective Characterization of Higher-Order Transformations

Whereas the state structure abstracts any higher-order object, the definition of a deterministic functional (Definition 3.3.2) and its projective characterization (Theorem 3.3.2) can be seen as the abstract generalization of the single-partite process functional that was presented in Subsection 1.3.5. This generalization is induced by the abstraction of a quantum operation (Definition 1.3.4) into a higher-order operation (Definition 3.3.4) since both the process and the deterministic functional are a mapping from their corresponding notion of an operation to a probability. Following the logic of the first chapter, the next thing to be defined after and characterized after the functional is then the transformation between two state structures, so that it abstracts the notion of admissible higher-order quantum transformations presented in Section 1.4 (Definition 1.4.2) to the notion of admissible transformation between state structures (Definition 3.4.3).

Indeed, a state-and-effect dual pair like $\mathscr{A}$ and $\overline{\mathscr{A}}$ only represent a specific kind of intervention where the party prepares then destructively measures a higher-order object whose state is an element of the state structure $\mathscr{A}$. But in between, the system often evolves, so its state changes. This evolution, as it sends an element of $\mathscr{A}$ to another, is a map $\mathcal{M} : \mathscr{A} \to \mathscr{A}$. More generally, this evolution may represent the intervention of another party in between the two stages in a similar fashion to a quantum channel. This kind of deterministically controlled modification of the evolution is represented by a mapping on the state structure $\mathscr{A}$ to itself, i.e. a *superoperator*[27], whose resolution can represent the probabilistic intervention of the in-between party in a similar fashion to a quantum instrument. Also, nothing prevents it from outputting a system in a state of a different state structure. This is precisely what the preparation and measurement are: as mentioned in the introduction (and as is explained in the discussion of the graphical methods in Appendix A.3; see Figure A.1b), a preparation (respectively, a measurement) is a *transformation* from an element of the state structure of the trivial system[28] 1 to an element of a state structure $\mathscr{A}$, $\mathscr{A} \ni V_A : 1 \to \mathscr{A}$ (respectively, from $\mathscr{A}$ to the state structure of the trivial system, $\overline{\mathscr{A}} \ni N_A : \mathscr{A} \to 1$).

27: As was presented in Section 1.4.

28: Which is the 0-dimensional set $\{1\}$ defined over the 1-dimensional operator system $\mathbb{R}$, see Definition 3.2.3.

Therefore, any element of a state structure is interpretable as a map from the trivial state structure to the element. Following this logic, maps from one state structure to another state structure are maps on maps, i.e. higher-order maps. In that sense, the notion of an order is relative. All state structures represent a higher-order mapping of some kind. Once the state structures associated with the local parties have been defined, the higher-order maps *relative to the local state structures* are the maps that relate two local state structures together. Note that 'local state structures' will be after that called 'base state structures' to avoid ambiguities with quantum non-locality[29]: a **base state structure** is a non-composite state structure representing all the deterministic interventions a local party can perform in a single round. For example, the base state structures in a bipartite process can be $\mathscr{A}$ and $\mathscr{B}$, where $\mathscr{A}$ is a set of quantum states that Alice can prepare in her local lab and $\mathscr{B}$ is a set of quantum channels that Bob can realize in his own local lab. The process itself, modeling their environment, is a higher-order map relating these two state structures.

29: For example, by writing a confusing statement like 'a non-local state from a local state structure', since nothing forbids a priori a party to prepare a bipartite system in their lab. If the base state structure of a party is the set of 4-dimensional quantum states, its states represent the preparation procedure of a 4-dimensional quantum system. Yet, the resulting quantum system can have a non-local quantum state with respect to a 2-dimensional partitioning (see References [119, 120] for instance).

[119]: Zanardi (2001), *Virtual Quantum Subsystems*.
[120]: Zanardi et al. (2004), *Quantum Tensor Product Structures are Observable Induced*.

In the following, it will be shown how sets of admissible higher-order maps can be built and characterized entirely from the two state structures they relate. The interest in working with state structures is that higher-order maps are state structures themselves, so the characterization of higher-order maps is recursive: a third-order map is characterized by the second-order maps it relates using the exact same formula for how these second-order maps are characterized by the first-order maps they relate. Once this general formula is known, i.e. Theorem 3.4.1, any set of higher-order maps can be built out of the knowledge of the lower-order maps it relates.

3.4.1. Higher-Order Maps as Admissible Transformations

But what is an admissible higher-order transformation between state structures? As a recursive application of the operational definition of a map [14, 53, 72, 76, 121, 122] refined and formalized in more recent works [5, 8, 10, 11, 13, 33, 46, 60, 82–84, 88, 123], the short answer given by Definition 1.4.2 in Section 1.4 is "a completely CP-preserving TP-preserving linear map between state structures". From the mathematical viewpoint, this definition makes sense since the completely positive maps are the appropriate morphisms (transformations) between state structures [103], and trace-preservation is required to keep normalized probabilities.

In terms of the statistical interpretation, a broad interpretation compatible with all these sources[30] is "any map that preserves the probabilistic structure between state structures, even locally". By this, it is meant that starting from a set of local parties, say A, B, and C, each of which have the ability to prepare and measure systems in a state taken from their base state structure, say $\mathscr{A} \subset \mathcal{L}\left(\mathcal{H}^A\right)$, $\mathscr{B} \subset \mathcal{L}\left(\mathcal{H}^B\right)$, and $\mathscr{C} \subset \mathcal{L}\left(\mathcal{H}^C\right)$, a higher-order transformation is admissible if it relates two local parties through the generalized Born rule, no matter their choice of operations, say A and B. In symbols, $\mathcal{M}$ sends a state of A to one of B; its input must be compatible with any choice of state in $\mathscr{A}$ and its output with any choice of measurement in $\overline{\mathscr{B}}$. This implies that for all choices of resolution of $\mathscr{A}$ and $\overline{\mathscr{B}}$ conditioned by arbitrary settings x, y of Alice and Bob, respectively noted $\{V_{a|x}\}$ and $\{N_{b|y}\}$, the following is a well-defined probability distribution:

$$p(a, b|x, y) = \left(N_{b|y}\, ,\, \mathcal{M}\left(V_{a|x}\right)\right)_B \; . \tag{3.51}$$

The action of such maps $\mathcal{M}$ should preserve the set of deterministic operations on each side. This requires that the marginals

$$p(a|x, y) = \sum_b \left(N_{b|y}\, ,\, \mathcal{M}\left(V_{a|x}\right)\right)_B \; , \tag{3.52a}$$

$$p(b|x, y) = \sum_a \left(N_{b|y}\, ,\, \mathcal{M}\left(V_{a|x}\right)\right)_B \; , \tag{3.52b}$$

have to be well-defined probability distributions as well. This condition can be shown to be equivalent to requiring $\mathcal{M}$ to be linear [5, 8, 13, 123]. Because $\mathcal{M}$ is linear, its adjoint can be defined,

$$p(a, b|x, y) = \left(N_{b|y}\, ,\, \mathcal{M}\left(V_{a|x}\right)\right)_B = \left(\mathcal{M}^*\left(N_{b|y}\right)\, ,\, V_{a|x}\right)_A \; , \tag{3.53}$$

[14]: Davies et al. (1970), *An operational approach to quantum probability*.
[53]: Hardy (2001), *Quantum Theory From Five Reasonable Axioms*.
[72]: Kraus (1983), *States, Effects, and Operations: Fundamental Notions of Quantum Theory*.
[76]: Ozawa (1984), *Quantum measuring processes of continuous observables*.
[121]: Busch et al. (1995), *Operational Quantum Physics*.
[122]: Fuchs (2002), *Quantum Mechanics as Quantum Information (and only a little more)*.
[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.
[8]: Chiribella et al. (2008), *Transforming quantum operations: Quantum supermaps*.
[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.
[13]: Chiribella et al. (2008), *Quantum Circuit Architecture*.
[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.
[46]: Milz et al. (2022), *Resource theory of causal connection*.
[60]: Shrapnel et al. (2018), *Updating the Born rule*.
[82]: Wilson et al. (2023), *Quantum Supermaps are Characterized by Locality*.
[83]: Jenčová (2012), *Generalized channels: Channels for convex subsets of the state space*.
[84]: Wilson et al. (2022), *A Mathematical Framework for Transformations of Physical Processes*.
[88]: Apadula et al. (2022), *No-signalling constrains quantum computation with indefinite causal structure*.
[123]: Barnum et al. (2005), *Influence-free states on compound quantum systems*.
[103]: Choi et al. (1977), *Injectivity and operator spaces*.

30: See especially Reference [82] for a comprehensive discussion; the current discussion is an original yet informal rephrasing of their argument so to give intuition and motivation for Definition 3.4.3.

and the above can be rephrased in the CJ picture: $\mathcal{M} \mapsto M_{AB}$ by Definition 2.2.1 so that

$$p(a,b|x,y) = \text{Tr}\left[M_{AB}^{\dagger} \cdot \left(V_{a|x} \otimes N_{b|y}^{T}\right)\right] = \text{Tr}\left[\left(M_{AB}^{T}\right)^{\dagger} \cdot \left(V_{a|x}^{T} \otimes N_{b|y}\right)\right] . \tag{3.54}$$

As the resolutions $\left(V_{a|x} \otimes N_{b|y}^{T}\right)$ have support on the entirety of the space $\mathcal{H}^{A} \otimes \mathcal{H}^{B}$ and are positive, $\mathcal{M}$ must be positive. This means that M_{AB} is positive on pure tensors (POPT; see Definition A.1.2), so the dagger can be omitted in the inner products. The above can then be concisely written using the link product (see also Proposition 4.4 in [33]):

[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

$$p(a,b|x,y) = N_{b|y} * M_{AB} * V_{a|x}^{T} . \tag{3.55}$$

The requirements (3.52) that the marginals are well-defined become[31]

31: The transpose in Equation (3.55) has been swallowed into the definition of $V_{a|x}$ since these equations must be true for all $V_{a|x}$, which is the set of positive operators, hence closed under the transposition.

$$p(a|x,y) = \left(\sum_{b} N_{b|y}\right) * M_{AB} * V_{a|x} , \tag{3.56a}$$

$$p(b|x,y) = N_{b|y} * M_{AB} * \left(\sum_{a} V_{a|x}\right) . \tag{3.56b}$$

$V_{a|x} \in \mathcal{L}\left(\mathcal{H}^{A}\right)$ and $N_{b|y} \in \mathcal{L}\left(\mathcal{H}^{B}\right)$ are arbitrary resolutions of $\mathscr{A}$ and $\overline{\mathscr{B}}$ that respectively send the positive operators $\left(\sum_{b} N_{b|y}\right) * M_{AB}$ and $M_{AB} * \left(\sum_{a} V_{a|x}\right)$ to a probability through the inner product. Because of that, the conditions of Theorem 3.3.2 are met: these equations are verified provided that, respectively, $\left(\sum_{b} N_{b|y}\right) * M_{AB} \in \overline{\mathscr{A}}$ and $M_{AB} * \left(\sum_{a} V_{a|x}\right) \in \mathscr{B}$. Moreover, since $\sum_{a} V_{a|x} \in \mathscr{A}$ and $\sum_{b} N_{b|y} = N_{|y} \in \overline{\mathscr{B}}$ are arbitrary elements of state structures, (3.52) can be further simplified into the requirements

$$\forall N_B \in \overline{\mathscr{B}}, \qquad N_B * M_{AB} \in \overline{\mathscr{A}} , \tag{3.57a}$$

$$\forall V_A \in \mathscr{A}, \qquad M_{AB} * V_A \in \mathscr{B} . \tag{3.57b}$$

By the linearity of the map $\mathcal{M}$ and the uniqueness of the adjoint, these two conditions are a roundabout way of saying that the preservation of probabilistic structure requires that $\mathcal{M}$ maps an element of $\mathscr{A}$ to one of $\mathscr{B}$ [32],

[32]: Castro-Ruiz et al. (2018), *Dynamics of Quantum Causal Structures*.

$$\forall V \in \mathscr{A}, \quad \mathcal{M}\left(V\right) \in \mathscr{B} . \tag{3.58}$$

It remains to see what the preservation of probabilities *even locally* entails. Remark that Equation (3.55) is compatible with arbitrary extension by the state of some other, non-involved local party C through the tensor product. This is because of the associativity[32] of the link product,

32: See property 2 in Proposition 2.2.2.

$$\forall V_A \in \mathscr{A}, \forall U_C \in \mathscr{C},$$
$$M_{AB} * V_A * U_C = \left(M_{AB} * V_A\right) \otimes U_C = M_{AB} * \left(V_A \otimes U_C\right) . \tag{3.59}$$

In terms of linear maps, this is but the statement that the extension by a tensor product is a natural transformation[33]:

33: See References [124, 125]. In categorical terms, this amounts to assuming that the category of State Structures is a monoidal sub-category of CPM [126, 127]. See in particular the Caus[$\mathcal{C}$] construction in [33].

[124]: Roman (2017), *An Introduction to the Language of Category Theory*.
[125]: Heunen et al. (2019), *Categories for Quantum Theory: An Introduction*.

[126]: Selinger (2007), *Dagger Compact Closed Categories and Completely Positive Maps: (Extended Abstract)*.
[127]: Coecke (2008), *Axiomatic Description of Mixed States From Selinger's CPM-construction*.
[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

$$\forall V_A \in \mathscr{A}, \forall U_C \in \mathscr{C}, \quad \mathcal{M}\left(V_A\right) \otimes U_C = \left(\mathcal{M} \otimes \mathcal{I}_C\right) \circ \left(V_A \otimes U_C\right) , \tag{3.60}$$

meaning that 'applying the map, then extending by U_C' is equivalent to 'extending by U_C then applying the map'.

In the same way that two local parties can share entangled quantum states, one can postulate that parties A and C may share a non-separable joint state $W_{AC} \in \mathscr{A} \otimes \mathscr{C} \subset \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^C\right)$, so that $\mathcal{M}$ is compatible with acting *locally* on the A's part of the shared state:

$$\forall W_{AC} \in \mathscr{A} \otimes \mathscr{C}, \quad M_{AB} * W_{AC} \in \mathscr{B} \otimes \mathscr{C} \tag{3.61}$$

In the above equation, a notion of parallel composition of two state structures was used, $\mathscr{B} \otimes \mathscr{C}$. For now, it is taken as the following definition, but its meaning and interpretation will be clarified in the following.

Definition 3.4.1 (Tensor Composition of State Structures) *Let $\mathscr{A}$ and $\mathscr{B}$ be two state structures as in Equations 3.15, their* ***tensor*** *composition $\mathscr{A} \otimes \mathscr{B} \subset \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ is the set of all operators W characterized by the following constraints:*

$$W \in \mathscr{A} \otimes \mathscr{B} \iff$$

$$W \geq 0\,, \tag{3.62a}$$

$$Tr\left[W\right] = c_A c_B\,, \tag{3.62b}$$

$$\left(\mathcal{P}_A \otimes \mathcal{P}_B\right)\{W\} = W\,. \tag{3.62c}$$

Consequently, $\mathscr{A} \otimes \mathscr{B}$ is the intersection of the linear span of $\mathscr{A}$ and $\mathscr{B}$ with the cone of positive operators in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ and with a hyperplane for trace-normalization.

Notice that this definition, like Theorem 3.3.2, involves a new state structure which characterization is obtained by applying an operation on known state structures. In the case of the dual state structure, it was the bar operation; in this case, it is the tensor product of projectors, whose definition simply consists of using the definition of a tensor product of linear maps. Again, this operation results in a valid projector on state structures as in Definition 3.2.7. Moreover, the tensor product preserves commutation, meaning that if $\mathcal{P}_A$ and $\mathcal{P}_A^{'}$ as well as $\mathcal{P}_B$ and $\mathcal{P}_B^{'}$ commute with each other, then $\mathcal{P}_A \otimes \mathcal{P}_B$ commute with $\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}$, this will be shown in Appendix D.3.1.

In terms of linear maps, Equation (3.61) reads

$$\forall W \in \mathscr{A} \otimes \mathscr{C}, \quad \left(\mathcal{M} \otimes \mathcal{I}_C\right)\{W\} \in \mathscr{B} \otimes \mathscr{C}\,. \tag{3.63}$$

It is the requirement that when the state space of a subsystem is locally modified, the state space of the other subsystems is left untouched: $\mathrm{Tr}_A\left[W\right] \in \mathscr{C}$ and $\mathrm{Tr}_B\left[\left(\mathcal{M} \otimes \mathcal{I}_C\right)\{W\}\right] \in \mathscr{C}$. Importantly, however, this does not prevent an influence in the sense that the reduced state on Charlie's side, while belonging to the same state space, can be different after the map $\mathcal{M}$ has acted on Alice's side:

$$\exists \mathcal{M}: \quad \mathrm{Tr}_A\left[W\right] \neq \mathrm{Tr}_B\left[\left(\mathcal{M} \otimes \mathcal{I}_C\right)\{W\}\right]\,. \tag{3.64}$$

Collecting the different requirements that have been presented so far, one obtains the definition of an admissible higher-order map (first formalized in [9]).

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.

Definition 3.4.2 (Higher-Order Transformation) *Let $\mathcal{M}$ be a map from $\mathcal{L}\left(\mathcal{H}^A\right)$ to $\mathcal{L}\left(\mathcal{H}^B\right)$, let $\mathscr{A} \subset \mathcal{L}\left(\mathcal{H}^A\right)$ and $\mathscr{B} \subset \mathcal{L}\left(\mathcal{H}^B\right)$ be state structures. The map $\mathcal{M}$ represents a higher-order transformation between the set of transformations $\mathscr{A}$ and $\mathscr{B}$ only if it is a* ***structure-preserving map*** *between $\mathscr{A}$ and $\mathscr{B}$. I.e., if and only if 1) it is linear; 2) it maps any element of state structure $\mathscr{A}$ to one in $\mathscr{B}$, that is,*

$$\forall V \in \mathscr{A}, \quad \mathcal{M}(V) \in \mathscr{B}\,. \tag{3.65}$$

3) it keeps this property when these state structures are embedded in larger systems. That is, for any state structure $\mathscr{C}$, the map $\mathcal{M} \otimes \mathcal{I}_C \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^C\right), \mathcal{L}\left(\mathcal{H}^B \otimes \mathcal{H}^C\right)\right)$ should map any element of $\mathscr{A} \otimes \mathscr{C}$ to one of $\mathscr{B} \otimes \mathscr{C}$.

However, this definition for state structures is not enough to entail admissibility as in Definition 1.4.2 as generalized complete positivity (Definition 1.4.1) is not guaranteed to hold by this definition alone[34]. Remark that the definition instead entails that $M_{AB} * W_{AC}$ as in Equation (3.61) to be a well-defined functional on some state structure $\overline{\mathscr{B} \otimes \mathscr{C}}$, hence that

$$\forall N_{BC} \in \overline{\mathscr{B} \otimes \mathscr{C}},\ \forall W_{AC} \in \mathscr{A} \otimes \mathscr{C}:\ N_{BC} * M_{AB} * W_{AC} = 1\,. \tag{3.66}$$

Using the properties of the link product, this is equivalent to

$$\forall N_{BC} \in \overline{\mathscr{B} \otimes \mathscr{C}},\ \forall W_{AC} \in \mathscr{A} \otimes \mathscr{C}:\ M_{AB} * (W_{AC} * N_{BC}) = 1\,. \tag{3.67}$$

While this last equation holds, it may be tempting to conclude that the set of all $(W_{AC} * N_{BC})$ is actually equivalent to $\mathscr{A} \otimes \overline{\mathscr{B}}$. However, this is not the case, for instance when $\mathscr{C}$ is one-dimensional this set is made of all pure tensors, i.e., $\{W \otimes N | W \in \mathscr{A}, N \in \overline{\mathscr{B}}\}$. Nevertheless, if one considers the resolutions of this set, these are made of all positive operators in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$[35]. Therefore, if N_{BC} and W_{AC} are assumed to be under the control of parties that can act probabilistically, M_{AB} must be able to send these resolutions to a well-defined probability. In symbols, let $\{N_b\}$ be a resolution of an element of $\overline{\mathscr{B} \otimes \mathscr{C}}$ and let $\{W_a\}$ be one of $\mathscr{A} \otimes \mathscr{C}$, then

$$\forall \{W_a\}, \forall \{N_b\},\ M_{AB} * (W_a * N_b) \in [0, 1]\,. \tag{3.68}$$

Because $(W_a * N_b)$ can be proportional to any positive operator in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$, this requirement is equivalent to requiring that

$$\forall V \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right) :\ V \geq 0,\ \mathrm{Tr}\left[V \cdot M_{AB}\right] \geq 0\,. \tag{3.69}$$

This, in turn, amounts to restricting M_{AB} to the set of positive operators.

Summarizing, by the definition of a resolution of a state structure, and assuming that parties could locally extend their operation on each side of the map $\mathcal{M}$, the Choi operator of $\mathcal{M}$ is forced to be positive. This condition is actually sufficient[36] to enforce generalized CP. This leads to the definition of an admissible transformation for state structures:

34: I am grateful to the reviewers for pointing out this mistake in an earlier version of the manuscript.

35: A simple way to see it is to notice that the identity $\mathbb{1}_{AB}$ is a pure tensor, $\mathbb{1}_{AB} = \mathbb{1}_A \otimes \mathbb{1}_B$.

36: Whether it is necessary is an open question left for future work. This question is linked to the remark Appendix C.3.1 about the absence of operational justification for the definition of a resolution of a state structure; if this condition is not necessary, this would be further evidence that the definition of a resolution is 'too strong' with respect to the operational interpretation: imposing probabilistic behaviors to be represented by any collection of positive operators add an extra, not operationally justified, constrain on top of complete CP-preservation.

Definition 3.4.3 (Admissible Higher-Order Transformation (between state structures)) *Let $\mathcal{M}$ be a map from $\mathcal{L}\left(\mathcal{H}^A\right)$ to $\mathcal{L}\left(\mathcal{H}^B\right)$, let $\mathscr{A} \subset \mathcal{L}\left(\mathcal{H}^A\right)$ and $\mathscr{B} \subset \mathcal{L}\left(\mathcal{H}^B\right)$ be state structures. The map $\mathcal{M}$, is an **admissible transformation** between state structures $\mathscr{A}$ and $\mathscr{B}$ if and only if 1) it is linear; 2) it maps any element of state structure $\mathscr{A}$ to one in $\mathscr{B}$; 3) its Choi operator is positive.*

3.4.2. Projective Characterization of Transformations

The following theorem characterizes admissible higher-order transformations [2, 10, 11, 32, 33, 37, 83][37].

Definition 3.4.4 (Transformations between State Structures) *Let $\mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right)$ be an admissible higher-order transformation between state structures $\mathscr{A}$ and $\mathscr{B}$ as in Definition 3.4.3. Let $M \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ be the Choi-Jamiołkowski representation of this map. This operator is a **transformation between state structures** $\mathscr{A}$ and $\mathscr{B}$. The set $\{M\}$ of all such operators is noted $\mathscr{A} \to \mathscr{B}$.*

Theorem 3.4.1 (Characterization the Transformations between State Structures) *The set $\mathscr{A} \to \mathscr{B}$ of all transformations between state structures $\mathscr{A}$ and $\mathscr{B}$ is a state structure characterized by the following conditions:*

$$M \in \mathscr{A} \to \mathscr{B} \iff$$

$$M \geq 0\,, \tag{3.70a}$$

$$\mathrm{Tr}\left[M\right] = c_{\overline{A}} c_B = \frac{c_B}{c_A} d_A\,, \tag{3.70b}$$

$$\mathcal{P}_{A\to B}\{M\} = M\,, \tag{3.70c}$$

where

$$\mathcal{P}_{A\to B} := \mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{P}_A \otimes \mathcal{I}_B + \mathcal{P}_A \otimes \mathcal{P}_B - \mathcal{P}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \tag{3.71}$$

is a projector on operator system.

The proof is delayed to Appendix C.2.2, because it requires Lemma 3.5.1 derived below.

As with Theorem 3.3.2, the projector in Equation (3.71) can be concisely defined as an operation on projectors $\mathcal{P}_A$ and $\mathcal{P}_B$ like $\mathcal{P}_{A\to B} = \mathcal{P}_A \to \mathcal{P}_B$, the algebraic connector relating the two projectors into a new one is called the **transformation** and represented by $\to$. A transformation between state structure represented by M_{AB} can be seen as the bipartite functional $\widetilde{\mathcal{M}}$ such that

$$\mathrm{Tr}\left[M_{AB}\left(\cdot_A \otimes \cdot_B\right)\right] = \left(\cdot_B\,, \mathcal{M}\left(\cdot_A\right)\right)_B = \widetilde{\mathcal{M}}(\cdot_A\,, \cdot_B)\,. \tag{3.72}$$

This functional is normalized on states from $\mathscr{A}$ in tensor product with effects from $\overline{\mathscr{B}}$ as is implicit in Equation (3.51) and the discussion below it:

$$\forall V_A \in \mathscr{A}\, \forall N_B \in \overline{\mathscr{B}}, \quad \mathrm{Tr}\left[M_{AB}\left(V_A \otimes N_B\right)\right] = 1\,. \tag{3.73}$$

[2]: Hoffreumon et al. (2022), *Projective characterization of higher-order quantum transformations*.
[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.
[32]: Castro-Ruiz et al. (2018), *Dynamics of Quantum Causal Structures*.
[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.
[37]: Milz et al. (2023), *Transformations between arbitrary (quantum) objects and the emergence of indefinite causality*.
[83]: Jenčová (2012), *Generalized channels: Channels for convex subsets of the state space*.

37: This generalizes many results: a non-projective version of this theorem appears in References [83], [10] (Lemma 2 and Theorem 5) and [11] as well as [33] ([83] presents a singular approach compared to the other sources as the theorems are derived for convex sets on operator systems instead of state structures, meaning that the trace normalization is replaced by a convexity requirement); a projective version in the case of the Process Matrices was considered in Ref. [32], although it misses the last two terms in Equation (3.71) as pointed out in my work [2]. The same result was independently derived in Ref. [37].

(It is also the key observation for the proof of Theorem 3.4.1.) As a consequence, the projector characterizing transformations can be expressed in terms of projectors to dual state structure. That is, it actually can be defined from the tensor product of supermaps and the negation operation under the following combination:

$$\mathcal{P}_{A\to B} = \overline{\mathcal{P}_A \otimes \overline{\mathcal{P}}_B} \,. \tag{3.74}$$

Thus, it can be defined as an operation on $\mathcal{P}_A$ and $\mathcal{P}_B$, derived from $\overline{\cdot}$ and $\cdot \otimes \cdot$ and noted with an arrow $\to$:

$$\mathcal{P}_A \to \mathcal{P}_B := \overline{\mathcal{P}_A \otimes \overline{\mathcal{P}}_B} \,. \tag{3.75}$$

This is a connector whose direction matters. In order not to change the order in the tensor factorization of the spaces, the reversed symbol, $\leftarrow$, is defined accordingly:

$$\mathcal{P}_{B\to A} = \mathcal{P}_A \leftarrow \mathcal{P}_B := \overline{\overline{\mathcal{P}}_A \otimes \mathcal{P}_B} \,, \tag{3.76}$$

and will be used whenever it allows sorting the tensor factors in lexical order without overloading the notation with parenthesis.

3.5. Characterization of Bipartite State Structures

The set of admissible transformations between two state structures $\mathscr{A}$ and $\mathscr{B}$ is itself a state structure, noted $\mathscr{A} \to \mathscr{B}$. As the notation hints, this new state structure can be seen as a special kind of composition between two state structures. This builds on the idea that a channel $\mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right)$ can always be decomposed as[38]

$$\mathcal{M}(\cdot) = \sum_i \sigma_i \mathrm{Tr}\left[\eta_i \,\cdot\right] \tag{3.77}$$

where $\{\sigma_i\} \subset \mathcal{L}\left(\mathcal{H}^B\right)$ and $\{\eta_i\} \subset \mathcal{L}\left(\mathcal{H}^A\right)$ are sets of operators constrained by the definition of the channel. Its CJ representation is the bipartite positive[39] operator $M \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ such that

$$M_{AB} = \sum_i \eta_i \otimes \sigma_i^T \,. \tag{3.78}$$

The span of allowed combinations of η_i's and σ_i's so that M_{AB} belongs to the span of valid transformations is exactly what the projector $\mathcal{P}_{A\to B}$ involved in Theorem 3.4.1 characterizes. The extra conditions in the theorem then require this decomposition to form a positive operator which is correctly trace-normalized.

Putting these conditions aside, the projective requirement enforces in particular Equations (3.57), stating that the reduced operator seen at one side of the transformation after something has been applied on the other side must always be an element of the dual state structure on this side. For instance, if Bob is at the output of the transformation and measures the states coming out of it by a deterministic measurement in $\overline{\mathscr{B}}$, then the 'reduced state' seen by Alice at the input side is a deterministic

38: This is due to the Hilbert spaces isomorphism $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right) \cong \mathcal{L}\left(\mathcal{H}^A\right)^* \otimes \mathcal{L}\left(\mathcal{H}^B\right)$, where $*$ denotes the algebraic dual.

39: Because $\mathcal{M}$ is CP.

measurement on the input, i.e. an element in $\overline{\mathscr{A}}$. Because the elements can, in particular, be the identity operator, i.e., $V_A = c_A \frac{\mathbb{1}}{d_A} \in \mathscr{A}$ and $N_B = c_{\overline{B}} \frac{\mathbb{1}}{d_B} \in \overline{\mathscr{B}}$, these equations imply that[40]

$$c_{\overline{B}} \frac{\mathbb{1}_B}{d_B} * M_{AB} = \sum_i \frac{\operatorname{Tr}[\sigma_i]}{c_B} \eta_i = \frac{1}{c_B} \operatorname{Tr}_B [M_{AB}] \in \overline{\mathscr{A}} \,; \tag{3.79a}$$

$$M_{AB} * c_A \frac{\mathbb{1}_A}{d_A} = \sum_i \frac{\operatorname{Tr}[\eta_i]}{c_{\overline{A}}} \sigma_i^T = \frac{1}{c_{\overline{A}}} \operatorname{Tr}_A [M_{AB}] \in \mathscr{B} \,. \tag{3.79b}$$

40: Where the definition

$$c_{\overline{A}} := \frac{d_A}{c_A} \tag{3.31b}$$

is used to simplify the scalar quantity.

Thus, that M_{AB} is a particular way to combine valid elements of $\overline{\mathscr{A}}$ with valid elements of $\mathscr{B}$. According to this insight, the operator system spanned by the set of transformations $\mathscr{A} \to \mathscr{B}$ is obtained as a special way of combining the operator systems respectively spanned by $\overline{\mathscr{A}}$ and $\mathscr{B}$. But how is this composite span obtained?

A special case of transformations can be understood as a measurement followed by a repreparation like $M_{AB} = R_A \otimes U_B^T$ where $R_A \in \overline{\mathscr{A}}$ and $U_B \in \mathscr{B}$. It may be conjectured that $\mathscr{A} \to \mathscr{B}$ is spanned by affine combinations of such measurements, meaning that[41]

41: The transpose over subsystem B has been swallowed in the definition of U_i for conciseness.

$$\begin{gathered}\forall M_{AB},\ \exists \Omega_i : \ \forall i \in \Omega_i : \ \exists q_i \in \mathbb{R},\ \sum_i q_i = 1,\ \exists R_i \in \overline{\mathscr{A}},\ \exists U_i \in \mathscr{B} : \\ M_{AB} = \sum_i q_i \, R_i \otimes U_i \,.\end{gathered} \tag{3.80}$$

However, this decomposition only recovers Definition 3.4.1. The tensor product of two state structures does indeed belong to the affine span of pure tensor products of $\mathscr{A}$ and $\mathscr{B}$.

Lemma 3.5.1 *Any element of a tensor product state structure can be decomposed as an affine sum of tensor products of elements from the composed state structures, i.e.,*

$$\begin{gathered}\forall W \in \mathscr{A} \otimes \mathscr{B}, \quad W = \sum_i q_i \, V_i \otimes N_i, \\ \textit{where}, \forall i,\ V_i \in \mathscr{A},\ N_i \in \mathscr{B},\ q_i \in \mathbb{R},\ \textit{and} \sum_i q_i = 1 \,.\end{gathered} \tag{3.81}$$

The proof is presented in Appendix C.2.3. As a consequence, the intersection of the affine hull of the tensor product of operators in state structures $\overline{\mathscr{A}}$ and $\mathscr{B}$ with the cone of positive operators in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ is exactly the tensor product state structure $\overline{\mathscr{A}} \otimes \mathscr{B}$.

Yet, $\mathscr{A} \to \mathscr{B}$ cannot belong to this affine span since the operator system it spans is strictly bigger. This can be seen since the projectors $\mathcal{P}_A \to \mathcal{P}_B$ and $\overline{\mathcal{P}}_A \otimes \mathcal{P}_B$ commute[42], hence the intersection of their respective images is characterized by

42: As both are defined in terms of linear combinations, negations, and tensors, which all preserve commutation.

$$(\mathcal{P}_A \to \mathcal{P}_B) \circ \left(\overline{\mathcal{P}}_A \otimes \mathcal{P}_B\right) = \overline{\mathcal{P}}_A \otimes \mathcal{P}_B \neq \mathcal{P}_A \to \mathcal{P}_B \,. \tag{3.82}$$

Therefore, the bipartite composition using the transformation connector and the tensor connectors are different[43].

43: Actually, remark that this inequality breaks in certain limiting cases like for example in the quantum channel case when $\mathcal{P}_A = \mathcal{I}_A$ and $\mathcal{P}_B = \mathcal{I}_B$. The consequence of this will be explored in Section 5.3.

As implied with the CJ representation, the 'input' and 'output' sides of this composite state are differentiated by a partial transpose. This is because the 'input' side is interpreted as a functional related to $\overline{\mathscr{A}}$ whereas the 'output' side is a state like $\mathscr{B}$. If the 'input' of the transformation was reversed, i.e. if the two single-partite state structures $\overline{\mathscr{A}}$ and $\mathscr{B}$ involved in the definition of $\mathscr{A} \to \mathscr{B}$ were interpreted as two states (diagrammatically, as top facing wires), then the transformation is a way of defining a set of bipartite states out of two states structures, noted as $\overline{\mathscr{A}} \parr \mathscr{B}$.

Thus, two bipartite compositions have been considered so far, $\mathscr{A} \otimes \mathscr{B}$ and $\mathscr{A} \parr \mathscr{B}$. While the former has been postulated, the latter is the admissible functionals normalized on it in the sense of Equation (3.73). It follows from the proof and interpretation of Theorem 3.4.1 as well as Equation (3.72) that this parr product of state structures is the composition defined using the transformation between two state structures.

Definition 3.5.1 (Parr Composition of State Structures) *The bipartite state structure defined through the transformation, i.e. the CJ representation of admissible transformations from a dual state structure $\overline{\mathscr{A}}$ to a state structure $\mathscr{B}$, is called the parr composition of $\mathscr{A}$ and $\mathscr{B}$. In symbols:*

$$\mathscr{A} \parr \mathscr{B} := \overline{\mathscr{A}} \to \mathscr{B} \tag{3.83}$$

This state structure is directly characterized by combining Theorem 3.3.2 and Theorem 3.4.1.

Corollary 3.5.2 *The parr composition of two state structures is exactly the dual of the tensor composition of the duals of each state structure composing it,*

$$\mathscr{A} \parr \mathscr{B} = \overline{\overline{\mathscr{A}} \otimes \overline{\mathscr{B}}}\,. \tag{3.84}$$

Remark that this equivalence appears once again at the level of operations on the projectors: because of Equation (3.75), it is direct to check that

$$\overline{\mathcal{P}}_A \to \mathcal{P}_B = \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B}\,. \tag{3.85}$$

So the 'parr' composition of projectors can be defined as

$$\mathcal{P}_A \parr \mathcal{P}_B := \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B}\,. \tag{3.86}$$

Remark that this new way of composing projectors is, like the tensor of projectors, a symmetric operation[44]. Hence, the two ways of constructing a bipartite state structure, $\mathscr{A} \otimes \mathscr{B}$ and $\mathscr{A} \parr \mathscr{B}$, are directly related. But what actually distinguishes them?

44: I.e., $\mathcal{P}_A \parr \mathcal{P}_B \cong \mathcal{P}_B \parr \mathcal{P}_A$. This was not the case with the transformation, $\mathcal{P}_A \to \mathcal{P}_B \neq \mathcal{P}_A \leftarrow \mathcal{P}_B \cong \mathcal{P}_B \to \mathcal{P}_A$, since $\mathcal{P}_A \to \mathcal{P}_B = \overline{\mathcal{P}_A \otimes \overline{\mathcal{P}}_B} \neq \overline{\overline{\mathcal{P}}_A \otimes \mathcal{P}_B} = \mathcal{P}_A \leftarrow \mathcal{P}_B$.

Remark: Parallel with entanglement. The mathematical theory of entanglement provides an analog question under the problem of how to build the sets of join quantum states on the tensor product space from the definition of the state space on each tensor factor (see References [74, 128, 129] for instance).

[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.
[128]: Kläy et al. (1987), *Tensor products and probability weights*.
[129]: Aubrun et al. (2021), *Entangleability of cones*.

At a purely geometric level, the difference between $\mathscr{A} \otimes \mathscr{B}$ and $\mathscr{A} \parr \mathscr{B}$ lies in choosing how to define a bipartite state structure, which is a hyperplane of positive operators in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$, from two similarly defined hyperplanes $\mathscr{A} \subset \mathcal{L}\left(\mathcal{H}^A\right)$ and $\mathscr{B} \subset \mathcal{L}\left(\mathcal{H}^B\right)$. In the case of entanglement, a similar question has been studied: how to form a joint convex cone in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ out of two convex cones, which are the quantum state spaces in $\mathcal{L}\left(\mathcal{H}^A\right)$ and $\mathcal{L}\left(\mathcal{H}^B\right)$ [128].

[128]: Kläy et al. (1987), *Tensor products and probability weights*.

Let $\mathscr{A}_{quant.}$ be the state structure of quantum states on $\mathcal{H}^A$ and $\mathscr{B}_{quant.}$ the one on $\mathcal{H}^B$. The *algebraic dual* of these state spaces, noted $\mathscr{A}^*_{quant.}$ and $\mathscr{B}^*_{quant.}$ and defined by

$$E \in \mathscr{A}^*_{quant.} \iff \forall \rho \in \mathscr{A}_{quant.},\ (E\ ,\ \rho) \geq 0\ , \tag{3.87}$$

are the set of all (representation of) positive functionals on quantum states. That is, the set of unormalized quantum effects. To define the set of join states, there are actually two relevant ways to be considered[45]. The first is the minimal tensor product, consisting of the convex hull of all product states,

45: See e.g., Section 5.1 in Reference [79].

[79]: Plávala (2021), *General probabilistic theories: An introduction*.

$$\mathscr{A}_{quant.} \otimes_{min} \mathscr{B}_{quant.} := \mathrm{Conv}\{\rho_A \otimes \sigma_B | \rho_A \in \mathscr{A};\ \sigma_B \in \mathscr{B}\}\ . \tag{3.88}$$

This is the set of separable states. The second way, the maximal tensor product, is obtained as the dual in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ of the tensor product of each dual in their respective spaces,

$$\mathscr{A}_{quant.} \otimes_{max} \mathscr{B}_{quant.} := \left(\mathscr{A}^*_{quant.} \otimes_{min} \mathscr{B}^*_{quant.}\right)^*\ . \tag{3.89}$$

In the case of state structures, Definition 3.3.3 provides another form of duality, $\overline{\mathscr{A}}$, 'with respect to the deterministic operations' according to the defining condition (3.29) in Theorem 3.3.2. This duality is at the level of the linear spaces spanned by the state structures rather than at the level of cones of positive operators. In the same fashion as above, the tensor product of state structures can be defined as the minimal tensor product, since it is the minimal closure of all tensor products of elements from the two sets that result in a valid state structure. And in the same fashion still, the parr composition can be defined as the maximal tensor product with respect to the $\overline{\cdot}$ duality, i.e. it can also be defined as the 'dual of the tensor product of the duals'.

3.5.1. No-signaling as Local Quasi-Orthogonality

The difference between state structures $\mathscr{A} \otimes \mathscr{B}$ and $\mathscr{A} \parr \mathscr{B}$ lies in their signaling relations. Or, more precisely, whether they allow for a deterministic transmission of information between the parties sharing a state of these bipartite state structures.

To formulate this statement precisely, assume two parties Alice and Bob trying to measure a shared bipartite state structure. Each locally sees their own measurement operation as an effect state structure, respectively $\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$. Their measurements are locally disconnected and thus assumed in tensor product[46]. Thus, both their probabilistic and deterministic joint operations are assumed to factorize into a product state. In symbols, their joint deterministic operations are represented by the set

46: Otherwise, they can trivially be signaling to one another just by using the pre-existing correlations featured in their non-product operation.

$$\{V \otimes N | V \in \overline{\mathscr{A}}, N \in \overline{\mathscr{B}}\}\ . \tag{3.90}$$

This set is contained in state structure $\overline{\mathscr{A}} \otimes \overline{\mathscr{B}}$. The greatest admissible set of states they can jointly measure is the dual, which is $\mathscr{A} \parr \mathscr{B} = \overline{\overline{\mathscr{A}} \otimes \overline{\mathscr{B}}}$.

Hence, in general, the two parties share an environment[47] (a state) $W \in \mathscr{A} \parr \mathscr{B}$ and measure locally so that the probability distribution of their outcomes is obtained via

$$p(a, b|x, y) = \mathrm{Tr}\left[\left(V_{a|x} \otimes N_{b|y}\right) \cdot W\right] \ . \tag{3.91}$$

47: Remember that it can be anything outside their control that respects their local operations: if both party's operations are quantum measurements, W is a bipartite quantum state; if they are quantum instruments, W is a bipartite process matrix; if Alice is preparing a quantum state and Bob is measuring, W is a channel; etc.

The question reduces to finding what kind of shared state W Alice and Bob can locally measure so that their outcome distributions are guaranteed to be no-signaling for all choices of operation.

In the previous section, it was claimed that Definition 3.3.1 imposes a sort of no-deterministic influence of the choice of state on the choice of effect that was interpreted as a kind of no-signaling. This property stems from the quasi-orthogonality relation (3.48), which can be phrased as a projective condition (3.31c). Hence, a pair of deterministic preparation and measurement procedures applied on the same system cannot influence each other if the operator systems they respectively span are quasi-orthogonal. This is the content of the state/unit effect duality of Definition 3.3.3.

This property can be turned into a subsystem-wise property. Enforcing that a shared state W is no-signaling can be phrased as a sort of local quasi-orthogonality condition as is now shown. Indeed, imposing the no-signaling conditions[48] (1.6) on Equation (3.91) yields

$$\forall a, x, y, y', \quad \mathrm{Tr}\left[\left(V_{a|x} \otimes N_{|y}\right) \cdot W\right] = \mathrm{Tr}\left[\left(V_{a|x} \otimes N_{|y'}\right) \cdot W\right] \ ; \tag{3.93a}$$
$$\forall b, x, x', y, \quad \mathrm{Tr}\left[\left(V_{|x} \otimes N_{b|y}\right) \cdot W\right] = \mathrm{Tr}\left[\left(V_{|x'} \otimes N_{b|y}\right) \cdot W\right] \ . \tag{3.93b}$$

48: Defined in Definition 1.2.1 as

$$\forall a, b, x, y, y', \quad \sum_b p(a, b|x, y) = \sum_b p(a, b|x, y') \ ; \tag{1.6a}$$

$$\forall a, b, x, x', y, \quad \sum_a p(a, b|x, y) = \sum_a p(a, b|x', y) \ . \tag{1.6b}$$

These equations can be reduced into a more concise condition that resembles Equation (3.29).

Consider Equation (3.93a), since it should hold for any y, y' the choice of a particular setting is no longer needed, and one can simply consider different unit effects $N, N' \in \overline{\mathscr{B}}$. Rewriting it as

$$\forall N, N', \ \mathrm{Tr}_A\left[V_{a|x} \cdot \mathrm{Tr}_B\left[(\mathbb{1} \otimes N) \cdot W\right]\right] = \mathrm{Tr}_A\left[V_{a|x} \cdot \mathrm{Tr}_B\left[(\mathbb{1} \otimes N') \cdot W\right]\right] \ , \tag{3.94}$$

one can simplify further by noticing that the possible $V_{a|x}$ actually range over the whole of $\mathcal{L}\left(\mathcal{H}^A\right)$ by definition of a resolution, so that only the trace over B part[49] is relevant:

49: Which can be interpreted as the (unnormalized) reduced state seen by Alice.

$$\forall N, N', \ \mathrm{Tr}_B\left[(\mathbb{1} \otimes N) \cdot W\right] = \mathrm{Tr}_B\left[(\mathbb{1} \otimes N') \cdot W\right] \ . \tag{3.95}$$

Finally, as $\mathbb{1}/c_B$ is a valid element of $\overline{\mathscr{B}}$, it can replace N' to obtain the shortened

$$\forall N, \quad \mathrm{Tr}_B\left[(\mathbb{1} \otimes N) \cdot W\right] = \frac{1}{c_B} \mathrm{Tr}_B\left[(\mathbb{1} \otimes \mathbb{1}) \cdot W\right] \ . \tag{3.96}$$

Remark that this form gives the operational interpretation of local no-signaling: once Bob has done his operation, the reduced state seen on Alice's side, $W_A = \mathrm{Tr}_B\left[(\mathbb{1} \otimes N_B) \cdot W\right]$, is independent of the unit effect applied by Bob[50]. Alice cannot distinguish, even probabilistically, whether Bob has applied a given N or the maximally mixed operation $c_{\overline{B}}\mathbb{1}$. Using Proposition 3.3.2, observe that $1/c_B = \text{Tr}\,[N]\,/d_B$ because of Equation (3.31b), so $\text{Tr}\,[N]$ can be put in the right-hand side so to reach the desired formulation that resembles quasi-orthogonality. Doing the same reasoning for condition (1.6b), the following rephrasing of no-signaling (1.6) is reached:

$$\forall N_B, \quad \text{Tr}_B\left[(\mathbb{1}\otimes N_B)\cdot W\right] = \frac{\text{Tr}\,[N_B]\,\text{Tr}_B\,[W]}{d_B}\ ; \tag{3.97a}$$

$$\forall\, V_A, \quad \text{Tr}_A\left[(V_A\otimes\mathbb{1})\cdot W\right] = \frac{\text{Tr}\,[V_A]\,\text{Tr}_A\,[W]}{d_A}\ , \tag{3.97b}$$

Hence, no-signaling (1.6) can be recast into the conditions (3.97), which amounts to requiring quasi-orthogonality for only one of the tensor factors of an operator.

Moreover, and like the global quasi-orthogonality, local quasi-orthogonality is a projective constraint as well.

Lemma 3.5.3 *Let W be an operator in $\mathcal{L}\left(\mathcal{H}^A\otimes\mathcal{H}^B\right)$, let V be one in the operator system spanned by $\overline{\mathscr{A}}$ and let N be one in $\mathcal{L}\left(\mathcal{H}^B\right)$. Then, a necessary and sufficient condition for*

$$Tr_A\left[(V\otimes N)\cdot W\right] = \frac{Tr\,[V]}{d_A}\cdot N\cdot Tr_A\,[W]\ , \tag{3.98}$$

to hold for all V and N is that

$$(\mathcal{P}_A\otimes\mathcal{I}_B)\,\{W\} = W\ . \tag{3.99}$$

That is to say, that W belongs to the subspace spanned by $\mathscr{A}\otimes\mathcal{L}\left(\mathcal{H}^B\right)$.

See Appendix C.2.4 for the proof.

3.5.2. The Projective Characterization of Bipartite State Structures

The no-signaling constraints can be used to impose the conditions (3.97) on the bipartite composition of state structures obtained by higher-order transformation. These two conditions define three new subsets of the composite state structure. Let $\mathscr{A}\parr\mathscr{B} = \overline{\overline{\mathscr{A}}\otimes\overline{\mathscr{B}}}$ be the bipartite state structure obtained as the set of CJ representation of higher-order transformations between state structure $\overline{\mathscr{A}}$ and $\mathscr{B}$, i.e., the parr composition $\mathscr{A}\parr\mathscr{B} = \overline{\mathscr{A}}\to\mathscr{B}$ as in Definition 3.5.1. This state structure has $\mathscr{A}$ and $\mathscr{B}$ as *reduced state structures*, meaning that $\forall W_{AB}\in\mathscr{A}\parr\mathscr{B}$,

$$\forall V_A\in\overline{\mathscr{A}}, \quad W_{AB}*V_A\in\mathscr{B}\ , \tag{3.100a}$$

$$\forall N_B\in\overline{\mathscr{B}}, \quad W_{AB}*N_B\in\mathscr{A}\ , \tag{3.100b}$$

as implied by Definition 3.4.3 and which can be checked from Theorem 3.4.1.

Because this state structure has $\mathscr{A}$ and $\mathscr{B}$ as reduced state structures, it must have been obtained as a composition of $\mathscr{A}$ with $\mathscr{B}$. And, because

50: When applied to quantum processes like channels, this no-signaling criterion is sometimes called *causality* of quantum channels in the literature [50, 78, 81, 130] as in the sense of 'no-signaling from the future'; see the discussions in Section 1.2 and Subsection 5.3.1.

[50]: Chiribella et al. (2010), *Probabilistic theories with purification*.
[78]: D'Ariano et al. (2017), *Quantum Theory from First Principles: An Informational Approach*.
[81]: Coecke et al. (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*.
[130]: Kissinger et al. (2017), *Equivalence of relativistic causal structure and process terminality*.

it verifies neither of conditions (3.97), this composition is the two-way signaling one. Three other compositions can be defined accordingly.

Definition 3.5.2 *The bipartite state structure $\mathscr{A} \parr \mathscr{B}$ is called the* ***two-ways-signaling composition*** *of state structures $\mathscr{A}$ and $\mathscr{B}$. Accordingly:*
The subset of $\mathscr{A} \parr \mathscr{B}$ obeying condition (3.97a) *is called their* ***A-to-B one-way-signaling composition****;*
The subset of $\mathscr{A} \parr \mathscr{B}$ obeying condition (3.97b) *is called their* ***B-to-A one-way-signaling composition****;*
And the subset of $\mathscr{A} \parr \mathscr{B}$ obeying both of conditions (3.97) *is called their* ***no-signaling composition****.*

One of the key results of the characterization is that these four sets are state structures.

Proposition 3.5.4 (One-Way Signaling Composition of State Structures) *Let $\mathscr{A}$ and $\mathscr{B}$ be two state structures as in Equations* (3.15)*, their A-to-B one-way signaling composition is the state structure $\mathscr{A} \prec \mathscr{B} \subset \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ consisting of all operators W characterized by the following conditions:*

$$W \geq 0\,, \tag{3.101a}$$
$$Tr\,[W] = c_A c_B\,, \tag{3.101b}$$
$$(\mathcal{P}_A \prec \mathcal{P}_B)\,\{W\} = W\,, \tag{3.101c}$$

where

$$\mathcal{P}_A \prec \mathcal{P}_B := \mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\,. \tag{3.102}$$

The same way, their B-to-A one-way signaling composition is the analogously defined state structure $\mathscr{A} \succ \mathscr{B} \subset \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ but instead using the projector

$$\mathcal{P}_A \succ \mathcal{P}_B := \mathcal{P}_A \otimes \mathcal{I}_B - \mathcal{D}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B\,. \tag{3.103}$$

Proof. The positivity and trace conditions are inherited from W being a valid transformation. The projector condition is obtained directly by taking the intersection of the subspace of valid transformations with the subspace of operators that are no-signaling from B to A, defined by By Theorem 3.4.1, the former is characterized by projector $\overline{\mathcal{P}}_A \to \mathcal{P}_B$ and by Lemma 3.5.3 the latter is characterized by $\mathcal{I}_A \otimes \mathcal{P}_B$. Since these projectors commute, the intersection of the subspace they define is equivalent to their composition. This composition is also a projector on an operator system (these two statements will be proven in the next chapter). A bit of algebra yields the projector (3.102),

$$\begin{aligned}&\left(\overline{\mathcal{P}}_A \to \mathcal{P}_B\right) \circ (\mathcal{I}_A \otimes \mathcal{P}_B)\\ &\quad= \left(\mathcal{I}_A \otimes \mathcal{I}_B - \overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) \circ (\mathcal{I}_A \otimes \mathcal{P}_B)\\ &\quad\quad= \mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\,.\end{aligned} \tag{3.104}$$

This required that $\overline{\mathcal{P}}_B \circ \mathcal{P}_B = \mathcal{D}_B$ which is the case since the only place two quasi-orthogonal spaces intersect is at the span of the identity, therefore the composition of their respective projectors yields the projector onto the span of the identity which is $\mathcal{D}_B$. The proof for the other composition is the same. □

Proposition 3.5.5 (No-signaling Composition of State Structures) *Let $\mathscr{A}$ and $\mathscr{B}$ be two state structures as in Equations* (3.15)*, their no-signaling composition is the tensor product of state structures $\mathscr{A} \otimes \mathscr{B}$ as in Definition 3.4.1.*

Proof. Similar to the proof above, the non-trivial part is to show that

$$\left(\overline{\mathcal{P}}_A \to \mathcal{P}_B\right) \circ (\mathcal{I}_A \otimes \mathcal{P}_B) \circ (\mathcal{P}_A \otimes \mathcal{I}_B) = \mathcal{P}_A \otimes \mathcal{P}_B \,. \tag{3.105}$$

From Equation (3.104), the left-hand side is equal to

$$\begin{aligned} \left(\mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) \circ (\mathcal{P}_A \otimes \mathcal{I}_B) &= (\mathcal{P}_A \otimes \mathcal{P}_B - \mathcal{D}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B) \\ &= \mathcal{P}_A \otimes \mathcal{P}_B \,. \end{aligned} \tag{3.106}$$

And therefore the left-hand side is equal to the right-hand side. (The proof that these three projectors commute and that their composition is a well-defined projector on operator systems is again delayed for the discussion in the next chapter.) □

As will be shown in the next chapter, with the addition of the state structure of deterministic functional, Theorem 3.3.2, these five ways of combining state structures are enough to characterize every family of operations considered in the literature so far, and they extend the hierarchy of higher-order maps of Bisio and Perinotti [10, 11] by allowing a composition that 'block signaling' in one direction[51].

51: Be aware that this hierarchy, as well as some of its blind spots that the methods developed in this thesis intent to overcome, will be reviewed in the next chapter, Section 4.3.

3.6. Example: Biased Quantum Theory

The novelty of the state structure approach compared to previous approaches like References [10, 11] is that the base state structure –the state structure upon which the hierarchy is built– does not have to be the set of quantum states. In that regard, the projective approach recovers what the categorical treatment of References [33, 36] was able to do. For example, in a state structure like $\mathscr{A} \to \mathscr{B}$ the base state structures $\mathscr{A}$ and $\mathscr{B}$ can be any valid state structure. For comparing the resulting higher-order process theories, one can consider various sets of possible base state structures for a fixed number of subsystems associated with a fixed number of parties. Say Alice's system is known to be bipartite and quantum, then her possible base state structures can be, for example, a tensor composition, $\mathscr{A} = \mathscr{A}_0 \otimes \mathscr{A}_1$, or a transformation $\mathscr{A} = \mathscr{A}_0 \to \mathscr{A}_1$ and the same reasoning applies to Bob.

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.
[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.
[36]: Simmons et al. (2022), *Higher-order causal theories are models of BV-logic*.

Nonetheless, the set of base state structures in consideration does not actually have to be restricted to only those that can be built from the quantum states (corresponding to the identity projector) and the various compositions presented in this chapter. Any state structure built from a projector on operator systems would work. So what about projectors that cannot be built from the identity and the various composition rules?

A physically motivated example is the set of diagonal operators with respect to a fixed basis. These correspond to the dephased states obtained after a measurement in this basis, hence to classical systems to an extent. The projector to a diagonal subspace, Δ, happens to obey the properties (3.20) of a projector on an operator system[52]. The set of base state structures for the states and effects of a single-partite system, in this case, is characterized by, respectively, projectors Δ and $\overline{\Delta}$ instead of $\mathcal{I}$ and $\mathcal{D}$.

52: See the example presented in the methods, Appendix C.1.4.

Inspired by this example, and in order to present an original theory that has not been studied before but which can be built using the projective characterization methods, the set of base state structures, i.e. the state space of the subsystems of each party, can be taken to be characterized by projector $\overline{\Delta}$. Simply put, the purpose of this section is to present the construction of a higher-order theory based on a subset of states that is only generated by off-diagonal matrices for a given basis, so that their diagonal elements will all happen to be $1/d_A$. This will illustrate the utilization of the theorems so as to study the properties of the multipartite state structures built out of it. The interest of this toy model is that it has similar signaling properties to higher-order quantum transformations, especially the bipartite process matrices, although it is represented on a Hilbert space of much smaller dimension.

For simplicity, all subsystems are assumed two-dimensional. The spaces of operators on each subsystem will be expressed in the Pauli basis[53] $\{\mathbb{1}, X, Y, Z\}$. Let $\mathcal{P} = \overline{\Delta}$ be the projector that restricts the basis elements to $\{\mathbb{1}, X, Y\}$ in the Pauli basis so that the base state structure on system A is $\mathscr{A}$ defined over the operator system $\mathrm{Span}\left\{\mathbb{1}^A, X^A, Y^A\right\}$, associated with projector $\mathcal{P}_A$. The normalization is assumed to be $c_A = 1$ so to look like a subspace of the quantum states.

53: Where, from Definition 2.3.4, $\mathbb{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Definition 3.6.1 (Biased Quantum Theory) *In the space of operators over a Hilbert space of dimension two, the elements of the following state structure are called the* ***biased quantum states****:*

$$V \geq 0\,; \tag{3.107a}$$

$$Tr\,[V] = 1\,; \tag{3.107b}$$

$$\overline{\Delta}(V) := \frac{1}{2}\left(Tr\,[V]\,\mathbb{1} + Tr\,[X \cdot V]\,X + Tr\,[Y \cdot V]\,Y\right) = V\,. \tag{3.107c}$$

Any state structure that uses the biased states or the dual state structure to the biased states as the base state structure of all its parties is referred to as a ***biased quantum theory****.*

By Proposition 3.3.2, the dual state structure to the biased quantum states in $\mathcal{L}\left(\mathcal{H}^A\right)$, i.e. the state structure of unit effects on $\mathscr{A}$, is $\overline{\mathscr{A}}$ defined over $\mathrm{Span}\left\{\mathbb{1}^A, Z^A\right\}$, associated with $\overline{\mathcal{P}}_A$, and normalized to $c_{\overline{A}} = \frac{d_A}{c_A} = 2$. Compare it to quantum theory in 2 dimensions: the normalization is the same but $\mathscr{A}_{quant.}$ is defined over $\mathrm{Span}\{\mathbb{1}, X, Y, Z\}$ and associated with $\mathcal{I}_A$ whereas $\overline{\mathscr{A}}_{quant.}$ is defined over $\{\mathbb{1}\}$ and associated with $\mathcal{D}_A$.

The theory characterized by $\mathcal{P}$ can be called a *biased* quantum theory in the sense that the measurement can be 'biased' towards an arbitrary element of $\overline{\mathscr{A}}$. Let the deterministic operations like $V_{|x} \in \mathscr{A}$ have a normalization taken so that $c_A = 1$. By Theorem 3.3.2, a measurement is the set of deterministic operations $N_{|y} \in \overline{\mathscr{A}}$ resolved into sets $\{N_{b|y}\}$. The probability rule reads

$$p(b|x,y) = \left(N_{b|y}\,,\,V_{|x}\right)\;; \tag{3.108}$$

this can be interpreted in regular quantum mechanics as a measurement of POVM element $N_{b|y} \in \mathcal{L}\left(\mathcal{H}^A\right)$ followed by a projection onto the 'state' $N_{|y} \in \overline{\mathscr{A}} \subset \mathrm{Span}\,\{\mathbb{1}, Z\}$. It is almost like quantum theory, but quantum theory is restricted to projecting onto the 'state' $\mathbb{1}$. In other words, while the post-measurement state is traced out at the end of a destructive measurement in quantum theory, in this theory it is first projected onto a selected $N_{|y}$ and only then traced out.

For that reason, the state and effect pair characterized by $\{\mathcal{P}_A, \overline{\mathcal{P}}_A\}$ is similar to a quantum theory for states in $\mathcal{L}\left(\mathcal{H}^A\right)$ for which there exists an inherent deterministic postselection of the measurement –here in the computational basis $\{\mathbb{1}, Z\}$. Nevertheless, the theory by itself does not allow for the usual counter-logical behaviors encountered in theories with postselection (see e.g. [131, 132]) because it is inherently constraining the allowed states into a basis that is quasi-orthogonal to the one of the postselection (see also [133, 134]). Another way to picture it is that the theory allows for a postselection but in a basis that is by construction mutually unbiased [74] with respect to the basis of the state: in the example, Alice can in general postselect in any state of the form $\mathbb{1} + p\, Z$, where p real and $p^2 \leq 1$ because of positivity[54]. That is, she can choose to project the state into a mixture of projectors $|0\rangle\!\langle 0|$ and $|1\rangle\!\langle 1|$, but the state is itself built by superposing the eigenstates of the X operator, $|\pm\rangle$, with the ones of the Y, $|\pm i\rangle$, and these are the two mutually unbiased bases w.r.t. the computational. Therefore her postselection cannot be used to distinguish from the maximally mixed state $\mathbb{1}/2$. Equivalently, she cannot distinguish between states by choosing $p \neq 0$, which explains how the theory is operationally equivalent (in the sense of 'tomographically indistinguishable' [78, 79]) to a qubit quantum theory with one 'forbidden' axis of the Bloch sphere.

[131]: Aaronson (2005), *Quantum computing, postselection, and probabilistic polynomial-time*.

[132]: Lloyd et al. (2011), *Closed Timelike Curves via Postselection: Theory and Experimental Test of Consistency*.

[133]: Silva et al. (2014), *Pre- and postselected quantum states: Density matrices, tomography, and Kraus operators*.

[134]: Silva et al. (2017), *Connecting processes with indefinite causal order and multi-time quantum states*.

[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.

54: See Lemma C.1.2 presented in the methods.

[78]: D'Ariano et al. (2017), *Quantum Theory from First Principles: An Informational Approach*.

[79]: Plávala (2021), *General probabilistic theories: An introduction*.

In other words, despite its postselected aspect, Theorem 3.3.2 guarantees the pair $\mathscr{A}, \overline{\mathscr{A}}$ to be a perfectly well-behaved preparation and measurement pair. Meaning that no matter the settings x and y, the choices of $V_{|x}$, $N_{|y}$ and its resolution $\{N_{b|y}\}$, by construction no incoherence like overnormalized or negative probabilities can be found in the distribution of the outcomes. This impossibility of incoherence is the fact that y cannot get correlated to x for any choice of operation in Equation (3.108); in Subsection 3.3.3 this was referred to as the *statistical independence on average* of a state and effect dual pair.

Interpreting the signaling in a one-party example is of course trivial. Yet, as soon as more than one party is allowed, for example by associating a party with the state and another one with the effect, the possibility of deterministically sending a signal from one party to another will coincide with this kind of allowed postselection. This behavior also appears in quantum instrument formalism[55], where the choice $\mathcal{M}_{|y}$ of *which* quantum channel the elements of an instrument $\{\mathcal{M}_{b|y}\}$ are summing up to also amounts to deterministically inducing a bias in the outcome probability, $p(b|x,y) \neq p(b|x,y')$.

55: As presented in the motivating example.

3.6.1. Bipartite Biased Quantum Theory

In the bipartite case, there are now two copies of the state structures associated with the systems of two parties $\mathscr{A} = \mathscr{B} = \mathrm{Span}\{\mathbb{1}, X, Y\}$. Following Section 3.5, the scenario under consideration involves two parties A and B sharing a bipartite system whose state belongs to a composition of state structures $\mathscr{A}$ and $\mathscr{B}$. The 'measurements' performed by these parties on their shared states are resolutions of $\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$ in a tensor product, where $\overline{\mathscr{A}} = \overline{\mathscr{B}} = \mathrm{Span}\{\mathbb{1}, Z\}$.

The difference between a bipartite no-signaling composite state, meaning an element of $\mathscr{A} \otimes \mathscr{B}$, and the more general bipartite compositions $\mathscr{A} \prec \mathscr{B}$, $\mathscr{A} \succ \mathscr{B}$ and $\mathscr{A} \parr \mathscr{B}$ becomes apparent. The set of all valid states normalized on the local effects resolving $\overline{\mathscr{A}} \otimes \overline{\mathscr{B}}$ is $\overline{\overline{\mathscr{A}} \otimes \overline{\mathscr{B}}} =: \mathscr{A} \parr \mathscr{B}$ which, according to Proposition 3.3.2 is made of the following 13 basis elements:

$$\begin{aligned}\mathscr{A} \parr \mathscr{B} \subset \mathrm{Span}\Big\{&\mathbb{1}^A \otimes \mathbb{1}^B, \mathbb{1}^A \otimes X^B, \mathbb{1}^A \otimes Y^B, X^A \otimes \mathbb{1}^B, X^A \otimes X^B, X^A \otimes Y^B, X^A \otimes Z^B, \\ &Y^A \otimes \mathbb{1}^B, Y^A \otimes X^B, Y^A \otimes Y^B, Y^A \otimes Z^B, Z^A \otimes X^B, Z^A \otimes Y^B\Big\}.\end{aligned} \tag{3.109}$$

These were obtained by applying the projector $\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} = \mathcal{I}_A \otimes \mathcal{I}_B - \overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B$, where $\overline{\mathcal{P}}$ is the projector on the dual operator system, spanned by $\{\mathbb{1}, Z\}$.

On the other hand, the no-signaling composition of $\mathscr{A}$ and $\mathscr{B}$, $\mathscr{A} \otimes \mathscr{B}$ as by Definition 3.4.1, is actually a subspace of $\mathscr{A} \parr \mathscr{B}$, its no-signaling subspace, but only made of 9 elements. The four missing elements are

$$\mathscr{A} \parr \mathscr{B} \backslash \mathscr{A} \otimes \mathscr{B} \subset \mathrm{Span}\Big\{X^A \otimes Z^B, Y^A \otimes Z^B, Z^A \otimes X^B, Z^A \otimes Y^B\Big\}, \tag{3.110}$$

which are exactly the elements containing a Z term. The consequence of this observation is that if Alice and Bob share a bipartite no-signaling state $W \in \mathscr{A} \otimes \mathscr{B}$, they may observe nonlocal entanglement effects on their outcome distributions[56], but these correlations will obey no-signaling constraints (1.6) in both directions. If, however, they share a general bipartite state $W \in \mathscr{A} \parr \mathscr{B}$, they may use it to achieve deterministic signaling: for certain states, they will be able to signal perfectly in one direction, i.e. Alice can perfectly send a message to Bob, and vice-versa. These allow (deterministic) signaling between A and B because they do not satisfy the conditions (3.97) so Lemma 3.5.3 applies.

56: Actually, the smallness of the dimension makes it so that all states in $\mathscr{A} \otimes \mathscr{B}$ are separable. The proof is provided as a comment, Appendix C.3.5.

Indeed, these basis elements are those which are quasi-orthogonal globally, meaning that the operators $W \in \mathscr{A} \parr \mathscr{B}$ that contains some of them will verify $\forall N^A \in \overline{\mathscr{A}}, \forall N^B \in \overline{\mathscr{B}}$

$$\mathrm{Tr}\left[\left(N^A \otimes N^B\right) \cdot W\right] = \frac{1}{d_A d_B} \mathrm{Tr}\left[\left(N^A \otimes N^B\right)\right] \mathrm{Tr}\left[W\right], \tag{3.111}$$

because they satisfy Theorem 3.3.2 and therefore Equation (3.47). But some of these W will not obey quasi-orthogonality with respect to a local measurement, meaning that they will fail to satisfy at least one of Equations (3.97). For example, the subset $\mathscr{A} \prec \mathscr{B} \subset \mathscr{A} \parr \mathscr{B}$ defined in

Definition 3.5.2 and characterized by Proposition 3.5.4 has elements W such that

$$\mathrm{Tr}_A\left[\left(N^A \otimes \mathbb{1}\right)\cdot W\right] \neq 1/d_A \mathrm{Tr}_A\left[N^A\right]\mathrm{Tr}_A\left[W\right] \ . \tag{3.112}$$

This means Alice can use W to signal to Bob. Observe moreover that these four extra terms allowing for signaling split into either set $\mathscr{A} \prec \mathscr{B}$,

$$\mathscr{A} \prec \mathscr{B} \backslash \mathscr{A} \otimes \mathscr{B} \subset \mathrm{Span}\left\{X^A \otimes Z^B, Y^A \otimes Z^B\right\} , \tag{3.113}$$

obeying only condition (3.97a). Or into set $\mathscr{A} \succ \mathscr{B}$,

$$\mathscr{A} \succ \mathscr{B} \backslash \mathscr{A} \otimes \mathscr{B} \subset \mathrm{Span}\left\{Z^A \otimes X^B, Z^A \otimes Y^B\right\} , \tag{3.114}$$

obeying (3.97b)[57].

57: The fact that the union of sets $\mathscr{A} \prec \mathscr{B}$ and $\mathscr{A} \succ \mathscr{B}$ has the same span as set $\mathscr{A} \parr \mathscr{B}$ is actually a defining feature of the algebra of projectors that underly the characterization methods. This point will be explored in Chapter 5.

For instance, consider the task where Alice receives a classical bit x and wants to communicate it to Bob so that his outcome b has the same value as her setting, $b = x$. Without any resources, Bob can only guess and thus succeeds with $p(b = x) = 1/2$. Now if they measure a shared state in $\mathscr{A} \parr \mathscr{B}$, they can pick the following state:

$$W_{A \prec B} = \frac{1}{4}\left(\mathbb{1}^A \otimes \mathbb{1}^B + Z^A \otimes X^B\right) \ , \tag{3.115}$$

and choose to do the following: Alice 'steers' her measurement towards $|0\rangle$ or $|1\rangle$ depending on x,

$$N^A_{|x} = \mathbb{1}^A + (-1)^x \ Z^A \ , \tag{3.116}$$

while Bob is measuring an unbiased $N^B = \mathbb{1}$ resolved into a measurement in the $|\pm\rangle$ basis,

$$N^B_b = \frac{1}{2}\left(\mathbb{1}^B + (-1)^b \ X^B\right) \ , \tag{3.117}$$

where $b = 0, 1$ so that his probabilistic effects sum up to a unit effect, $N^B_0 + N^B_1 = N^B = \mathbb{1}^B$. One can check that they are effectively properly normalized positive operators belonging to the proper state structures, $N^A_{|x} \in \overline{\mathscr{A}}$, $N^B \in \overline{\mathscr{B}}$, despite $N^B_0, N^B_1 \notin \overline{\mathscr{B}}$. The measurement results in the following probability distribution:

$$p(b|x) = \mathrm{Tr}\left[\left(N^A_{|x} \otimes N^B_b\right)\cdot W_{A \prec B}\right] \ . \tag{3.118}$$

Injecting the above expressions into it yields

$$p(b|x) = \frac{1}{2}\left(1 + (-1)^{x+b}\right) \ , \tag{3.119}$$

which gives 0 when $x \neq b$ and 1 when $x = b$; Alice's setting is perfectly correlated to Bob's outcome. A bit was deterministically sent from A to B, $p(x = b) = 1$. A similar thing can be done in the reverse direction, e.g.

$$W_{A \succ B} = \frac{1}{4}\left(\mathbb{1}^A \otimes \mathbb{1}^B + X^A \otimes Z^B\right) \tag{3.120}$$

will allow a strategy in which the setting of Bob is equivalent to the outcome of Alice.

One can also try to construct a state in $\mathscr{A} \parr \mathscr{B}$ superposing the two signaling directions like

$$W_{A \parr B} = \frac{1}{4}\left(\mathbb{1}^A \otimes \mathbb{1}^B + p\left(X^A \otimes Z^B + Z^A \otimes X^B\right)\right) . \tag{3.121}$$

with $p \in \mathbb{R}$. Such a state is the analog of the OCB Process Matrix (PM) [5] in this toy theory: similar to how Alice and Bob can tune the channels they plug in the OCB PM so to signal in one direction or the other, here Alice and Bob can tune their measurement to signal in one direction or the other.

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

The direct follow-up question is whether such a state can be used to violate a causal inequality. It is actually known that quantum theories with a specific kind of linear post-selection restricted to avoid paradoxes can reproduce all process matrices and, in particular, those that violate causal inequalities [47, 134–136]. Since biased quantum theory is very similar to such theories, the question makes sense. But this is where bipartite biased quantum theory diverges from bipartite process matrix formalism: it cannot. Positivity implies that the weight p obeys $|p| \leq 1/2$, whence the state is separable into two one-way signaling states,

[47]: Oreshkov et al. (2016), *Operational quantum theory without predefined time*.
[134]: Silva et al. (2017), *Connecting processes with indefinite causal order and multi-time quantum states*.
[135]: Araújo et al. (2017), *Quantum computation with indefinite causal structures*.
[136]: Guryanova et al. (2019), *Exploring the limits of no backwards in time signalling*.

$$W_{A \parr B} = \frac{1}{2} W_{A \prec B} + \frac{1}{2} W_{A \succ B} , \tag{3.122}$$

with $W_{A \prec B}$ and $W_{A \succ B}$ given by Equations (3.115) and (3.120), respectively.

As a matter of fact, it can be shown that any state in state structure $\mathscr{A} \parr \mathscr{B}$ is decomposable as a convex mixture of one in $\mathscr{A} \prec \mathscr{B}$ with one in $\mathscr{A} \succ \mathscr{B}$.

Proposition 3.6.1 *The two-way signaling bipartite composition of the states spaces of biased quantum theory, $\mathscr{A} \parr \mathscr{B}$, is equivalent to the convex hull of the one-way compositions $\mathscr{A} \prec \mathscr{B}$ and $\mathscr{A} \succ \mathscr{B}$.*

Proof. Suppose it was not the case, then any non-trivial combination would contain a term of the form $qU \otimes Z + pZ \otimes V$, where U and V are weighted combinations of X and Y of the form like $U = aX + bY$ where $a^2 + b^2 = 1$. Such a non-trivial operator W would read

$$W = \frac{1}{4}\left(\mathbb{1}^A \otimes \mathbb{1}^B + qU^A \otimes Z^B + pZ^A \otimes V^B + \ldots\right) . \tag{3.123}$$

By Lemma C.1.2, the weights q, p must obey at least $|p| + |q| > 1$ if they were to prevent a convex decomposition like

$$W = |q|\frac{1}{4}\left(\mathbb{1}^A \otimes \mathbb{1}^B + \frac{q}{|q|}U^A \otimes Z^B + \ldots\right) + |p|\frac{1}{4}\left(\mathbb{1}^A \otimes \mathbb{1}^B + \frac{p}{|p|}Z^A \otimes V^B + \ldots\right) . \tag{3.124}$$

However, it is always possible to form a maximally entangled state of the form

$$\rho = \frac{1}{4}\left(\mathbb{1}^A \otimes \mathbb{1}^B - \frac{|q|}{q}U^A \otimes Z^B - \frac{|p|}{p}Z^A \otimes V^B - \frac{|q|}{q}\frac{|p|}{p}(U \cdot Z)^A \otimes (Z \cdot V)^B\right) . \tag{3.125}$$

As if the Pauli basis in Alice's side was redefined[58] under $X \mapsto U$, $Y \mapsto i\,U \cdot Z$ and $Z \mapsto Z$ while at the same it is redefined in Bob's side with $Z \mapsto V, Y \mapsto i\,Z \cdot V$ and $X \mapsto Z$. Such a state, whose third term $\frac{|q|}{q}\frac{|p|}{p}(U \cdot Z)^A \otimes (Z \cdot V)^B$ is outside of the support of $\mathscr{A} \parr \mathscr{B}$, has an inner product with W equal to

$$(W\ ,\ \rho) = 1/16\,(1 - (|p| + |q|))\ . \tag{3.126}$$

Therefore, if p and q were not convex weights, then $|p| + |q| > 1$, so the inner product would be negative meaning that the operator W would not be positive, a contradiction. □

58: One can check that the algebraic properties and normalization are indeed unchanged; these are still mutually unbiased observables but in a different eigenbasis.

This is at odds with the OCB PM which is not a convex mixture of one-way signaling terms but an affine one. Because of that, if the parties were trying to use the bipartite state to violate a causal inequality similar to the one presented in [5] they would never do better than a mixture of one-way strategies, so they would not violate the inequality.

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

For example, consider the game where it is required that Alice's setting is equivalent to Bob's outcome and vice-versa, $x = b$ and $a = y$. Assuming all variables to be uniformly distributed classical bits, the probability in the no-signaling case is $p(a = y, b = x) = 1/4$, and it cannot be improved using non-locality [59]. However, if A and B share the state (3.121), they can both do 'half' of the one-way signaling strategy, with the effects

$$N^A_{a|x} = \frac{1}{2}\left(\mathbb{1}^A + \frac{1}{\sqrt{2}}(-1)^a\, X^A + \frac{1}{\sqrt{2}}(-1)^x\, Z^A\right)\ , \tag{3.127a}$$

$$N^B_{b|y} = \frac{1}{2}\left(\mathbb{1}^B + \frac{1}{\sqrt{2}}\,(-1)^b\, X^B + \frac{1}{\sqrt{2}}(-1)^y\, Z^B\right)\ , \tag{3.127b}$$

59: Beside note 56, notice this is a 'Guess Your Neighbour Input (GYNI)' game with uniformly distributed input. It is known that neither classical nor quantum no-signaling strategies can outperform random guessing, see Reference [97].

[97]: Almeida et al. (2010), *Guess Your Neighbor's Input: A Multipartite Nonlocal Game with No Quantum Advantage*.

which are adding up to

$$N^A_{|x} = \mathbb{1}^A + \frac{1}{\sqrt{2}}(-1)^x\, Z^A\ , \tag{3.128a}$$

$$N^B_{|y} = \mathbb{1}^B + \frac{1}{\sqrt{2}}(-1)^y\, Z^B\ . \tag{3.128b}$$

This specific choice of operations result in the probability distribution $p(a, b|x, y) = \mathrm{Tr}\left[\left(N^A_{a|x} \otimes N^B_{b|y}\right) \cdot W\right]$. Explicit computation gives

$$p(a, b|x, y) = \frac{1}{4}\left(1 + \frac{1}{4}\left((-1)^{a+y} + (-1)^{b+x}\right)\right)\ . \tag{3.129}$$

Hence, the probability of Alice guessing correctly individually is $\sum_b p(a = y, b|x, y) = 5/8$, and the same holds for Bob, this is slightly better than the purely random case. Yet, while the probability of both of them correctly guessing each other's input is $p(a = y, b = x) = 3/8$ which is better than the no-signaling (i.e. purely random) case, this is worse than a strategy using one-way signaling in a pre-decided direction followed by a random guess for the other, resulting in a probability of $p_{one-way}(a = y, b = x) = 1/2$.

The bottom line of this first example is that compared to quantum theory, there exist states in $\overline{\overline{\mathscr{A} \otimes \mathscr{B}}}$ which allow to deterministically beat either one of the no-signaling constraints (1.6) and at the same time there exist states that allow for two-ways probabilistic signaling as well. However, the biased quantum theory is too simple to beat a causal inequality: as the two-way signaling states are convex sums of one-way signaling strategy, their probability distribution can always be explained in terms of a mixture of one-way signaling correlations.

Remark that with respect to every Bell-kind scenario with state structures characterized by the projective methods, quantum theory plays a special role. This is the only state structure whose bipartite states normalized on no-signaling composite effects are automatically no-signaling. It can be checked from Equation (3.70c) that the projector $\overline{\overline{\mathcal{I}}_A \otimes \overline{\mathcal{I}}_B}$ associated with the 'fully signaling' bipartite quantum states $\mathscr{A}_{quant.} \parr \mathscr{B}_{quant.}$ is equivalent to the projector $\mathcal{I}_A \otimes \mathcal{I}_B$ associated with the 'no-signaling' subset of bipartite states, $\mathscr{A}_{quant.} \otimes \mathscr{B}_{quant.}$. Hence, bipartite quantum states are automatically no-signaling, $\mathscr{A}_{quant.} \parr \mathscr{B}_{quant.} = \mathscr{A}_{quant.} \otimes \mathscr{B}_{quant.}$. This may not be a surprise[60], but compared to the general properties of the state structures, this is an oddity.

60: Especially for those knowledgeable of the categorical presentation of quantum theory. This equality is the fact that the quantum processes are the morphisms of a Compact Closed category [137]. On the contrary, general higher-order quantum transformations are morphisms of a $*-$autonomous category [33, 36], for which $\mathscr{A} \parr \mathscr{B} \supsetneq \mathscr{A} \otimes \mathscr{B}$.

[137]: Abramsky et al. (2004), *A Categorical Semantics of Quantum Protocols*.
[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.
[36]: Simmons et al. (2022), *Higher-order causal theories are models of BV-logic*.

The particular behavior of quantum theory is the reason why a different theory (the biased quantum theory) was chosen as the illustrating example for this section. The study of quantum theory and its higher-order generalizations in the state structures language will happen in a later chapter, after the algebraic properties of the projectors have been presented.

3.6.2. Biased Quantum Channel Theory as a Toy Model for OCB Correlations.

While most of the important features of the bipartite state structures were present in the bipartite state case, it was not possible to violate a causal inequality with it. But can it work for the transformation as in Definition 3.4.3? After all, this is another case of bipartite state structure that can be built in biased quantum theory. In this case, the signaling properties are not with respect to parties 'on the left' or 'on the right' of the shared 'state' W; but rather with respect to the party 'at the input' or 'at the output' of the shared 'channel' M.

This kind of state structure can for example represent the evolution of states of the biased quantum theory. This evolution is by the no-restriction hypothesis all the structure-preserving maps from the biased theory to itself. Theorem 3.4.1 characterize these as the set $\mathscr{A} \to \mathscr{B}$ where the state structure of the input and output are the same, $\mathscr{A} = \mathscr{B} \subset \mathrm{Span}\,\{\mathbb{1}, X, Y\}$; its basis elements are

$$\begin{aligned}\mathscr{A} \to \mathscr{B} \subset \mathrm{Span}\Big\{&\mathbb{1}^A \otimes \mathbb{1}^B, \mathbb{1}^A \otimes X^B, \mathbb{1}^A \otimes Y^B, X^A \otimes X^B, X^A \otimes Y^B, Y^A \otimes X^B, \\ &Y^A \otimes Y^B, Z^A \otimes \mathbb{1}^B, Z^A \otimes X^B, Z^A \otimes Y^B, Z^A \otimes Z^B\Big\}.\end{aligned} \tag{3.130}$$

Again, out of these eleven terms the ones allowing signaling can be singled out, and then can be further split with respect to the direction of signaling they allow,

$$\mathscr{A} \to \mathscr{B} \backslash \overline{\mathscr{A}} \otimes \mathscr{B} \subset \mathrm{Span}\left\{X^A \otimes X^B, X^A \otimes Y^B, Y^A \otimes X^B, Y^A \otimes Y^B, Z^A \otimes Z^B\right\} ; \tag{3.131a}$$

$$\overline{\mathscr{A}} \prec \mathscr{B} \backslash \overline{\mathscr{A}} \otimes \mathscr{B} \subset \mathrm{Span}\left\{X^A \otimes X^B, X^A \otimes Y^B, Y^A \otimes X^B, Y^A \otimes Y^B\right\} ; \tag{3.131b}$$

$$\overline{\mathscr{A}} \succ \mathscr{B} \backslash \overline{\mathscr{A}} \otimes \mathscr{B} \subset \mathrm{Span}\left\{Z^A \otimes Z^B\right\} . \tag{3.131c}$$

Here, the fact that a transformation between state structures is a composition that does not forbid signaling in any direction is more striking as the deterministic signaling from output to input can be interpreted as a 'backward-in-time' influence. Suppose Alice and Bob are sharing the channel

$$M_{A \prec B} = \frac{1}{2}\left(\mathbb{1}^A \otimes \mathbb{1}^B + X^A \otimes X^B\right) \tag{3.132}$$

from state structure $\overline{\mathscr{A}} \prec \mathscr{B}$ as defined by Proposition 3.5.4. Alice can perfectly signal to Bob by encoding her setting x in the X basis, $V_{|x} = 1/2(\mathbb{1} + (-1)^x X)$, and if Bob measures in the same basis, they effectively have a perfect single bit channel, $p(b = x) = 1$.

On the other hand, suppose they share the channel

$$M_{A \succ B} = \frac{1}{2}\left(\mathbb{1}^A \otimes \mathbb{1}^B + Z^A \otimes Z^B\right) \tag{3.133}$$

from state structure $\overline{\mathscr{A}} \succ \mathscr{B}$. Now it is Bob who can perfectly signal to Alice: Alice has to use an ancilla so that she can prepare the same joint state as the bipartite example, Equation (3.115),

$$W_{A' \succ A} = \frac{1}{4}\left(\mathbb{1}^{A'} \otimes \mathbb{1}^A + X^{A'} \otimes Z^A\right) . \tag{3.134}$$

She sends the A part through the channel and keeps the A' part as her ancilla. Bob can then apply the measurement $N^B_{|y} = \mathbb{1} + (-1)^y Z$ at the outcome of the channel, depending on the variable y he wishes to send. Alice can then finally measure her ancilla in the X basis, $N^{A'}_a = 1/2(\mathbb{1} + (-1)^a X)$, leading to her performing the probabilistic operation $V^A_a := N^{A'}_a * W_{A' \succ A}$

$$V^A_a = \mathrm{Tr}_{A'}\left[\left(N^{A'}_a \otimes \mathbb{1}^A\right) W_{A' \succ A}\right]^T = \frac{1}{4}\left(\mathbb{1}^A + (-1)^a Z^A\right) . \tag{3.135}$$

The outcome distribution is then the distribution obtained from $V^A_a * M_{A \succ B} * N^B_{|y} = p(a|y)$,

$$p(a|y) = \mathrm{Tr}\left[\left(V^A_a \otimes \left(N^B_{|y}\right)^T\right) \cdot M_{A \succ B}\right] , \tag{3.136}$$

and she will get perfect correlation with Bob setting, $p(a = y) = 1$ exactly like in the bipartite example. Therefore, in the state structure $\mathscr{A} \to \mathscr{B}$, there are transformations allowing perfect signaling from Alice to Bob, like $M_{A \prec B}$ as well as from Bob to Alice, like $M_{A \succ B}$.

Contrastingly with the bipartite state case, a channel in biased quantum theory can be represented by an operator M that does not admit a convex decomposition like $M = pM_{A \prec B} + (1 - p)M_{A \succ B}$. The simplest example of this is the identity channel $\mathcal{I}$ from A to B. It has the following CJ representation:

$$M_{\mathcal{I}} = \frac{1}{2}\left(\mathbb{1}^A \otimes \mathbb{1}^B + X^A \otimes X^B - Y^A \otimes Y^B + Z^A \otimes Z^B\right) \, , \quad (3.137)$$

which is a valid element of the set of transformations $\mathscr{A} \to \mathscr{B}$ as can be checked using Theorem 3.4.1. This kind of channel can be used by the party to send information back and forth. It is this kind of shared channel that violates a causal inequality in the same manner that the OCB does. As for the bipartite process matrix, the game is a GYNI conditioned by a referee. Alice and Bob's settings x and y are evenly distributed bits, and they additionally receive a shared bit z sent by a referee. The game is won if Bob guesses Alice's setting correctly when $z = 0$ or if Alice guesses Bob's when $z = 1$. Encoding the guess of each player on their outcomes, the game-winning probability is equal to $\sum_{a,b} 1/2p(a = y, b|x, y, z = 0) + 1/2p(a, b = x|x, y, z = 1)$.

A classical strategy will conclude that the winning probability is $3/4$ since Alice can always send her setting to Bob, who is assumed at the output of the channel, hence in her causal future, but since Alice is at the input side, she cannot obtain the future setting of Bob y any differently than by guessing. But this is classical reasoning – this bound of $3/4$, which is the causal inequality of the OCB example, is violated using biased channel theory. Here is how: at her side of the channel, Alice can choose to do the following operation

$$V_{a|x} = 1/2\left(\frac{1}{2}\left(\mathbb{1} + \frac{1}{\sqrt{2}}\left((-1)^x X + (-1)^a Z\right)\right)\right) \, , \quad (3.138)$$

and Bob can choose to either measure or send depending on z:

$$N_{b|y,z} = \delta_{z,0}\left(\frac{1}{2}\left(\mathbb{1} + (-1)^b X\right)\right) + \delta_{z,1}\left(\mathbb{1} + (-1)^y Z\right) \, , \quad (3.139)$$

The probability is then $p(a, b|x, y, z) = V_{a|x} * M_{\mathcal{I}} * N_{b|y,z}$ given by

$$p(a, b|x, y, z) = \delta_{z,0}\frac{1}{2}\left(1 + \frac{(-1)^{b+x}}{\sqrt{2}}\right) + \delta_{z,1}\frac{1}{2}(1 + \frac{(-1)^{a+y}}{\sqrt{2}}) \, . \quad (3.140)$$

This yields the same success probability as in the case of the OCB, $\sum_{a,b} 1/2p(a, b = x|x, y, z = 0) + 1/2p(a = y, b|x, y, z = 1) = 1/2(1 + 1/\sqrt{2}) \approx 0.85$.

Starting from a toy theory to illustrate the use of the projective methods, the most well-known example of indefinite causal order violating a causal inequality was recovered. The biased quantum channel theory indeed reproduces the behavior of the bipartite process matrix formalism to a certain extent. This hints that the projective characterization of state structure allows for the abstraction of signaling properties, independently of the considered theory. The important ingredients needed to violate a bipartite causal inequality were 1) a scenario involving two local parties sharing an object of a state structure $\mathscr{A} \parr \mathscr{B}$; 2) that two-way signaling composition of state structures was non-trivial, $\mathscr{A} \parr \mathscr{B} \neq \mathscr{A} \otimes \mathscr{B}$; 3) that the state structure possess elements outside of the convex hull of the two one-way signaling compositions $\mathscr{A} \prec \mathscr{B}$ and $\mathscr{A} \succ \mathscr{B}$. The second reason is why quantum theory cannot violate a causal inequality[61] and the third

61: First proven in [5] then in [138].

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

[138]: Purves et al. (2021), *Quantum Theory Cannot Violate a Causal Inequality*.

reason is why the bipartite biased quantum theory of last section was not able to violate one.

Abstracting these last two points is the study of how the signaling relations are encoded in the composition rules of state structures, this is the topic of the next chapter. Looking at these rules alone can thus already give a lot of information about the signaling structure of processes, even if the local state structures $\mathscr{A}, \mathscr{B}$ associated with each party are left unspecified. As will be shown, these relations are themselves encoded in the algebra of the compositions of projectors. On top of knowing how the algebra works allows for a quick characterization of the possible signaling directions of a given higher-order transformation, in this chapter it will also be shown that this algebra is a known model of logic whose simplicity allows for automated proofs.

Before concluding this example, there remain a few observations that are worth mentioning.

First, remark that Alice has no procedure involving preparations and measurements that can result in her operation being the resolution of Equation (3.138). Again because of Proposition 3.6.1, if Alice is restricted to preparations and measurements in the biased theory she cannot obtain (3.138) as the combination of preparing a bipartite state and measuring one half of it like $V^A_{a|x} = N^{A'}_{a|x} * W^{A'A}_{|x}$ where[62] $N^{A'}_{a|x}$ is a resolution of $\overline{\mathscr{A}}$ and $W^{A'A}_{|x}$ is a bipartite state in $\mathscr{A}' \parr \mathscr{A}$. If she was allowed to share quantum entanglement with her ancillary system, then she could do it by teleportation [139], for example by taking $V^A_{a|x}$ of Equation (3.138) as a resolution of $\overline{\mathscr{A}}'$ applied to a maximally entangled state of subsystems A' and A. This is one of the main drawbacks of the 'every resolution is an operation' hypothesis underlying Definition 3.2.4 on which the whole formalism is based. As discussed in Appendix C.3.1, sometimes it seems that some resolutions are not feasible without breaking out of the state structure. Studying the extent of this problem as well as the definition of a resolution is a direction left open for future research.

62: $\mathscr{A}'$ is a copy of $\mathscr{A}$ in an ancillary subsystem A' of Alice.

The second observation, again on the resolution (3.138), is its link with the 2-to-1 Quantum Random Access Code (QRAC) [140, 141]. The 'state' seen by Bob after Alice's transmission is indeed of the form $\frac{1}{2}\left(\mathbb{1} + \frac{1}{\sqrt{2}}\left((-1)^x X + (-1)^a Z\right)\right)$ for fixed a and x. As discussed in the sources mentioned, this state is known to be the best encoding of two bits on a qubit with respect to the task of randomly accessing the value of either. The operation of Bob amounts to deciding which bit he wants to access by choosing between measuring basis X or Z. The success bound of 85% then makes sense as it is the optimal bound for encoding two bits on a qubit. The difference with a QRAC is that if he accesses the Z basis he is not reading but encoding on x. Hence, by choosing a basis the parties can choose the directionality of the flow of classical information. As only one bit is accessed at once, this is a sort of delocalized random access code: by his choice of operation, Bob decides whether Alice gets to see his setting or if he gets to see hers. Interpreting the maximal violation of causal inequality of the OCB-kind as optimizing a delocalized random access code is also a direction left open for future research.

[140]: Ambainis et al. (1999), *Dense quantum coding and a lower bound for 1-way quantum automata*.
[141]: Ambainis et al. (2002), *Dense Quantum Coding and Quantum Finite Automata*.

The third observation[63], related to the second, is that interpreting the bipartite biased quantum state as a measure-and-prepare scenario may

63: For which I am grateful to Ravi Kunjwal for pointing out dimension witnesses.

[142]: Brunner et al. (2013), *Dimension Witnesses and Quantum State Discrimination*.

lead to the elaboration of new dimension witnesses [142]. The ability to violate a causal inequality indeed requires a minimal dimension of the joint bipartite system, as shown in the bipartite example compared to the OCB process matrix. Hence, it is tempting to postulate that it can lead to device-independent bounds on the dimension of a system. However, the biased quantum channel theory, while of a smaller dimension than the bipartite biased theory[64], can violate a causal inequality. So, the interpretation as input and output also appears to play a role in the device-independent bounds implied by causal inequalities. This is yet another direction open for future prospects.

64: Equations (3.109) and (3.130) show that the first has 11 basis elements while the second has 13.

3.7. Summary

This chapter presented the concept of a *state structure* gradually so as to keep track of the assumptions. The central object defined in it were the state structure, Definition 3.2.2, and the projector on an operator system Definition 3.2.7. These two tools have been used to derive the basic rules for building and characterizing classes of higher-order processes as state structures defined out of known state structures. These rules are summarized below in the tables 3.1 and 3.2.

The characterization has been conducted in two steps: first, the single-partite characterization of 'state and effect' pairs of state structures, and second, the characterization of bipartite composite state structures from their interpretation as transformations.

The first step was done by defining the equivalent of process functionals on operations that resolved state structures. A single-partite characterization. To do so, starting from a state structure, Definition 3.2.2, the concept of a resolution has been defined in Definition 3.2.4. In doing so, the element of a state structure can be interpreted as the deterministic operation of a party, so that her probabilistic operations are represented by the resolution of the element in the state structure. Here, a first major assumption has been made: that all elements of all resolutions of a state structure actually correspond to a probabilistic operation that some party can do in her lab. Elements discussing this hypothesis are presented in Appendix C.3.1.

Note that the two definitions have been made out of formal analogy. The characterization, as will be shown next, recovers the results of the literature on higher-order processes. Therefore, discussing the assumptions underlying these definitions is only done for completeness as well as to propose some future research directions. These have no consequences in the characterization of signaling in higher-order processes per se.

From the definition of a state structure, the concept of a frame function was subsequently defined in Definition 3.3.1. The frame function generalizes Gleason-kind proofs to resolutions of state structure by postulating a functional from the probabilistic operations to a probability. This is where a second major assumption has been made: the existence of such functionals obeying Equations (3.24) requires the hypothesis of generalized non-contextuality and homogeneity. Elements discussing these are presented in Appendix C.3.2.

With resolutions and frame functions, the dual state structure was characterized in Theorem 3.3.2. This dual state structure is exactly the CJ representation of the set of frame functions, that is, of deterministic functionals on a state structure.

Using this result, the characterization methods have been applied to the admissible transformations between state structures as in Definition 3.4.3.

Here, another assumption was made in the formalism: that the parties on each side of a transformation between two state structures can, in principle, implement any no-signaling bipartite operation and their resolutions. But this ignores the eventuality that interaction is sometimes needed to achieve the composite object, as is the case for no-signaling bipartite channels [61]. Hence, when the transformation is interpreted as time evolution this 'hidden' interaction may be backward in time. As with the other two, it is not clear operationally why this assumption should hold *a priori*, and some elements of this hypothesis are presented in Appendix C.3.4. But as with the other two and as mentioned in the margin, this non-justified hypothesis does lead to the generalization of previous characterizations obtained in the literature, so discussing its validity is left for future work.

[61]: Beckman et al. (2001), *Causal and localizable quantum operations*.

The final characterization step was obtained by realizing that transformations can be phrased as bipartite states. This is a consequence of assuming that every object is the CJ representation of some (super)map. As such, the common characterization of maps and bipartite states has been guided by requiring certain no-signaling constraints between the parties. It was recognized that the transformation is a composition that allows for two-way signaling while the tensor product of state structures is a composition allowing for none. Lemma 3.5.3 has been the key for figuring out the intermediate one-way signaling composition in Proposition 3.5.4. This allowed the sorting of the bipartite compositions into four classes: no-signaling, one-way signaling from A to B, from B to A, and two-way signaling as in Definition 3.5.2.

In addition, all these characterization methods were shown to be essentially the characterization of the subspace on which the state structures are defined. As such, the characterization can be reduced by studying the algebraic properties of composing the projectors to define the composite state structures. These rules are summarized in Table 3.2. In Chapter 5, a more in-depth study of this algebra will be conducted.

Finally, the biased quantum theory has been presented as the concluding example of the section. This is a proof of concept of the characterization methods since it is based on a postselected toy model, which can be phrased as a state structure. This toy model was used to present how concretely state structures encode signaling directions and even went up to reproduce the OCB correlations.

Structure	Name	Space	Condition: $W \geq 0 \in$ Structure $\iff$
$\overline{\mathscr{A}}_{quant.}$	Qu. Unit Effects	$\mathcal{L}\left(\mathcal{H}^A\right)$	$W = \mathbb{1}_A$
$\mathscr{A}_{quant.}$	Qu. States	$\mathcal{L}\left(\mathcal{H}^A\right)$	$\mathrm{Tr}\left[W\right] = 1$
$\mathscr{A}$	State	$\mathcal{L}\left(\mathcal{H}^A\right)$	$\overline{\mathscr{A}}_{quant.} \subseteq \mathscr{A} \subseteq \mathscr{A}_{quant.}$
$\overline{\mathscr{A}}$	Unit Effect	$\mathcal{L}\left(\mathcal{H}^A\right)$	$\forall M \in \mathscr{A}: \quad \mathrm{Tr}\left[W \cdot M\right] = 1$
$\mathscr{A} \parr \mathscr{B}$	two-way sign. composite	$\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$	$\forall M_A \in \overline{\mathscr{A}}: \quad \mathrm{Tr}_A\left[W \cdot (M_A \otimes \mathbb{1}_B)\right] \in \mathscr{B}$ $\forall M_B \in \overline{\mathscr{B}}: \quad \mathrm{Tr}_B\left[W \cdot (\mathbb{1}_A \otimes M_B)\right] \in \mathscr{A}$
$\mathscr{A} \prec \mathscr{B}$	one-way sign. composite	$\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$	$\forall M_A \in \overline{\mathscr{A}}: \quad \mathrm{Tr}_A\left[W \cdot (M_A \otimes \mathbb{1}_B)\right] \in \mathscr{B}$ $\forall M_B \in \overline{\mathscr{B}}: \quad \mathrm{Tr}_B\left[W \cdot (\mathbb{1}_A \otimes M_B)\right] = c_B \mathrm{Tr}_B\left[W\right] \in \mathscr{A}$
$\mathscr{A} \otimes \mathscr{B}$	no-signaling composite	$\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$	$\forall M_A \in \overline{\mathscr{A}}: \quad \mathrm{Tr}_A\left[W \cdot (M_A \otimes \mathbb{1}_B)\right] = c_A \mathrm{Tr}_A\left[W\right] \in \mathscr{B}$ $\forall M_B \in \overline{\mathscr{B}}: \quad \mathrm{Tr}_B\left[W \cdot (\mathbb{1}_A \otimes M_B)\right] = c_B \mathrm{Tr}_B\left[W\right] \in \mathscr{A}$

Table 3.1.: Summary of the basic state structures and how to combine them with respect to some defining properties. (Qu. = Quantum; sign. = signaling.)
Each line reads “The <Name> state structure <Struct.> has support on <Space>. It is defined as the set of positive and trace-normalized operators that respect <Condition>.”.

Struct.	Characterization	Projector rule		
$\overline{\mathscr{A}}_{quant.}$	Equation (3.22)	$\mathcal{D}_A$	$\overset{(3.18)}{:}$	$\mathcal{D}_A(V_A) := \frac{\mathbb{1}_A}{d_A}\mathrm{Tr}\left[V_A\right]$
$\mathscr{A}_{quant.}$	Equation (3.21)	$\mathcal{I}_A$	$\overset{(3.16)}{:}$	$\forall V_A,\ \mathcal{I}_A(V_A) = V_A$
$\mathscr{A}$	Proposition 3.2.1	$\mathcal{P}_A$	:	Definition 3.2.7
$\overline{\mathscr{A}}$	Theorem 3.3.2	$\overline{\mathcal{P}}_A$	$\overset{(3.33)}{:=}$	$\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A$
$\mathscr{A} \to \mathscr{B}$	Theorem 3.4.1	$\mathcal{P}_A \to \mathcal{P}_B$	$\overset{(3.75)}{:=}$	$\mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{P}_A \otimes \mathcal{I}_B + \mathcal{P}_A \otimes \mathcal{P}_B - \mathcal{P}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B$
$\mathscr{A} \otimes \mathscr{B}$	Definition 3.4.1	$\mathcal{P}_A \otimes \mathcal{P}_B$	:	$(\mathcal{P}_A \otimes \mathcal{P}_B)\left\{\sum_i c_i\, V_i \otimes N_i\right\} = \sum_i c_i\, \mathcal{P}_A\left\{V_i\right\} \otimes \mathcal{P}_B\left\{N_i\right\}$
$\mathscr{A} \prec \mathscr{B}$	Proposition 3.5.4	$\mathcal{P}_A \prec \mathcal{P}_B$	$\overset{(3.102)}{:=}$	$\mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B$.
$\mathscr{A} \parr \mathscr{B}$	$\overline{\mathscr{A}} \to \mathscr{B}$	$\overline{\mathcal{P}}_A \to \mathcal{P}_B$	$\overset{(3.85)}{=}$	$\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B}$

Table 3.2.: Summary of the characterization of basic state structures and the associated projector rules.

Intermezzo: Towards the Projector Algebra

4.

> The only tangible reality about sense is the way it is written, the formalism; but the formalism remains an unaccommodating object of study, without true structure, a piece of *soft camembert*.
>
> Girard (1989), *Proofs and Types* [143]

With the characterization methods developed in the previous chapters, it looks like the problem of signaling in higher-order quantum theory can be addressed. A process involving a certain amount of subsystems $A_0, A_1, A_2, \ldots$ can be systematically probed for a fixed signaling order by exhaustively searching if it belongs to a state structure with fixed causal order like $((\mathscr{A}_0 \prec \mathscr{A}_1) \prec \mathscr{A}_2) \prec \ldots$. Using Proposition 3.5.4, this can be done by applying projector $((\mathcal{P}_{A_0} \prec \mathcal{P}_{A_1}) \prec \mathcal{P}_{A_2}) \prec \ldots$ on the process matrix representing it. If the process had a fixed causal order, the projectors may catch it by exhausting all possible permutations of the factors. This is implicitly what was done in the biased quantum theory example of the previous chapter: every time a new class of composite objects was defined, its support was split into the terms allowing for signaling in certain fixed directions.

However, some examples of state structures and constructions may not be so simple to decompose. Worse, with the increasing number of nodes, the number of projectors to test grows factorially, which is far from efficient. It can be asked whether the algebraic properties of the bipartite compositions found in the previous chapter, i.e., $\otimes$, $\prec$, and $\parr$, could help simplify these issues. To do so, the next chapter will conduct a systematic study of the algebra of projectors under these connectives.

In this section, some selected examples from the literature will be used to motivate some of the questions the projector algebra aims to answer. In particular, some peculiar behaviors of the state structures will be observed. Finding an explanation for these behaviors will motivate some aspects of the projector algebra while at the same time providing more examples of how to use the characterization methods. Remark that this section is facultative and can be skipped by a reader who already knows these works or who is in a hurry to get to the results. In this chapter, the base state structures $\mathscr{A}, \mathscr{B}, \ldots$ associated with each party in the following examples are assumed to be the set of quantum states, to which corresponds projector $\mathcal{I}$.

4.1. Quantum Theory and Isomorphisms

Here, the same examples of state spaces constructed for the Biased Quantum Theory in Section 3.6 are constructed for Quantum Theory. As announced, for this base state structure, some composite state spaces happen to be isomorphic. This illustrates the first reason for a general study of the compositions of state structure: finding isomorphisms.

4.1.1. Single Party Quantum Theory

This is the simplest example of a dual pair of state structures that can be built assuming Theorem 3.3.2. In this case, $\mathscr{A} = \mathscr{A}_{quant.}$ is the state space of finite-dimensional quantum theory; its underlying operator system has support over the whole of $\mathcal{L}\left(\mathcal{H}^A\right)$. To it is associated its negation, $\overline{\mathscr{A}}$, interpreted as 'the need of a state of $\mathscr{A}$ to obtain a probability of 1', i.e. the deterministic measurement of a state or *unit effects*. Concretely, the states are the regular notion of the quantum state in density matrix form, as defined in Equation (3.21). And to it corresponds the regular notion of unit effect as $\overline{\mathscr{A}} \equiv \{\mathbb{1}\}$ because Theorem 3.3.2 implies that the valid unit effect in $\overline{\mathscr{A}}$ have to obey Equation (3.22) since $\overline{\mathcal{I}} = \mathcal{I} - \mathcal{I} + \mathcal{D} = \mathcal{D}$. That is to say, any collection of effects should resolve $\mathbb{1}$.

Reformulating the POVM example of Section 3.1, it starts with a base state structure $\mathscr{A} \subset \mathcal{L}\left(\mathcal{H}^A\right)$ with projector $\mathcal{I}_A$ and a trace of 1, and the formalism consists of the state and effect dual pair in complementary (i.e. quasi-orthogonal) state structures, $(\rho, \mathbb{1}) \in \mathscr{A} \times \overline{\mathscr{A}}$, linked by the normalization of the probability rule

$$1 = (\mathbb{1}\,,\,\rho)_A = \mathrm{Tr}\left[\mathbb{1}\cdot\rho\right]\ . \tag{4.1}$$

Probabilistic assignments are obtained by resolving the effect state structure by a collection of positive operators $\{E_b\}$ indexed by outcome b. The deterministic unit effect they sum up to has to be $\{\mathbb{1}\}$, so the setting on the effect side, y, is irrelevant. However, the setting on the state side, x, is relevant for determining which average state $\rho_{|x}$ has been prepared. The probability rule (Born rule) reads

$$p(b|\rho_{|x}, \mathbb{1}) = p(b|x) = \mathrm{Tr}\left[E_b\cdot\rho_{|x}\right]\ , \tag{4.2}$$

This is a state structure derivation of the POVM formalism.

4.1.2. Bipartite Quantum Theory

Using Definition 3.4.1, the bipartite quantum theory is an example of no-signaling bipartite composition of state structures. A bipartite scenario involving a joint system whose state is described in some space $\mathcal{L}\left(\mathcal{H}\right)$ induces a bipartition $\mathcal{L}\left(\mathcal{H}\right) \cong \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ so that the (Einstein) local measurements of party Alice and Bob only act on, respectively, subsystem A and B [120].

[120]: Zanardi et al. (2004), *Quantum Tensor Product Structures are Observable Induced*.

The one-party example has shown that the local measurements of, say, Alice resolve the state structure $\overline{\mathscr{A}}$ as in Equation (3.22). Assume Bob has a similarly defined $\overline{\mathscr{B}}$ so that the joint local measurement of Alice and Bob are resolving $\overline{\mathscr{A}} \otimes \overline{\mathscr{B}}$. If they are moreover allowed to perform joint but no-signaling operations, by Proposition 3.5.5 their deterministic operations are exactly represented by the set $\overline{\mathscr{A} \otimes \mathscr{B}}$ characterized by

$$M \geq 0\,, \tag{4.3a}$$
$$\mathrm{Tr}_{AB}\left[M\right] = d_A d_B\ , \tag{4.3b}$$
$$\left(\mathcal{D}_A \otimes \mathcal{D}_B\right)\{M\} = M\ . \tag{4.3c}$$

It is not hard to see from $\frac{\mathbb{1}_A}{d_A}\mathrm{Tr}_A\left[\frac{\mathbb{1}_B}{d_B}\mathrm{Tr}_B\left[M\right]\right] = \frac{\mathbb{1}_A}{d_A} \otimes \frac{\mathbb{1}_B}{d_B}\mathrm{Tr}_{AB}\left[M\right]$ that also in the bipartite case, the effects of quantum theory resolve a single element, $\{M = \mathbb{1} = \mathbb{1}_A \otimes \mathbb{1}_B\}$. Note, however, that the effects can now in general be entangled in the sense that there are resolutions $\{M_i\}$ for which there is no possibility to find a decomposition $\{M_i = \sum_i q_i E_i^A \otimes F_i^B\}$ where $q_i \geq 0$, $\sum_i q_i = 1$ and for all q_i and i, with E_i (F_i) a valid effect resolving an element of $\overline{\mathscr{A}}$ ($\overline{\mathscr{B}}$). A Bell measurement is an instance of such a collection of entangled effects resolving $\mathbb{1}$ in the $d_A = d_B = 2$ case.

Using Theorem 3.3.2 to characterize the valid states, an operator $W \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ is a valid state if it belongs to the dual state structure $\overline{\overline{\mathscr{A}} \otimes \overline{\mathscr{B}}} = \mathscr{A} \parr \mathscr{B}$, i.e.

$$W \geq 0\,, \tag{4.4a}$$

$$\mathrm{Tr}_{AB}\left[W\right] = 1\,, \tag{4.4b}$$

$$\overline{(\mathcal{D}_A \otimes \mathcal{D}_B)}\{W\} = W\,. \tag{4.4c}$$

As mentioned in Subsection 3.6.1, the projective constraint can be simplified into

$$\overline{(\mathcal{D}_A \otimes \mathcal{D}_B)}\{W\} \overset{(3.31c)}{=} \left(\mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{D}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right)\{W\} = \left(\mathcal{I}_A \otimes \mathcal{I}_B\right)\{W\} = W\,. \tag{4.5}$$

This projector is the same as the one involved in a no-signaling composition using Definition 3.4.1. This means that the two-way signaling composition $\mathscr{A}_{quant.} \parr \mathscr{B}_{quant.}$ of state structures of density matrices is actually the same composition as their no-signaling composition, $\mathscr{A}_{quant.} \otimes \mathscr{B}_{quant.}$. In other words, in quantum theory, the set of valid states normalized on local measurements is exactly the set of no-signaling composite states: this recovers the Einsteinian intuition that it is impossible for parties measuring a part of a shared quantum state to signal to the other one. Contrastingly, states from arbitrary bipartite state structures can be used for signaling, as shown for the Biased Quantum Theory in Subsection 3.6.1. What is more surprising is that it is actually the only theory having this property, as will be proven in Section 5.3 (Lemma 5.3.1). Generally, the set of states normalized on a pair of local measurements contains terms that can signal in one direction or the other. For instance, this ability to signal will also be present in the set of bipartite process matrices presented below.

4.1.3. Quantum Channel Theory and Reformulating the Example of Section 3.1

The evolution between quantum states is characterized by Theorem 3.4.1; the obtained set is the CJ representation of the set of quantum channels; its probabilistic resolutions yield the quantum instrument formalism; and the dual state structure yields the set of single-partite process matrices. This example is, therefore, the rephrasing of the introductory example of Section 3.1 in the language of state structures.

The introductory example can be reduced to the description of a party A preparing a quantum system according to her setting x and passing it on to a party B measuring it and obtaining outcome b. Its interest was to motivate the following successive rephrasings of the Born rule:

$$p(b|x) \overset{(3.1)}{=} \left(E_b\,,\,\rho_{|x}\right) \overset{1)}{=} \left(\mathbb{1}\,,\,\mathcal{M}_b\left(\rho_{|x}\right)\right) \overset{2)}{=} \left(M_b\,,\,\rho_{|x}\otimes\mathbb{1}\right) \overset{3)}{=} \left(M_b\,,\,W_{|x}\right)\ . \tag{4.6}$$

Step 1), passing from (3.1) to (3.2), consisted in lifting Bob's POVM measurement $\{E_b\}$ into a quantum instrument $\{\mathcal{M}_b\}$. The state space is no longer a state and effect pair, but rather two snapshots tracking the evolution of the state prepared by Alice $\rho_{|x} \in \mathscr{A}_{quant.}$ into the Bob's post-measurement state $\mathcal{M}_b\left(\rho_{|x}\right) \in \mathcal{L}\left(\mathcal{H}^B\right)$. In accordance with Definition 3.4.3, the instrument elements sum up to an admissible mapping $\sum_b \mathcal{M}_b = \mathcal{M}$ from states in $\mathscr{A}_{quant.}$ to $\mathscr{B}_{quant.}$. This is the set of all CPTP maps $\{\mathcal{M}\}$ from A to B

Step 2), passing from (3.2) to (3.4), consisted in going to the CJ picture, $\mathcal{M}_b \mapsto M_b$. By preservation of the linear structure, $\mathcal{M} = \sum_b \mathcal{M}_b \mapsto \sum_b M_b = M$; these are the M such that for all quantum state $\rho_A \in \mathscr{A}_{quant.} \subset \mathcal{L}\left(\mathcal{H}^A\right)$, there exists a state $\sigma_B \in \mathscr{B}_{quant.} \subset \mathcal{L}\left(\mathcal{H}^A\right)$ so that

$$\left[\mathrm{Tr}_A\left[M\cdot\left(\rho_A\otimes\mathbb{1}\right)\right]\right]^T = \sigma_B\ . \tag{4.7}$$

Here, Theorem 3.4.1 can be used to characterize the set of all such operators $M \in \mathcal{L}\left(\mathcal{H}^A\otimes\mathcal{H}^B\right)$ which are the CJ representation of the admissible transformations as $\mathscr{A}_{quant.} \to \mathscr{B}_{quant.}$. By Theorem 3.4.1, they must satisfy

$$M \geq 0\,, \tag{4.8a}$$

$$\mathrm{Tr}\left[M\right] = d_A\ , \tag{4.8b}$$

$$\left(\mathcal{I}_A \to \mathcal{I}_B\right)\{M\} = M\ , \tag{4.8c}$$

to be valid. Remark that the projective condition,

$$\left(\mathcal{I}_A \to \mathcal{I}_B\right)\{M\} := \overline{\mathcal{I}_A\otimes\mathcal{D}_B}\{M\} = \left(\mathcal{I}_A\otimes\mathcal{I}_B - \mathcal{I}_A\otimes\mathcal{D}_B + \mathcal{D}_A\otimes\mathcal{D}_B\right)\{M\}\,, \tag{4.9}$$

which is equivalently seen as a two-way signaling composition,

$$\begin{aligned}\left(\mathcal{I}_A \to \mathcal{I}_B\right)\{M\} \equiv \left(\mathcal{I}_A \parr \overline{\mathcal{I}}_B\right)\{M\} &: = \left(\mathcal{I}_A\otimes\mathcal{I}_B - \mathcal{I}_A\otimes\overline{\mathcal{I}}_B + \mathcal{D}_A\otimes\mathcal{D}_B\right)\{M\} \\ &= \left(\mathcal{I}_A\otimes\mathcal{I}_B - \mathcal{I}_A\otimes\mathcal{D}_B + \mathcal{D}_A\otimes\mathcal{D}_B\right)\{M\}\,,\end{aligned} \tag{4.10}$$

is actually equivalent to the one-way composition since the projector coincide:

$$\begin{aligned}\left(\overline{\mathcal{I}}_A \prec \mathcal{I}_B\right)\{M\} &: = \left(\mathcal{I}_A\otimes\mathcal{I}_B - \overline{\overline{\mathcal{I}}}_A\otimes\mathcal{D}_B + \mathcal{D}_A\otimes\mathcal{D}_B\right)\{M\} \\ &= \left(\mathcal{I}_A\otimes\mathcal{I}_B - \mathcal{I}_A\otimes\mathcal{D}_B + \mathcal{D}_A\otimes\mathcal{D}_B\right)\{M\}\,.\end{aligned} \tag{4.11}$$

From Equation (4.8b) and

$$\left(\mathcal{D}_A\otimes\mathcal{D}_B\right)\{M\} = \frac{\mathrm{Tr}_{AB}\left[M\right]}{d_A d_B}\,\mathbb{1}_{AB}\ , \tag{4.12}$$

these two conditions are equivalent to the more common quantum '1-comb' condition: $\mathrm{Tr}_B\left[M\right] = \mathbb{1}_A$ (see the discussion in Subsection 5.3.1

where this will be shown explicitly). Again, quantum channels are peculiar: seeing the elements of $\mathscr{A}_{quant.} \to \mathscr{B}_{quant.}$ as composite systems in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$, the M's can be interpreted as composite states on this space, but compared to biased quantum channels, Equation (3.131), the support of the A-to-B one-way signaling composition is equal to the two-way signaling composition, i.e.,

$$\mathscr{A}_{quant.} \to \mathscr{B}_{quant.} = \overline{\mathscr{A}}_{quant.} \prec \mathscr{B}_{quant.} \,. \tag{4.13}$$

This is because the part in B is normalized on quantum effects (3.22), so there is only one deterministic effect at the output of a quantum channel (i.e. $\{\mathbb{1}_B\}$); hence it is impossible to deterministically steer the input side from the output side as there are no two different deterministic effects N and N' such that: $\text{Tr}_B\left[M \cdot (\mathbb{1} \otimes N)\right] \neq \text{Tr}_B\left[M \cdot (\mathbb{1} \otimes N')\right]$. This absence of deterministic influence can be interpreted as a no-signaling from the output to the input condition, i.e. a form of *causality* according to the literature (see, e.g., References [50, 81]). This particularity will be revisited thoroughly in Section 5.3.

[50]: Chiribella et al. (2010), *Probabilistic theories with purification*.
[81]: Coecke et al. (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*.

Back to the motivating example of Section 3.1: Step 3), passing from (3.4) to (3.5), consisted in completing the set of 'states' $\rho_{|x} \otimes \mathbb{1}$ into all 'higher-order states'. Since the whole set is dual to the 'higher-order effects' characterized by Equation (4.8), the corresponding set of states is obtained through Theorem 3.3.2[1].

1: Notice this is actually working the proof of Theorem 3.4.1 backward; see Appendix C.2.2.

$$W \geq 0\,, \tag{4.14a}$$

$$\text{Tr}\left[W\right] = d_B\,, \tag{4.14b}$$

$$\overline{\mathcal{I}_A \to \mathcal{I}_B}\{W\} = W\,. \tag{4.14c}$$

This is the set of single-partite process matrices by construction, as it is the set of functionals normalized on quantum instruments. Indeed, any resolution $\{M_b\}$ of a channel leads to a probability rule of the form (3.5):

$$p(b|W_{|x}, M) = \left(M_b\,,\, W_{|x}\right)\,. \tag{4.15}$$

The set of valid W's is characterized by the projector

$$\overline{\mathcal{I}_A \to \mathcal{I}_B} = \overline{\overline{\mathcal{I}_A \otimes \overline{\mathcal{I}}_B}} = \mathcal{I}_A \otimes \overline{\mathcal{I}}_B\,; \tag{4.16}$$

as a consequence, it is the state structure $\mathscr{A}_{quant.} \otimes \overline{\mathscr{B}}_{quant.}$, characterized by $\mathcal{I}_A \otimes \mathcal{D}_B$. By Lemma 3.5.1, it is supposed to be the affine span of the objects of the form $\rho_A \otimes \mathbb{1}_B$. Yet, the remarkable thing here is that there is only one element in $\overline{\mathscr{B}}_{quant.}$, $\mathbb{1}_B$. Therefore, there is no need to take the affine span as $\mathscr{A}_{quant.}$ is convex closed. In other words, this recovers the known result that a single partite process matrix reduces to inputting a quantum state and tracing out the output (see the supplementary material of Reference [5])[2].

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.
2: Remark that the reservations about the definition of a resolution expressed in Appendix C.3.1 reappear in this result: by assuming that the resolutions of single-partite process matrices represent all probabilistic operations between the input and the output of a channel, some resolutions will require side-channels to be realized, e.g., $W_a = \rho_a \otimes \rho_a^T$ such that $\sum_a W_a = W = \rho_A \otimes \mathbb{1}_B$. However, the on average deterministic process matrix they sum up to does not require a side-channel to be implemented since it factors as input states ρ_A in tensor product with a destructive measurement $\mathbb{1}_B$. The deterministic objects appear to require fewer resources to be implemented than their probabilistic resolutions; again, the structure of the theory is based on definitions obtained by formal analogy (in this case Definition 3.2.4), and the discussion thereof is left open for future work.

4.1.4. The No-Signaling Subset of a Quantum Channel.

The previous example showed that some state structures characterized by the methods developed in the previous chapter are equivalent in some cases. This is, therefore, a reason why studying how the compositions interact with each other: avoiding drawing misled conclusions (e.g., "quantum theory allows faster than light signaling") because of a missed equality ($\mathscr{A}_{quant.} \parr \mathscr{B}_{quant.} = \mathscr{A}_{quant.} \otimes \mathscr{B}_{quant.}$). However, finding equality is not the only outcome of a general study of the algebra that one may expect: inequalities are equally interesting and often more revealing. Since state structures are sets, inequalities are better phrased as inclusion relations.

A first example. From the above considerations, it can be guessed that the set of elements of the form $M = \mathbb{1}_A \otimes \sigma_B^T$, which is $\overline{\mathscr{A}}_{quant.} \otimes \mathscr{B}_{quant.}$ is a subset of the valid quantum channels,

$$\overline{\mathscr{A}}_{quant.} \otimes \mathscr{B}_{quant.} \subseteq \overline{\mathscr{A}}_{quant.} \prec \mathscr{B}_{quant.} = \mathscr{A}_{quant.} \to \mathscr{B}_{quant.} \,. \quad (4.17)$$

And indeed they are the 'trace-and-replace' channels such that input ρ_A is traced out and replaced by σ_B:

$$\left(\mathrm{Tr}_A\left[\left(\mathbb{1}_A \otimes \sigma_B^T\right) \cdot \left(\rho_A \otimes \mathbb{1}_B\right)\right]\right)^T = \mathrm{Tr}_A\left[\rho_A\right] \left(\sigma_B^T\right)^T = \sigma_B \,. \quad (4.18)$$

Its resolutions factors into $M_i = E_i \otimes \sigma_i$ such that $\sum_i M_i = M$; also $\sum_i E_i = \mathbb{1}; \sum_i \sigma_i = \sigma$ since its marginals are well-defined, i.e., $\mathrm{Tr}_B\left[M\right] = \mathbb{1}_A$ and $\frac{1}{d_A}\mathrm{Tr}_A\left[M\right] \in \mathscr{B}_{quant.}$. Using the generalized Born rule, this leads to a probability rule:

$$p(b|W = \rho_{|x} \otimes \mathbb{1}, M = \mathbb{1} \otimes \sigma^T) = \mathrm{Tr}\left[\left(\rho_{|x} \otimes \mathbb{1}\right) \cdot \left(E_b \otimes \sigma_b\right)\right] = \mathrm{Tr}\left[\rho_{|x} \cdot E_b\right] \mathrm{Tr}\left[\sigma_b\right] = p(b|x) \,, \quad (4.19)$$

so they can be interpreted as the probabilistic 'measure-and-reprepare' scenario. In that sense, the transformation can indeed be seen as another composition of $\overline{\mathscr{A}}_{quant.}$ with $\mathscr{B}_{quant.}$ –the two-way signaling composition $\overline{\mathscr{A}}_{quant.} \parr \mathscr{B}_{quant.}$– that is bigger than $\overline{\mathscr{A}}_{quant.} \otimes \mathscr{B}_{quant.}$. But here, compared to the bipartite quantum theory and akin to the biased quantum channel case, the inclusion is strict, $\overline{\mathscr{A}}_{quant.} \otimes \mathscr{B}_{quant.} \subsetneq \overline{\mathscr{A}}_{quant.} \parr \mathscr{B}_{quant.}$; the 'trace-and-replace' channels are but a certain kind of channels.

Actually, $\overline{\mathscr{A}}_{quant.} \otimes \mathscr{B}_{quant.}$ is, by definition, the no-signaling composition, so neither the measurement nor the repreparation can have a deterministic influence over the other. To see it explicitly, $\overline{\mathscr{A}}_{quant.}$ and $\mathscr{B}_{quant.}$ are treated as if they were under the control of different parties so that the resolutions of M now depend on two classical variables: $b \mapsto (j,k)$ where j is the measurement outcome seen at A and k is the choice of repreparation at B. The factorization is turned into $M_{j,k} = E_j \otimes \sigma_k$ such that[3] $\sum_j \sum_k M_{j,k} = M; \sum_j E_j = \mathbb{1}; \sum_k \sigma_k = \sigma$. The probability distribution becomes

3: The downright assignment is obtained by $b \mapsto j = k$: $E_j = \delta_{j,b} E_b$ and $\sigma_k = \delta_{k,b}\sigma_b$. Thus, many of the E_j's and σ_k's are the zero operator.

$$p(j,k|x) := p(j,k|W_{|x}, M_{j,k}) = \mathrm{Tr}\left[\left(\rho_{|x} \otimes \mathbb{1}\right) \cdot \left(E_j \otimes \sigma_k\right)\right] \,, \quad (4.20)$$

so the joint probability distribution of j and k factors:

$$p(j,k|x) = \mathrm{Tr}\left[\rho_{|x} \cdot E_j\right] \mathrm{Tr}\left[\mathbb{1} \cdot \sigma_k\right] = p(j|\rho_{|x})p(k) = p(j|x)p(k) \,. \quad (4.21)$$

Hence, the conditional distributions are statistically independent; the knowledge of j cannot help determining k, and vice-versa. Therefore, in such a channel, no information can be deterministically passed from the input side at A to the output side at B. This is a concrete illustration of why the set $\overline{\mathscr{A}}_{quant.} \otimes \mathscr{B}_{quant.}$ is dubbed the 'no-signaling' composition of $\overline{\mathscr{A}}_{quant.}$ and $\mathscr{B}_{quant.}$, and what Lemma 3.5.3 really entails at the level of distributions. $\mathscr{A}_0 \to \mathscr{A}_1$ for this reason.

The extra thing that can be learned from this example is that the no-signaling composition, as expected, is a proper subset of the one- and two-way signaling compositions. One of the aims of studying the abstract mathematical structure of compositions of projectors will be to generalize this observation for any state structure, particularly in the multipartite setting. Indeed, already in a tripartite scenario, it is not straightforward to tell a channel allowing no signaling at all apart from one allowing signaling from Alice to Bob but none to Charlie, or one allowing signaling from Alice to Charlie and Bob but none at a single party. The general characterization of the no-signaling subset of any state structure will be achieved in Subsection 5.1.3 under Definition 5.1.2.

4.2. The Bipartite Process Matrix Formalism and Inclusions

Here is presented a second example of the inclusion of a state structure into another one based on a signaling heuristic. This is a more subtle example as it is four-partite and mixes transformation (two-way signaling composition) with parallel composition (no-signaling composition) –that is, Proposition 3.5.5 and Theorem 3.4.1. This example is the bipartite process matrix as first defined in [5] which is one of the "canonical" examples of higher-order quantum theory with indefinite causal order, see Subsection 2.3.3.

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

This example will show that the ordering of compositions is important. From there, that the ordering between $\otimes$ and $\to$ is important in a multipartite state structure like $\mathscr{A}_0 \to (\mathscr{A}_1 \otimes \mathscr{A}_2) \neq (\mathscr{A}_0 \to \mathscr{A}_1) \otimes \mathscr{A}_2$ (which is read: "the state structure of the transformation from state structure $\mathscr{A}_0$ to the no-signaling composition of the state structures $\mathscr{A}_1$ and $\mathscr{A}_2$ is inequivalent to the no-signaling composition of the state structure of the transformation from $\mathscr{A}_0$ to $\mathscr{A}_1$ with the state structure $\mathscr{A}_2$"). This semantic difference differentiates the bipartite channels and their no-signaling subset.

4.2.1. Bipartite Quantum Channels

Let there be 4 subsystems: the input of Alice A_0, her output A_1, the input of Bob B_0 and his output B_1, all with isomorphic base state structures $\mathscr{A}_0 \cong \mathscr{A}_1 \cong \mathscr{B}_0 \cong \mathscr{B}_1$ of finite-dimensional quantum states like in Equations (3.21) (the '$quant.$' subscript has been dropped for conciseness). A bipartite channel is a CPTP map $\mathcal{M} : \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{B_0}\right) \to \mathcal{L}\left(\mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1}\right)$, hence with state structure $(\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1)$. In the bipartite case, a bipartite channel is nothing more than the previous example of Subsection 4.1.3. The deterministic probability rule reads:

$$(\mathbb{1}_{A_1} \otimes \mathbb{1}_{B_1} \, , \, \mathcal{M}\left(\rho_{A_0 B_0}\right)) = 1 \, , \tag{4.22}$$

which holds for all $\mathbb{1}_{A_1} \otimes \mathbb{1}_{B_1} \in \overline{\mathscr{A}_1 \otimes \mathscr{B}_1}$ and all $\rho_{A_0B_0} \in \mathscr{A}_0 \otimes \mathscr{B}_0$. In the CJ picture, this is

$$(M\,,\, \rho_{A_0B_0} \otimes \mathbb{1}_{A_1} \otimes \mathbb{1}_{B_1}) = 1\,, \tag{4.23}$$

with $M \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{B_0} \otimes \mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1}\right)$[4], leading to the characterization of valid M's by applying Theorem 3.4.1 on (4.4):

$$M \geq 0\,, \tag{4.24a}$$

$$\text{Tr}\,[M] = d_{A_0} d_{B_0}\,, \tag{4.24b}$$

$$((\mathcal{I}_{A_0} \otimes \mathcal{I}_{B_0}) \to (\mathcal{I}_{A_1} \otimes \mathcal{I}_{B_1}))\,\{M\} = M\,. \tag{4.24c}$$

The projector is $(\mathcal{I}_{A_0} \otimes \mathcal{I}_{B_0}) \to (\mathcal{I}_{A_1} \otimes \mathcal{I}_{B_1}) = \mathcal{I}_{A_0} \otimes \mathcal{I}_{B_0} \otimes \mathcal{I}_{A_1} \otimes \mathcal{I}_{B_1} - \mathcal{I}_{A_0} \otimes \mathcal{I}_{B_0} \otimes \overline{\mathcal{I}_{A_1} \otimes \mathcal{I}_{B_1}} + \mathcal{D}_{A_0} \otimes \mathcal{D}_{B_0} \otimes \mathcal{D}_{A_1} \otimes \mathcal{D}_{B_1}$; it can be further simplified by noticing that $\overline{\mathcal{I}_{A_1} \otimes \mathcal{I}_{B_1}} = \mathcal{D}_{A_1} \otimes \mathcal{D}_{B_1}$. This simplification[5] then gives condition $M - \text{Tr}_{A_1B_1}\,[M] \otimes \mathbb{1}_{A_1} \otimes \mathbb{1}_{B_1} + \mathbb{1}_{A_0} \otimes \mathbb{1}_{B_0} \otimes \mathbb{1}_{A_1} \otimes \mathbb{1}_{B_1} = M$, which, using the trace condition, is equivalent to the usual condition $\text{Tr}_{A_1B_1}\,[M] = \mathbb{1}_{A_0} \otimes \mathbb{1}_{B_0}$. One indeed recovers the same 1-comb condition as the single partite case but applied on two subsystems.

4: The ordering of the tensor factor has been switched into numeral then alphabetic for simplicity.

5: Note that once again, $\overline{\mathcal{I}_{A_1} \otimes \mathcal{I}_{B_1}} = \mathcal{D}_{A_1} \otimes \mathcal{D}_{B_1}$ is a particularity of the identity and the depolarizing superoperators; it is not valid for general projectors: $\overline{\mathcal{P}_{A_1} \otimes \mathcal{P}_{B_1}} \neq \overline{\mathcal{P}}_{A_1} \otimes \overline{\mathcal{P}}_{B_1}$. This is yet again a particularity specific to the state structure of quantum theory.

4.2.2. No-Signaling Bipartite Channels.

Consider the swap channel, sending the input at A_0 into the output at B_1 and similarly B_0 to A_1. This channel can obviously not be realized if the situation does not allow signaling from Alice to Bob and vice-versa. This simple example indicates that the parallel composition of two single-partite channels is actually smaller than the bipartite channels:

$$(\mathscr{A}_0 \to \mathscr{A}_1) \otimes (\mathscr{B}_0 \to \mathscr{B}_1) \subsetneq (\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1)\ , \tag{4.25}$$

at least in the case of quantum channels. This is now shown using projective methods.

According to Definition 3.4.1, the left-hand side of the above is indeed the composition of two state structures of single-partite channels done in a manner forbidding a deterministic influence of each party on the other, Proposition 3.5.5, as indicated by the $\otimes$ symbol. This definition characterizes the set of operators obeying

$$M \geq 0\,; \tag{4.26a}$$

$$\text{Tr}\,[M] = d_{A_0} d_{B_0}\,; \tag{4.26b}$$

$$((\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \otimes (\mathcal{I}_{B_0} \to \mathcal{I}_{B_1}))\,\{M\} = M\,; \tag{4.26c}$$

equivalently, Lemma 3.5.1 indicates that this is (a trace-normalized slice of) the affine hull of a tensor product of single partite channels, i.e.,

$$M = \sum_i q_i\, M_i^A \otimes M_i^B\,, \tag{4.27}$$

where $M \geq 0$, $q_i \in \mathbb{R}$, $\sum_i q_i = 1$, in which each $M_i^A \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$ obeys single quantum channel conditions (4.8), and so does each $M_i^B \in \mathcal{L}\left(\mathcal{H}^{B_0} \otimes \mathcal{H}^{B_1}\right)$. In particular, these operators can be entangled because the parallel composition is taken as the no-signaling composition[6], Def-

6: In the context of a no-signaling quantum channel, separability is the analog of separable quantum states: the subset of channels whose decomposition like Equation (4.27) is a convex sum, for which each $q_i \in [0, 1]$. A milder form of decomposition is localizability: the possibility of obtaining the bipartite channel as two single-partite channels applied on a joint ancillary entangled state [61].
It is known that separable and localizable channels are subsets of the set of no-signaling channels. However, the characterization of these spaces is much more involved than working out their supporting subspaces, as their constraints are non-linear. As discussed in Section 3.5, the parallel composition is defined out of convenience as the no-signaling composition since it results in a state structure. See Appendix C.3.4 for a discussion on how big of an assumption it is to take the parallel composition as the no-signaling composition. Generalizing separability and localizability to higher-order quantum transformations is left open as a future research direction.
[61]: Beckman et al. (2001), *Causal and localizable quantum operations*.

inition 3.4.1, which is the maximal tensor product with respect to the inner product as discussed in Section 3.5.

The right-hand side of Equation (4.25) corresponds to the state structure of the bipartite quantum channels, as shown in the previous section. First, the inclusion can be proven by algebraic manipulations of projectors: notice that the set $(\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1)$ has a projector which can be simplified as

$$\overline{\mathcal{I}_{A_0} \otimes \mathcal{I}_{B_0} \otimes \overline{\mathcal{I}_{A_1} \otimes \mathcal{I}_{B_1}}} \cong \overline{(\mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1}) \otimes (\mathcal{I}_{B_0} \otimes \mathcal{D}_{B_1})}\,, \tag{4.28}$$

using the $\overline{\mathcal{I}_{A_1} \otimes \mathcal{I}_{B_1}} = \mathcal{D}_{A_1} \otimes \mathcal{D}_{B_1}$ identity.

On the other hand, the projector (4.26c) associated with $(\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1)$ can be rewritten as $\overline{\mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1}} \otimes \overline{\mathcal{I}_{B_0} \otimes \mathcal{D}_{B_1}}$ according to the discussion in Subsection 4.1.3 and Definition 3.4.1. By Proposition C.1.3, the inclusion can then be shown by computing their composition:

$$\left(\overline{(\mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1}) \otimes (\mathcal{I}_{B_0} \otimes \mathcal{D}_{B_1})}\right) \circ \left(\overline{\mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1}} \otimes \overline{\mathcal{I}_{B_0} \otimes \mathcal{D}_{B_1}}\right) = \overline{\mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1}} \otimes \overline{\mathcal{I}_{B_0} \otimes \mathcal{D}_{B_1}}\,, \tag{4.29}$$

and the converse is not true. Therefore,

$$\overline{(\mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1}) \otimes (\mathcal{I}_{B_0} \otimes \mathcal{D}_{B_1})} \supsetneq \overline{\mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1}} \otimes \overline{\mathcal{I}_{B_0} \otimes \mathcal{D}_{B_1}}\,, \tag{4.30}$$

Where the right-hand side of the above equation is none other than the projector characterizing the set $(\mathscr{A}_0 \to \mathscr{A}_1) \otimes (\mathscr{B}_0 \to \mathscr{B}_1)$, whereas the left-hand side characterizes $(\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1)$.

Yet, computing the intersection of the two projectors is quite a long and gruesome process. Worse, it is a specific instance, so to prove that the set of bipartite no-signaling channels of biased quantum theory is a subset of the bipartite biased channels requires computing another such intersection of projectors. However, all such computations could be skipped once the algebraic properties of the projectors are known. From their properties proven in Chapter 5, it is indeed direct to check that the identity $\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B \subseteq \overline{\mathcal{P}_A \otimes \mathcal{P}_B}$ is valid for any projector on operator systems (this is explicitly proven at Equation (D.51)). Hence, a knowledge of how the compositions $\{\otimes, \prec, \parr, \to\}$ interact together and with the dual $\overline{\cdot}$ allows to infer general properties of the state structure, thus the higher-order states, that are built.

According to the discussion of the previous Subsection 4.1.4, another general property that can be learned from the formula alone is that $(\mathscr{A}_0 \to \mathscr{A}_1) \otimes (\mathscr{B}_0 \to \mathscr{B}_1)$ is, as announced, exactly the set of no-signaling channels as defined in the literature [64, 65] (also known as causal channels, [61, 62]). That is, the subset of channels forbidding deterministic signaling from Alice's side to Bob's and vice-versa.

Again, this property stems from the support of the state structures, and thus could have been inferred by studying the projectors alone [7]. To prove it explicitly requires showing that the choice of input state on Alice's side cannot induce a deterministic influence on Bob's measurement and vice-versa. In equation, Alice and Bob share a bipartite channel M, each acting locally by state preparation and measurement. Call x and y the settings of respectively Alice and Bob, and a and b their outcomes. That is, depending on some input x, Alice prepares some state $\rho_{|x}$ that she

[64]: Piani et al. (2006), *Properties of quantum nonsignaling boxes*.
[65]: D'Ariano et al. (2011), *No Signaling, Entanglement Breaking, and Localizability in Bipartite Channels*.
[61]: Beckman et al. (2001), *Causal and localizable quantum operations*.
[62]: Eggeling et al. (2002), *Semicausal operations are semilocalizable*.

7: See in particular [64].

inputs in her side of the channel, A_0. Then, she measures at the output A_1 some POVM $\{E_{a|x}\}$, which choice can also depend on x, and sees outcome a. Bob does the same at B_0 with state $\sigma_{|y}$ and POVM $\{F_{b|y}\}$ at B_1. The probability rule then reads

$$p(a,b|x,y) = \mathrm{Tr}\left[M \cdot \left(\rho_{|x} \otimes \sigma_{|y} \otimes E^T_{a|x} \otimes F^T_{b|y}\right)\right] \ . \tag{4.31}$$

By Definition 1.2.1, Alice is no-signaling to Bob if her choice of setting x cannot deterministically influence his measurement outcome, Equation (3.97b) and Bob is no-signaling to Alice the converse holds, Equation (3.97a). In terms of the channel, these two conditions can be shown to be equivalent to

$\forall y, y'$,

$$\sum_b p(a,b|x,y) = \sum_b p(a,b|x,y'); \tag{1.6a}$$

$\forall x, x'$

$$\sum_a p(a,b|x,y) = \sum_a p(a,b|x',y). \tag{1.6b}$$

$$\mathrm{Tr}_{A_1}[M] = \mathbb{1}_{A_0} \otimes \mathrm{Tr}_{A_0A_1}[M] \ ; \tag{4.32a}$$

$$\mathrm{Tr}_{B_1}[M] = \mathbb{1}_{B_0} \otimes \mathrm{Tr}_{B_0B_1}[M] \ . \tag{4.32b}$$

which form a stronger constraint than the channel condition $\mathrm{Tr}_{BB_1}[M] = \mathbb{1}_A \otimes \mathbb{1}_{B_0}$. Indeed, computing Bob's outcome marginal distribution in (4.31), i.e. $p(b|x,y) = \sum_a p(a,b|x,y)$, and injecting (4.32a) shows that Bob's outcome is independent of x:

$$\begin{aligned}
p(b|x,y) &= \sum_a \mathrm{Tr}\left[M \cdot \left(\rho_{|x} \otimes \sigma_{|y} \otimes E^T_{a|x} \otimes F^T_{b|y}\right)\right] = \mathrm{Tr}\left[M \cdot \left(\rho_{|x} \otimes \sigma_{|y} \otimes (\mathbb{1}_{A_1}) \otimes F^T_{b|y}\right)\right] \\
&\overset{(4.32a)}{=} \mathrm{Tr}_{A_0B_0B_1}\left[(\mathbb{1}_{A_0} \otimes \mathrm{Tr}_{A_0A_1}[M]) \cdot \left(\rho_{|x} \otimes \sigma_{|y} \otimes F^T_{b|y}\right)\right] \\
&= \mathrm{Tr}_{A_0}\left[\rho_{|x}\right] \mathrm{Tr}_{B_0B_1}\left[\mathrm{Tr}_{A_0A_1}[M] \cdot \left(\sigma_{|y} \otimes F^T_{b|y}\right)\right] \\
&= \mathrm{Tr}_{B_0B_1}\left[\mathrm{Tr}_{A_0A_1}[M] \cdot \left(\sigma_{|y} \otimes F^T_{b|y}\right)\right] \\
&= p(b|y) \ .
\end{aligned} \tag{4.33}$$

The same thing can be shown for Alice, effectively proving that they are no-signaling to each other if they obey conditions (4.32). Proving this to be necessary requires an even longer proof found in References [9, 64].

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.
[64]: Piani et al. (2006), *Properties of quantum nonsignaling boxes*.

In contrast, when working with projectors and state structures, the necessity immediately follows from Lemma 3.5.3 and Proposition 3.5.5. What is more, this above equivalence of Definition 3.4.1 with (4.32) in the case of quantum theory can be shown in a general way from the algebraic properties of the projectors alone.

According to Proposition 3.5.5, the no-signaling conditions are exactly those encoded in the projector of the state structure $(\mathscr{A}_0 \to \mathscr{A}_1) \otimes (\mathscr{B}_0 \to \mathscr{B}_1)$, which reads $(\mathcal{I}_A \to \mathcal{I}_B) \otimes (\mathcal{I}_{B_0} \to \mathcal{I}_{B_1}) = \overline{\mathcal{I}_A \otimes \mathcal{D}_B \otimes \mathcal{I}_{B_0} \otimes \mathcal{D}_{B_1}}$. Indeed, this result states that no-signaling is the conjunction of the operator being a one-way signaling composition from A to B *and simultaneously* being one from B to A. By Lemma 3.5.3, the first is enforced by restricting the subspace to $(\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \otimes \mathcal{I}_{B_0} \otimes \mathcal{I}_{B_1}$ and the second by $(\mathcal{I}_{A_0} \otimes \mathcal{I}_{A_1} \otimes (\mathcal{I}_{B_0} \to \mathcal{I}_{B_1}))$. Notice that these are two projectors acting on different subspaces and thus commuting; their conjunction is then given by the intersection of both subspaces they define,

$$(\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \otimes (\mathcal{I}_{B_0} \to \mathcal{I}_{B_1}) = ((\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \otimes \mathcal{I}_{B_0} \otimes \mathcal{I}_{B_1}) \cap (\mathcal{I}_{A_0} \otimes \mathcal{I}_{A_1} \otimes (\mathcal{I}_{B_0} \to \mathcal{I}_{B_1})) \ , \tag{4.34}$$

and because they commute, the $\cap$ symbol is just the operator composition $\circ$ of the projectors. Hence, each of them enforces a condition independent of the other. The first one, $\left(\overline{\mathcal{I}_A \otimes \mathcal{D}_B} \otimes \mathcal{I}_{B_0} \otimes \mathcal{I}_{B_1}\right)\{M\} = M$, is explicitly

$$M - \mathbb{1}_B \otimes \mathrm{Tr}_B\left[M\right] + \mathbb{1}_A \otimes \mathbb{1}_B \otimes \mathrm{Tr}_{AB}\left[M\right] = M\ , \tag{4.35}$$

which is equivalent to condition (4.32a), and thus to statistical independence of Bob's outcome marginal $p(b|x,y)$ from Alice's setting x, as was shown above. The same way, $\left(\mathcal{I}_A \otimes \mathcal{I}_B \otimes \overline{\mathcal{I}_{B_0} \to \mathcal{D}_{B_1}}\right)$ can be shown equivalent to condition (4.32b) and thus the statistical independence of Alice's outcome marginal $p(a|x,y)$ from Bob's setting y.

The extra message conveyed by this example is that the algebra of projectors is homomorphic to signaling constraints. Not only does it simplify the proofs of equations (4.32), but it also gives signaling interpretation by looking at the projector expression alone: Equation (4.34) is read: "the parallel composition ($\otimes$) of the local quantum channels of Alice and Bob are equivalent $=$ to the set of bipartite channels that are one-way signaling from Alice to Bob *and simultaneously* ($\cap$) from Bob to Alice". In other words, it hides the general principle a no-signaling constraint is equivalent to a one-way signaling constraint in both directions. By the same token, that quantum channels are no-signaling from output to input, Equation (4.13), and that no-signaling from output to input quantum channels are its subset, Equation (4.17), are also relations that can be directly read from equivalences of projectors. In other words, the structure of signaling relations allowed by multipartite objects can be learned by decomposing the projector associated with the state structure, whence the interest in studying the algebra of projectors.

4.2.3. Bipartite No-Signaling Channel and Process Matrix Formalism.

One last interesting aspect that can be revealed by studying the algebra of projectors that appear in this example is duality relations. The characterization of bipartite process matrices can be recovered from the one of the base state structures of quantum states. With respect to the characterization of the state structure of no-signaling channels in the last section the set of bipartite process matrices are nothing short of the set of functionals normalized on this set [1, 5]. As with the construction of single-partite process formalism of Subsection 4.1.1, the construction of bipartite process formalism can be seen as successive rephrasing of a Born rule:

$$\begin{aligned}
p(a,b|x,y) &\overset{1)}{=} \left(E_{a|x} \otimes F_{b|y}\ ,\ \rho\right) \overset{2)}{=} \left(E_{a|x} \otimes F_{b|y}\ ,\ \mathcal{M}\left(\rho\right)\right) \\
&\overset{3)}{=} \left(E_{a|x} \otimes F_{b|y}\ ,\ \left(\mathcal{M}^A \otimes \mathcal{M}^B\right)\{\rho\}\right) \\
&\overset{4)}{=} \left(\mathbb{1}_A \otimes \mathbb{1}_B\ ,\ \left(\mathcal{M}^A_{a|x} \otimes \mathcal{M}^B_{b|y}\right)\{\rho\}\right) \overset{\mathfrak{C}}{=} \left(M^{A_0A_1}_{a|x} \otimes M^{B_0B_1}_{b|y}\ ,\ \rho_{A_0B_0} \otimes \mathbb{1}_{A_1} \otimes \mathbb{1}_{B_1}\right) \\
&\overset{5)}{=} \left(M_{a|x} \otimes M_{b|y}\ ,\ W\right)\ ,
\end{aligned} \tag{4.36}$$

where '$\mathfrak{C}$' indicates the Choi-Jamiołkowski isomorphism, Definition 2.2.1. This construction is now detailed step by step. It actually relates all the other structures built in this section; thus, it is a good summarizing example.

1) The construction begins similarly to that of the single-partite PM in Subsection 4.1.3 but assumes the induced dynamics to be bipartite to start with. Starting with the regular single-partite quantum theory as in Subsection 4.1.1

$$1 = \mathrm{Tr}\left[\mathbb{1} \cdot \rho\right] \ , \tag{4.37}$$

one first considers POVM effects in tensor product, $\exists\{E_{a|x}\} \subset \mathcal{L}\left(\mathcal{H}^{A_0}\right)$, $\exists\{F_{b|y}\} \subset \mathcal{L}\left(\mathcal{H}^{B_0}\right)$, controlled by local parties A and B to argue for bipartite states, $\rho \in \mathscr{A}_0 \parr \mathscr{B}_0 \subset \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{B_0}\right)$,

$$p(a,b|x,y) = \mathrm{Tr}\left[\left(E_{a|x} \otimes F_{b|y}\right) \cdot \rho\right] \ . \tag{4.38}$$

As shown in Subsection 4.1.2, a peculiarity of quantum theory is that these bipartite states are automatically no-signaling $\mathscr{A}_0 \parr \mathscr{B}_0 = \mathscr{A}_0 \otimes \mathscr{B}_0$. At this stage, the distribution $p(a,b|x,y)$ is a typical Bell scenario: two local parties A and B applying quantum measurements on a shared bipartite quantum system.

2) Then, this static bipartite quantum theory is extended by allowing evolution; quantum channels $\mathcal{M} : \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{B_0}\right) \to \mathcal{L}\left(\mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1}\right)$ are added to the picture[8]:

8: The state space at the output of the channel is assumed isomorphic to the state space at the input for simplicity.

$$p(a,b) = \mathrm{Tr}\left[\left(E^{A_1}_{a|x} \otimes F^{B_1}_{b|y}\right) \cdot \mathcal{M}\left(\rho_{A_0B_0}\right)\right] = \mathrm{Tr}\left[M \cdot \left(\rho \otimes E^{T}_{a|x} \otimes F^{T}_{b|y}\right)\right] . \tag{4.39}$$

These bipartite channels are characterized as in Section 4.2, so that these maps have state structure $(\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1)$.

3) To turn this Bell-like scenario into a bipartite process, the next step consists of restricting the set of valid bipartite channels to the no-signaling channels to guarantee that parties A and B are no-signaling even during the evolution. The heuristic is the following: if there are local effects, the evolution may be local, for example, if Alice and Bob's labs are space-like separated[9]. Put another way, Alice and Bob are each allowed to do any physically admissible transformation on their local share of the quantum system before measuring, but no global transformation is allowed. Both points of view conclude that the set of channels is restricted to its local subset spanned by the tensor product of local transformations:

9: There is a technicality ignored for the sake of following a similar argument to References [5, 123]: no-signaling is actually a weaker restriction than causal disconnection, see [61, 63, 65–67] for instance. Localizable channels, whose definition was briefly evoked in note 6, are the correct model of space-like separated evolution in this context. This assumes the operations to be in tensor product as in (4.40) but allowed to share entanglement. At the level of state structures, this changes nothing to the argument, as localizable and no-signaling channels share the same linear support because of Equation (3.81).
[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.
[123]: Barnum et al. (2005), *Influence-free states on compound quantum systems*.
[61]: Beckman et al. (2001), *Causal and localizable quantum operations*.
[63]: Popescu et al. (1994), *Quantum nonlocality as an axiom*.
[65]: D'Ariano et al. (2011), *No Signaling, Entanglement Breaking, and Localizability in Bipartite Channels*.
[66]: Perinotti (2021), *Causal influence in operational probabilistic theories*.
[67]: D'Ariano et al. (2014), *Determinism without causality*.

$$p(a,b|x,y) = \mathrm{Tr}\left[\left(M_{A_0A_1} \otimes M_{B_0B_1}\right) \cdot \left(\rho_{A_0B_0} \otimes E^{T_{A_1}}_{a|x} \otimes F^{T_{B_1}}_{b|y}\right)\right] . \tag{4.40}$$

Hence, the state structure of the maps, a bipartite composition, has been restricted to the no-signaling composition:

$$\begin{aligned} &M_{A_0A_1B_0B_1} \in (\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1) \\ &\overset{3)}{\mapsto} M_{A_0A_1} \otimes M_{B_0B_1} \in (\mathscr{A}_0 \to \mathscr{A}_1) \otimes (\mathscr{B}_0 \to \mathscr{B}_1) \ . \end{aligned} \tag{4.41}$$

The characterization of $M_{A_0A_1} \otimes M_{B_0B_1}$ is now obtained through Definition 3.4.1. This was done in Equation (4.26).

4) The probabilistic content of the operations is then passed to the channel side. This means that it is no longer the destructive measurements that are resolved but rather the channels, as in step 1) of Equation (4.6) in

the introductory example (see Subsection 4.1.3). Each channel in (4.40) is turned into an instrument. The probability rule becomes

$$p(a,b|x,y) = \mathrm{Tr}\left[\left(M_{a|x}^{A_0A_1} \otimes M_{b|y}^{B_0B_1}\right) \cdot \left(\rho_{A_0B_0} \otimes \mathbb{1}_{A_1} \otimes \mathbb{1}_{B_1}\right)\right] . \quad (4.42)$$

5) The final step is the same as in the single-partite PM case. It consists of extending 'states' $\rho_{A_0B_0} \otimes \mathbb{1}_{A_1} \otimes \mathbb{1}_{B_1}$ to any operator $W \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1} \otimes \mathcal{H}^{B_0} \otimes \mathcal{H}^{B_1}\right)$ normalized on the set of valid 'effects' that $M_{a|x}^{A_0A_1} \otimes M_{b|y}^{B_0B_1}$ are resolving. Since the state structure of the unit effects is $(\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1)$, the state structure of all 'higher-order states' W normalized on it is $\overline{(\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1)}$ characterized by applying Theorem 3.3.2 on Equation (4.26), which results in the projective characterization of bipartite process matrices W [35],

[35]: Araújo et al. (2015), *Witnessing causal nonseparability*.

$$W \geq 0\,; \quad (4.43a)$$

$$\mathrm{Tr}\left[M\right] = d_{A_1} d_{B_1}\,; \quad (4.43b)$$

$$\overline{(\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \otimes (\mathcal{I}_{B_0} \to \mathcal{I}_{B_1})}\{W\} = W\,. \quad (4.43c)$$

Remark that using Equation (4.13) and (5.2), the projector can be rephrased as

$$\overline{(\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \otimes (\mathcal{I}_{B_0} \to \mathcal{I}_{B_1})} = \overline{\mathcal{D}_{A_0} \prec \mathcal{I}_{A_1}} \parr \overline{\mathcal{I}_{B_0} \prec \mathcal{I}_{B_1}}\,. \quad (4.44)$$

By doing so, it has been inferred from signaling heuristics what is the characterization of the set of bipartite process matrices $\{W\}$ (the five steps were nothing else than a rephrasing of the steps of the original derivation in Reference [5]). Moreover, by manipulations on the projectors, it was shown that its state structure can be rewritten as

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

$$\overline{(\mathscr{A}_0 \to \mathscr{A}_1) \otimes (\mathscr{B}_0 \to \mathscr{B}_1)} = \overline{\overline{\mathscr{A}}_0 \prec \mathscr{A}_1} \parr \overline{\overline{\mathscr{B}}_0 \prec \mathscr{B}_1}\,. \quad (4.45)$$

From this rewriting, it becomes evident that the bipartite PM allows for signaling from Alice's side to Bob's and vice-versa, as it was explicitly written as a two-way signaling composition of state structures $\overline{\overline{\mathscr{A}}_0 \prec \mathscr{A}_1}$ with $\overline{\overline{\mathscr{B}}_0 \prec \mathscr{B}_1}$ in accordance to Definition 3.5.2.

From the example of the swap channel, it has been possible to infer the inclusion (4.17). Now, as the bipartite process matrix can signal in more directions than a state and effect pair [5], the inclusion has been reversed: the set of bipartite process matrix is a bigger state structure than the tensor product of bipartite quantum states with bipartite unit effects inasmuch as the set of no-signaling bipartite channels is a smaller state structure than the bipartite quantum channels. In symbols:

$$\begin{aligned} M_{A_0A_1B_0B_1} &\in (\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1) \supsetneq (\mathscr{A}_0 \to \mathscr{A}_1) \otimes (\mathscr{B}_0 \to \mathscr{B}_1) \ni M_{A_0A_1} \otimes M_{B_0B_1} \\ \rho_{A_0B_0} \otimes \mathbb{1}_{A_1} \otimes \mathbb{1}_{B_1} &\in \overline{(\mathscr{A}_0 \otimes \mathscr{B}_0) \to (\mathscr{A}_1 \otimes \mathscr{B}_1)} \subsetneq \overline{(\mathscr{A}_0 \to \mathscr{A}_1) \otimes (\mathscr{B}_0 \to \mathscr{B}_1)} \ni W_{A_0A_1B_0B_1} \end{aligned} \quad (4.46)$$

and constructing the bipartite process matrix consisted of going from the dual pair on the left-hand side of this pair of equations to the one on its right-hand side. The essence of the construction of bipartite process matrices then lies in restricting the set of 'effects' from the bipartite channels $M_{A_0A_1B_0B_1}$ to no-signaling channels $M_{A_0A_1} \otimes M_{B_0B_1}$

so that the set of 'states' is allowed to be extended from the prepare-and-measure $\rho_{A_0B_0} \otimes \mathbb{1}_{A_1} \otimes \mathbb{1}_{B_1}$ environment to the bipartite process matrices $W_{A_0A_1B_0B_1}$. Whence the name 'dual pair' in Definition 3.3.3: Theorem 3.3.2 encodes a duality in signaling: the less signaling the 'effects' are, the more the 'states' can be, and vice-versa. This duality is another reason for studying the algebraic relations between projectors on state structures. As will be shown in the following, this duality can be leveraged to simplify the study of the state structure within an object: instead of asking 'Which are the signaling directions allowed by this shared state?' the duality allows to ask the equivalent 'Which are the signaling direction forbidden by this shared effect?' and pick the easiest of the two to analyze.

4.3. Type Theory and Order

The rules defined in the previous chapter allow whole hierarchies of higher-order processes to be developed. For instance, one can consider the channels of channels between biased quantum theory by repeating the construction in Subsection 3.6.2, then channels of channels of channels, and so on... The idea of Perinotti and Bisio's works [10, 11] is to define a *type system* to classify these infinite hierarchies of nested maps[10].

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

10: Type systems are widely studied formal systems in logic and computer sciences, see Reference [143] for instance. The kind of type system devised by Perinotti and Bisio is a fragment of *linear type theory*, which generalizes simply typed λ-calculus into a programming language suitable for both classical and quantum computation. This type theory was already introduced in the context of quantum theory by various authors in the 2000s (see, for example, Reference [144]), but each time for a different purpose than studying nested quantum maps. Remark that the equivalence of the type-theoretic treatment of Perinotti and Bisio [10, 11] with the categorical treatment of Kissinger and Uijlen [33] can be explained simply because linear type theory is the internal language of $*-$autonomous categories, in the sense of a Curry-Howard-Lambek correspondence as shown by Seely and Ambler; see the review [145] and sources within.

[143]: Girard et al. (1989), *Proofs and Types*.
[144]: Selinger et al. (2004), *A lambda calculus for quantum computation with classical control*.
[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

[145]: Baez et al. (2010), *Physics, Topology, Logic and Computation: A Rosetta Stone*.

Given a global Hilbert space and a tensor partitioning of it, like $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \otimes \ldots$, one first defines a set of base types $\{A, B, \ldots\}$ as a collection of state spaces of quantum systems defined on spaces $\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right), \ldots$ and associated to parties A(lice), B(ob), ... The idea behind the type theory is then to associate a *type* to the global state space of the parties defined on $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \ldots\right)$. The type can be built using one occurrence of each base type and any combination of the rules $\{1, \to, (,)\}$ according to the following rules:

- All base types are valid types.
- 1 is a valid type called the trivial type; it is the state space of 1-dimensional quantum theory, i.e. the number 1.
- If $A \subset \mathcal{L}\left(\mathcal{H}^A\right)$ and $B \subset \mathcal{L}\left(\mathcal{H}^A\right)$ are valid types, then $(A \to B) \subset \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ is a valid type.

One recovers the hierarchy of Bisio and Perinotti by imposing the following postulates:

1. The base types are the state space of quantum theory.
2. The type $A \to B$ is the state space of the the CJ representation of the admissible mappings (in the sense of Definition 3.2.2) between A and B.

In terms of the projective characterization of the last chapter, the type system is built by assuming base types as being the state structures of density matrices: the set of base types corresponds to a set $\{\mathscr{A}, \mathscr{B}, \ldots\}$ of states structures each characterized by Equations 3.21. The type rule $\to$, called type *constructor*, corresponds to using Theorem 3.4.1 for defining new state structures like $\mathscr{A} \to \mathscr{B}$ for instance.

$$\rho \geq 0\,, \tag{3.21a}$$
$$\mathrm{Tr}\left[\rho\right] = 1\,, \tag{3.21b}$$
$$\mathcal{I}\{\rho\} = \rho\,. \tag{3.21c}$$

The type system can then be used recursively under splitting and grouping of parties/Hilbert space to increase and decrease the number of subsystems. For instance, several types can be associated with the same party when it is dealing with a composite system. In that case, subscript numbers will be used to differentiate between each part. For instance, if Alice has a tripartite system, base type A will split into a set of three base types A_0, A_1, and A_2 associated with each subsystem. Note that although two different parties A and B could have isomorphic state spaces, i.e. the same base type. In order to avoid a complicated notation, different letters are used for types instantiated on different systems – by convention the same letters as for the systems – and it will indicated by words, if needed, when the types are isomorphic. E.g., if parties A and B's systems are both qubits, their types will be noted as A and B, and it will be precised that the state space associated with these types are isomorphic since they are base types of the same dimension. However, the rules distinguish the base type on the 1-dimensional system as the *trivial type*, noted 1, which is the set of quantum states on a 1-dimensional Hilbert space –consisting of the number one– from the other base types, which are the sets of quantum states on any other finite-dimensional Hilbert spaces. This is important as any system A can be extended by the trivial type since $\mathcal{H}^A \otimes \mathbb{C} \cong \mathcal{H}^A$. As a consequence, in type notation, the trivial type will be noted 1 without reference to a party since it represents trivial systems (i.e. nothing).

The interest of the type system is to infer general properties on state space by the sole analysis of their type. In particular, the type system can be used to derive equivalence between types and, from there, rewrite rules to show equivalence between more complex types (meaning types instantiated on more subsystems).

The trivial type plays a special role in the construction and classification of valid types; indeed, it complexifies the situation already for single-partite types. Starting with a single base type $\{A\}$, one may have expected that the only valid type would be A, but actually many other valid types can be built using 1 like $1 \to A$, $A \to 1$, $(A \to 1) \to 1$, $1 \to (A \to 1)$, etc. Notice that $1 \to A$ is the set of transformations from the trivial system to A; this is a mapping from nothing to a quantum state, which is no different than state preparation, thus the state itself. Therefore, the first equivalence of types is

$$A = 1 \to A \; . \tag{4.48}$$

One may expect that all single-partite type is equivalent to the base type, but actually, there is another equivalence class. Consider $A \to 1$, i.e. the transformation from a state to the number 1: this describes the destructive measurements, exactly the set of *unit effects* or *deterministic functionals* characterized in Theorem 3.3.2. It is a different type than A: $\overline{A}$ is the type of all operators N that takes operators V of type A to the trivial type through rule (2.7): $\mathrm{Tr}_A [N \cdot V] = 1$. This set has the following special notation:

$$\overline{A} := A \to 1 \; . \tag{4.49}$$

Whence the choice of using $\overline{\mathscr{A}}$ to denote the dual state structure in the last chapter. Another special type in the single-partite setting is $(A \to 1) \to 1 = \overline{A} \to 1 = \overline{\overline{A}}$. By a direct interpretation of the formula, this is the type that sends the set of unit effects to the number one, which is the set of states, i.e.

$$\overline{\overline{A}} = A \; . \tag{4.50}$$

This type rule corresponds to Corollary 3.3.3. With this rule, all single partite types fall into two essential categories: states A or unit effects $\overline{A}$. This type constructor "$\rightarrow$", nicknamed **transformation**, is the key element of the type theory of higher-order transformations: each set can be seen as an abstract type, and new types can be defined out of existing ones using the transformation connector as a semantic rule. It is actually not associative: $(A \rightarrow 1) \rightarrow 1 \overset{(4.50)}{=} A$ whereas $A \rightarrow (1 \rightarrow 1) \overset{(4.48)}{=} A \rightarrow 1 \overset{(4.49)}{=} \overline{A}$. And this is how there can two inequivalent types for the single-party type.

For multipartite types, this non-associativity further splits types into 'orders', which convey the idea of 'how nested the map associated with the type is'. In the bipartite setting, there are two possibilities: one is the elementary type for two parties, $A \otimes B$, which is a special kind of transformation between them [10, Lemma 1]:

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.

$$(A \rightarrow (B \rightarrow 1)) \rightarrow 1 = \overline{A \rightarrow \overline{B}} := A \otimes B \, . \tag{4.51}$$

Generalizing this rule in the case where A and B are not base types yields the *parallel composition* of two types, defined as type $\overline{A \rightarrow \overline{B}} \equiv A \otimes B$ [11, Section V.E]. Since it corresponds to the set of all possible CJ representations of the parallel ($\otimes$) composition of two linear maps, the type $A \otimes B$ is 'not nested', so it is a *first order type*. The other possibility is the *second order type* $A \rightarrow B$, which is, by definition, the set of (admissible) linear maps from type A to type B. This is a transformation between types; hence, provided the types A and B are not base types, $A \rightarrow B$ represents a set of maps on the set of maps A, i.e. a nested map. Type $A \rightarrow B$ is 'nested' with respect to types A and B, so, by consequence, it is a *second order type*.

[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

For example, consider the case where A has the type of quantum channels; this is a composite type over subsystems A_0 and A_1 so that $A = A_0 \rightarrow A_1$, where A_0 is associated with the input space of the channel and A_1 the output. Let $B = B_0 \rightarrow B_1$ be similarly defined. Consequently, the Hilbert space is split between four subsystems for which $\{A_0, A_1, B_0, B_1\}$ are the base types. Then the first order type (with respect to the type of channel) $A \otimes B = (A_0 \rightarrow A_1) \otimes (B_0 \rightarrow B_1)$ coincides with the parallel composition of channels, that is the (CJ representation of) no-signaling channels [61, 64] from the joint input space $\mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{B_0}\right)$ to the joint output $\mathcal{L}\left(\mathcal{H}^{A_1} \otimes \mathcal{H}^{B_1}\right)$[11]. Whereas the second-order type $A \rightarrow B$ is a transformation (i.e., an admissible linear map) from a channel on Alice's side (between $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$ and $\mathcal{L}\left(\mathcal{H}^{A_1}\right)$) to a channel on Bob's side (between $\mathcal{L}\left(\mathcal{H}^{A_1}\right)$ and $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$). The second-order type is then the set of valid quantum 2-combs [9] (also called quantum supermap or superchannel [8]).

[61]: Beckman et al. (2001), *Causal and localizable quantum operations*.

[64]: Piani et al. (2006), *Properties of quantum nonsignaling boxes*.

11: This connection is explained in more detail in Subsection 4.2.2.

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.

[8]: Chiribella et al. (2008), *Transforming quantum operations: Quantum supermaps*.

Going down at the level of the base types, the situation is four-partite, and the order of the expressions increases. However, this fine-graining allows for the characterization of the no-signaling channels and 2-combs to be inferred from the characterization of the base type alone. What is more, the actual description of what is going on in this process can be read from the type formula. From the expression $A = A_0 \rightarrow A_1$, it is assumed that Alice is applying some quantum operation in between A_0 and A_1, which are base types. Then, one can infer that the set of allowed transformations to which she has access is of type $A_0 \rightarrow A_1$, a second-order type. This simple semantic statement is then translated into constraints to apply on the Hilbert space $\mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$ and yields (the Choi-Jamiołkowski representation of) the set valid quantum channels for Alice. The set of no-signaling channels shared by Alice and Bob then corresponds to type $A \otimes B = (A_0 \to A_1) \otimes (B_0 \to B_1)$. This is now of second order with respect to the base type, since it is a parallel composition of two second-order transformations. Again, this semantic statement can be translated into constraints to apply on four-partite operators $M_{A_0 A_1 B_0 B_1}$ for them to represent valid no-signaling bipartite channels. On the other hand, the set of quantum 2-combs corresponds to the different type $A \to B = (A_0 \to A_1) \to (B_0 \to B_1)$, which is a type of the third order since it is a transformation between second-order types. Again, from the semantic expression, the characterization follows.

Thus, starting from some postulates, a trivial type 1, base types $\{A, B, \ldots\}$, and the type constructor $\to$, all the higher-order generalizations of the quantum formalism based on nested maps (i.e., generalizing the *supermaps* of Reference [8]) can be defined and classified using the type system. These, in turn, yield the characterization constraints on the operators representing transformations of a given type [10, 11].

[8]: Chiribella et al. (2008), *Transforming quantum operations: Quantum supermaps*.

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.

[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

A more complex example is obtained by recovering the set of bipartite process matrices (PM) [5], which corresponds to the set of functionals normalized on the local quantum instruments of two parties, say Alice and Bob. Knowing that their local instruments sum up to quantum channels, i.e. they belong to types $(A_0 \to A_1)$ and $(B_0 \to B_1)$, the set of process matrices is the type that takes the parallel composition of these two types as input and outputs a trivial system. In the semantics, this statement corresponds to writing type $((A_0 \to A_1) \otimes (B_0 \to B_1)) \to 1$, from which the constraints for the characterization of the valid CJ operators representing bipartite PM directly ensue. From the type formula, one directly sees that the bipartite process matrix is the second-order type $\overline{(A_0 \to A_1) \otimes (B_0 \to B_1)}$ dual to the set of bipartite no-signaling channels.

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

A shortcoming of the type system is the absence of a systematic way to break down the signaling structure in an expression, which is one of the goals of this chapter. While $(A_0 \to A_1) \otimes (B_0 \to B_1)$ can be interpreted as the type of no-signaling channels by inspection, nothing can be told about its dual $\overline{(A_0 \to A_1) \otimes (B_0 \to B_1)}$, albeit it is known to be the type of bipartite process matrices hence objects that allow signaling from Alice's side to Bob's and vice-versa. A more problematic example of this kind is the result shown in [13] that a quantum comb, which is a transformation from a higher-order channel to a channel, always features a fixed signaling direction. Yet there is no direct way to see it from the type system, although it can be proven using some rewrite rules as was shown in [11].

[13]: Chiribella et al. (2008), *Quantum Circuit Architecture*.

[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

A supplementary issue raised by the proof of fixed signaling direction in quantum combs is that it requires reducing the order of their type to the first order. More generally, the non-associativity of the transformation connector, which is what allows for the definition of the order of types, can actually hide some equivalences. A concrete example of such non-associativity encountered in the proof is

$$(A_1 \to A_2) \to (A_0 \to A_3) \neq ((A_1 \to A_2) \to A_0) \to A_3\,, \tag{4.52}$$

in the case where the A_i's are the base types of four subsystems associated with Alice [10]. On the one hand, this is a comparison of a second-order type with a first-order type, so it makes sense that these are not equivalent, although the two types are sets of objects defined on $\mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2} \otimes \mathcal{H}^{A_3}\right)$. On the other hand, the following equivalence holds [9]:

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.

$$(A_1 \to A_2) \to (A_0 \to A_3) = ((A_0 \to A_1) \to A_2) \to A_3\,, \tag{4.53}$$

so there is an equivalence between a first-order type and a second-order type, despite non-associativity. The difference is that the ordering of subsystems is not the same; the right-hand side of Equation (4.52) can be rephrased into a second-order type so that the equation becomes:

$$(A_1 \to A_2) \to (A_0 \to A_3) \neq (A_2 \to A_0) \to (A_1 \to A_3)\,. \tag{4.54}$$

Both types are now of second order and represent sets of 2-combs (supermaps). However, in the left-hand side, this is the set of supermaps that transform an input channel between A_1 and A_2 to an output channel between A_o and A_3, whereas in the r.h.s., these are the supermaps that transform a channel between A_2 and A_0 to an output channel between A_1 and A_3. Therefore, a naive application of the notion of order may hide the equivalence between some types of transformations, as in Equation (4.53). Looking at the formula alone, one could have concluded that there is no use in comparing them because they do not feature the same base state structures at equivalent orders. But a naive application of the notion of order can also put incomparable transformations at the same level, as in Equation (4.54). Looking at the formula alone, one could have concluded that there is a way to rewrite one into the other.

There is therefore a reason to extend the type system into a wider set of rules that are more convenient to work with. There are moreover clear candidates for this extension: the rules for bipartite composition of state structures derived in the previous chapter and based on signaling relations. Therefore the aim of this chapter is to generalize the type theory using the state structure treatment and associated projective methods. This generalization is called the *algebra of state structures*.

[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

Type theory is dependent on a set of base types. In Reference [11], the first nontrivial types in the hierarchy, called the *elementary types*, are taken as the set of quantum states as in Equations (3.21). In terms of the projective characterization, this means that each base type is characterized with projector $\mathcal{I}$, and the elementary type on k subsystems is the set of quantum states characterized by projector $\mathcal{I}_\mathcal{A} \otimes \mathcal{I}_\mathcal{B} \otimes \ldots \mathcal{I}_\mathcal{K}$ (where party K is the $k-$th party). But within the framework of state structures, other base types can be considered: *any* projector on state structure defines a state structure that can be taken as a basis to construct a higher-order theory. What is more, the different parties can be associated with different base state structures. Nothing forbids a priori to consider party Alice to be able to prepare quantum states, whereas party Bob can only prepare classical states. What the algebra of state structure should be able to tell is how to link these two local operations into a joint global operation given a signaling relation. This is the first purpose of the algebra of state structure: by knowing what parties can do locally and how their signaling relations are constrained, using the algebra allows to quickly get an expression for the projector on their joint state structure, which in turn yields the characterization of the set of all joint operations the parties are allowed to perform as well as what the set of all environments they can share (the dual state structure).

The second purpose of studying the algebra is to abstract its structure. Knowing how the algebra works allows one to compare state structure simply by using the rewrite rules of the algebra in the same fashion as what was possible with type theory. This is a relevant question for two reasons: first, it can be used to classify the different families of operations. It should be no surprise that the set of bipartite channels allowing signaling only from Alice to Bob is a subset of the set of bipartite channels allowing signaling in the two directions. Remark that the state structures are essentially characterized by their projectors. In Section 3.5, it was shown in particular that the (on-average) signaling relations between the parts of a multipartite state structure are in correspondence with how the joint projector characterizing the overall state structure is built. Hence, the state structure algebra is actually homorphic to a projector algebra. Compared to types and state structures, the projector algebra is a concrete algebra on a Hilbert space, so the algebraic rules between state structures can be systematically uncovered by computations on projectors. This chapter will present relations between state structure through relations in the projectors. For example, an expression of the form[12]

$$\mathcal{P}_A \prec \mathcal{P}_B \subset \mathcal{P}_A \parr \mathcal{P}_B \;, \tag{4.55}$$

12: Where $\mathcal{P}_A$ and $\mathcal{P}_B$ characterize the set of valid single-partite channels of Alice and Bob, respectively.

encodes the fact that one-way signaling composition is a subset of two-way signaling composition. It may appear trivial, but as it turns out, these inclusion relations sometimes carry deep results. For example, that two-way signaling composition is (the affine hull of) the two one-way signaling compositions will be reduced to the projector expression

$$\mathcal{P}_A \parr \mathcal{P}_B = (\mathcal{P}_A \prec \mathcal{P}_B) \cup (\mathcal{P}_A \succ \mathcal{P}_B) \;. \tag{4.56}$$

Thus, abstracting the algebra is to give meaning to what the symbols like $\subset, \cup, =$ mean in terms of projectors, what they entail in terms of the subspaces these projectors characterize, and, therefore, what they mean in term of signaling relations.

In this regard, working out all equivalence relations for all compositions of projectors using the connectives in the algebra provides the systematic breakdown of the possible signaling relations composite state structure can present. For a fixed number of parties, thorough characterization consists of sorting them into equivalence classes in the same way that single partite types fall into two equivalence classes: either A or $\overline{A}$. Now, the issue with a systematic characterization of the algebra is its size. As will be shown, there are seven relevant operations in the algebra of projectors[13], $\{\overline{\cdot}, \cap, \cup, \otimes, \prec, \succ, \parr\}$. Hence, the number of different ways of combining just two projectors is already very large. Out of the very large amounts of expressions combining the seven connectives for two parties Alice and Bob, there are only four equivalence classes corresponding to the four possible signaling directions: none; Alice to Bob; Bob to Alice; and both. Lifting these operations to the case of more than two parties exponentially increases the number of equivalence classes (this number grows as powers of four): a systematic listing of all signaling structures

13: The transformation, $\rightarrow$ is omitted since it can be built out of the other relations.

is complicated.

This leads to the technical reason for studying the abstract structure of the algebra in this chapter: inferring general properties so as to infer general behaviors of signaling structure. In a sense, this is tackling the opposite problem of the systematic characterization: given two composite state structures that are believed to be equivalent, what general proof strategy can be used at the level of the algebra to prove it without exhaustively listing all equivalences for this given number of parties? This will result in the study of the lattice structure of the algebra, as well as the definition of a normal form for projectors. Putting two projectors in normal form indeed provides a systematic way to prove the equivalence of two state structures.

As a concluding example of Chapter 5, the issues of the type system raised around equations (4.52), (4.53), and (4.54) will be shown to disappear in a two-step example of the utilization of the algebra. First, it will be shown that quantum theory has a singular signaling behavior with respect to the algebra. This will explain why certain higher-order quantum maps do coincide with lower-order ones as in Equation (4.53). Second, an example of the use of the normal form will be used to extend this particular behavior of quantum channels to their full hierarchy: the result of References [9, 11] that quantum combs (which are nested transformations) are equivalent to quantum networks (which are successions of operations with a fixed signaling direction) will be rederived.

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

The Algebra of Projectors | 5.

L'algèbre n'est qu'une géométrie écrite, la géométrie n'est qu'une algèbre figurée.*

Sophie Germain (1896), *Oeuvres philosophiques*

In the previous chapter, the basic form of all sets of objects in the theory of higher-order processes was abstracted as a state structure in Definition 3.2.2. It was shown that all state structures are tied to their support on an operator system, Definition 3.2.1, characterized by a special kind of superoperator projector, Definition 3.2.7. It was moreover shown that composite state structures, i.e. the set of objects shared by several parties, are defined by the local state structures of each party as well as which signaling relations they permit between them. As a consequence, any composite state structure, like the transformation between state structure $\mathscr{A}$ and $\mathscr{B}$ or their tensor product for example, is characterized by a projector which is obtained from the projectors of the state structures it composes. With regard to that, the algebraic relation encoding how two projectors are composed encodes the signaling relations that the composite structure will feature.

Studying the algebra of projector compositions is thus a systematic way to study signaling in state structures. In this chapter, this will be done to obtain decompositions of composite state structures so as to infer the signaling relations they allow between their components.

This chapter relies heavily on previous results of Perinotti [10] and Bisio [11], as well as Kissinger and Uijlen [33]. In particular, the whole chapter starts by recognizing that Perinotti and Bisio's type system (reviewed in Section 4.3) can be expressed as structural rules on how projectors are composed using Theorem 3.4.1. From there, the type system is extended into an algebra of projector compositions by adding the various composition rules studied in the previous chapter. The chapter studies this extension, particularly the intrinsic logic of the obtained algebra as was pioneered in References [33, 36]. By construction, this algebra of projectors is homomorphic to the signaling relations between the different parts of a composite state structure. Consequently, studying the logic of the projector algebra provides a systematic way to study the logic of signaling in higher-order processes. In addition to the exhaustive study of this logic, the achievement of this chapter is to introduce a normal form for the projectors characterizing composite state structure. From this normal form, the signaling structure of any state structure, therefore of any set of higher-order objects, can be read.

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.

[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

[36]: Simmons et al. (2022), *Higher-order causal theories are models of BV-logic*.

5.1. Beyond Type Theory: the Projector Algebra

It was shown in the last chapter that, given a state structure, its dual state structure was characterized mainly through a rule on projectors,

*"Algebra is but a written geometry, geometry is but a figurative algebra."

the **negation** of a projector defined as

$$\overline{\mathcal{P}} := \mathcal{I} - \mathcal{P} + \mathcal{D}\,. \tag{5.1}$$

And that the corresponding state structure is defined over a subspace that is quasi-orthogonal to the subspace of the original state structure. This mathematical property was interpreted in Subsection 3.3.2 as a constraint on the influence that the states and effects can have on each other. Using this constraint as a no-signaling heuristic, four different bipartite composition rules for state structures were derived using the signaling relations. These composition rules are actually ways of composing the underlying operator systems of state structures, hence they can be encoded as ways of composing projectors. These are the **tensor** product of projectors, representing a no-signaling composition, which is simply the tensor product of linear maps; the **parr**, representing two-way signaling composition,

$$\mathcal{P}_A \parr \mathcal{P}_B := \mathcal{I}_A \otimes \mathcal{I}_B - \overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B\,, \tag{5.2}$$

(in which the negation rule has been used to shorten the expression); and the **prec**, representing a one-way signaling composition in which directionality matters so there are two of them:

$$\mathcal{P}_A \prec \mathcal{P}_B := \mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\,; \tag{5.3a}$$
$$\mathcal{P}_A \succ \mathcal{P}_B := \mathcal{P}_A \otimes \mathcal{I}_B + \mathcal{D}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B\,. \tag{5.3b}$$

With respect to these projective rules, the type system presented in Section 4.3 starts from the state structure of transformations, characterized by the **transformation** connector which is derived as a parr with negated input,

$$\mathcal{P}_A \to \mathcal{P}_B := \overline{\mathcal{P}}_A \parr \mathcal{P}_B\,. \tag{5.4}$$

Using this rule at the level of projectors instead of types, the relations obtained for types also hold for projectors:

$$\overline{A} = A \to 1 \Rightarrow \overline{\mathcal{P}}_A = \mathcal{P}_A \to 1\,; \tag{5.5a}$$
$$A \otimes B = \overline{A \to \overline{B}} \Rightarrow \mathcal{P}_A \otimes \mathcal{P}_B = \overline{\mathcal{P}_A \to \overline{\mathcal{P}}_B}\,. \tag{5.5b}$$

There is a correspondence types $\leftrightarrow$ projectors $\leftrightarrow$ state structures. However, the prec is outside of the definition of the original type system. Yet, it has a well-defined rule and interpretation. This is the limitation of the type system that the study of the algebra of projectors is meant to overcome: the formulae in the type system are abstract entities that require some translation to identify the set of operators they designate. Lowering it one step closer to the state structures, at the level of connection rules between projectors, allows to keep some of its abstract structure while at the same time rephrasing it as concrete mathematical objects (superoperator projectors) so that the translation is built-in.

The benefit of doing so is that abstract relations, like a rewrite rule between two types, can now be derived methodically from the study of the algebraic properties of superoperator projectors. In addition, because the projector rules ultimately represent ways of combining subspaces, two extra rules arise naturally for comparing subspaces: the union and intersection of subspaces. Actually, it is under these two rules that the projectors form an algebra. Phrasing the signaling relations and their abstract logic models in terms of this algebra is the genuinely new contribution of this work to the theory of higher-order quantum transformations [10, 11, 33, 36, 37, 88].

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.
[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.
[36]: Simmons et al. (2022), *Higher-order causal theories are models of BV-logic*.
[37]: Milz et al. (2023), *Transformations between arbitrary (quantum) objects and the emergence of indefinite causality*.
[88]: Apadula et al. (2022), *No-signalling constrains quantum computation with indefinite causal structure*.

The point of this chapter is to show that not only do the new connectors extend the type theory into an algebra but that the roles of the operations in the algebra are not arbitrary: they are actually well-known connectives that arise in models of logic. Hence, each added connector corresponds to a new way of defining state structure from signaling requirements, but at the same time, each added connector happens to refine the algebra of projectors into a more complex model of logic. These connectors will be reintroduced in an order such that the model of logic they define is a refinement of the previous one. Starting from the algebra of projectors defined under the union and the intersection, these refinements will be similar to the following models of logic: 1) Classical Logic (Boolean algebra), obtained by adding the negation (5.1); 2) Multiplicative Additive Linear Logic (MALL) [146], obtained by adding the tensor and the parr (5.2); 3) Pomset [147, 148] or BV-Logic [149], obtained by adding the prec (5.3).

[146]: Girard (1987), *Linear logic*.
[147]: Rétoré (1993), *Réseaux et Séquents Ordonnés*.
[148]: Retoré (2021), *Pomset Logic*.
[149]: Guglielmi (2007), *A System of Interaction and Structure*.

5.1.1. The Projector Algebra as a Lattice

The overarching question of this chapter can be put as "how are the signaling relations allowed by the state structure reflected on the algebraic properties of their characterizing projectors?". The first step is to identify what is the algebra and how to interpret it. By Definition 3.2.7, projectors on operator systems are a subset of the algebra of bounded linear maps on the space of operators on a Hilbert space. As such, arbitrary superoperators $\mathcal{S}, \mathcal{T}, \ldots$ inherit three operations 'for free' because of their linearity: the addition $+$, the composition $\circ$, and the scalar multiplication. With these three operations, the set of superoperators constitutes an *algebra over* $\mathbb{C}$ (in the abstract sense, see e.g. Chapter 7 of Reference [150]). But when the set of all superoperators is restricted to only the projectors on operator systems, does it still have the same abstract structure? Since these are projectors, they acquire the extra property that all elements of the subset are idempotent. However, this set of idempotent superoperators is no longer closed under any of the three operations.

[150]: Jacobson (2009), *Basic Algebra*.

The workaround to recover a closed set is to realize that state structures are trace-normalized, so scalar multiplication is an irrelevant feature that can be abandoned, and the field over which the projectors are defined can actually be restricted to the singleton $\{1\}$. The other realization is that the algebraic properties of the projectors will be used to compare the operator systems they define. But for a meaningful comparison to hold, these subspaces must be expressible in the same basis. A cornerstone of linear algebra is that to share the same basis, i.e. to be simultaneously diagonalizable, two operators must commute. **As a consequence, the working hypothesis for this chapter is that the elements of the algebra form a set of commuting projectors on operator systems.** This hypothesis will be shown to be verified along the presentation of the algebra as each newly introduced operation will be proven to preserve commutation.

Such a set of commuting idempotent projectors constitutes an abstract mathematical structure known as a *Boolean ring*[1]. In terms of the operator systems they define, comparison becomes straightforward as one can consider the *overlap* between the two such subspaces. For this purpose, one needs a new composition rule: the *intersection of projectors*[2] which is abstracted as a new operation noted $\cap$ and nicknamed 'cap'. It is defined by

$$\forall \mathcal{P}, \mathcal{P}', \quad \mathrm{Im}\left\{\mathcal{P} \cap \mathcal{P}'\right\} = \mathrm{Im}\{\mathcal{P}\} \cap \mathrm{Im}\left\{\mathcal{P}'\right\}, \tag{5.6}$$

That is, the intersection of two projectors is the projector defining the intersection of the spaces they define. For commuting projectors, the intersection of projectors is exactly the composition operation,

$$\mathcal{P} \cap \mathcal{P}' := \mathcal{P} \circ \mathcal{P}' . \tag{5.7}$$

The algebra is still not quite complete; a Boolean ring is almost a *Boolean algebra*[3] but it is not closed under the addition operation as it does not preserve idempotency. The interpretation in terms of the underlying operator systems can again guide how to change the definition: an addition of projectors should represent the union of the two spaces they characterize. The issue with using the addition to do that is that it counts twice the overlap of the two subspaces[4]; a correct addition is then obtained by turning the addition into the exclusive disjunction, or *union of projectors*, noted $\cup$ and nicknamed the 'cup'. It is defined by

$$\forall \mathcal{P}, \mathcal{P}', \quad \mathrm{Im}\left\{\mathcal{P} \cup \mathcal{P}'\right\} = \mathrm{Im}\{\mathcal{P}\} \,+\, \mathrm{Im}\left\{\mathcal{P}'\right\}, \tag{5.8}$$

where the '+' here refers to the Minkowski sum: $\mathrm{Im}\{\mathcal{P}\} \,+\, \mathrm{Im}\{\mathcal{P}'\} := \{x + y | x \in \mathrm{Im}\{\mathcal{P}\},\, y \in \mathrm{Im}\{\mathcal{P}'\}\}$. That is, the union of two projectors is the projector defining the joint span of the spaces they define. For commuting projectors, the union of projectors is obtained as

$$\mathcal{P} \cup \mathcal{P}' := \mathcal{P} + \mathcal{P}' - \mathcal{P} \cap \mathcal{P}' . \tag{5.9}$$

With these, a set of projectors on operator system defined over a space $\mathcal{L}(\mathcal{H})$ that are commuting pairwise can be extended into a set closed under the operations $\{\cup, \cap\}$. This abstract set forms a ring, where the $\cup$ plays the role similar to the addition $+$ whereas the $\cap$ plays the role of the multiplication $\times$. This is a special kind of ring whose elements are all idempotent, i.e. $\mathcal{P} \cap \mathcal{P} = \mathcal{P}$ called a Boolean ring. This ring can also be interpreted as a distributive lattice, a sublattice of simultaneously diagonalizable subspaces, where $\cap$ plays the role of the 'meet' and $\cup$ the 'join'. The images of the projectors in this ring/lattice are all the operator systems that can be defined for a given basis of a given Hilbert space. Hence, when ignoring normalization the elements of the Boolean ring of projectors are in one-to-one correspondence with the state structures on a given space.

This ambivalent ring/lattice is the 'algebra' this section is about[5]; this name is now made formal.

Definition 5.1.1 (Projector Algebra) *Let $\{\mathcal{P}, \mathcal{P}', \ldots\}$ be a set of superoperator projectors on operator systems like in Definition 3.2.7 so that they all act on the same space $\mathcal{L}(\mathcal{H})$. This set is a **commuting** set of projectors if its*

1: See Appendix D.2.1, for the technical details.

2: Which generalizes the inclusion of subspaces as in Proposition C.1.3.

3: See Appendix D.1.1 for the definition.

4: This set-theoretic intuition is proven explicitly in Appendix D.2.1.

5: The definition of a ring is reminded in Appendix D.1.1. Some technicalities: 1) The projectors under operations $\cup, \cap$ are actually not exactly a Boolean ring but a Boolean semiring naturally related to one. This simplification is made for the sake of exposition.
2) A Boolean ring is not only a ring but also a distributive lattice. This lattice can always be turned into a Boolean algebra by adding a uniquely defined extra operation called a 'negation' (which is done in the next section). However, as such a Boolean algebra is actually still a ring as it is not defined over a field but rather over the singleton $\{1\}$. In other words, the only scalar compatible with a Boolean algebra is the number 1, whereas the abstract definition of an algebra requires the set of scalars compatible with it to be a field. The name Boolean "algebra" is due to historical reasons and should not be understood as an algebra over a field. On the other hand, a Boolean algebra is a (distributive and complemented) lattice. For these reasons, the phrasing 'algebra of projectors' is used as a colloquial term to convey the idea that the operations $(\cup, \cap)$ are similar to the usual operations $(+, \times)$ of school children's algebra; whereas 'lattice of projectors' is used as a technical term, for example when dealing with definitions.

elements are pairwise commuting with respect to the composition $\circ$. This commuting set is ***complete*** *when no extra projector can be added to the set while preserving pairwise commutation.*

By a ***Projector Algebra*** *on a given space $\mathcal{L}(\mathcal{H})$, it is meant a complete commuting set of superoperator projectors on operator systems together with the operations $\{\cap, \cup\}$.*

When discussing projectors, it is always assumed that they belong to a given projector algebra, hereafter concisely referred to as 'the algebra'.

A Boolean ring is a special kind of lattice that has many useful properties that are now detailed[6]. First, one can prove that an operator system characterized by $\mathcal{P}^{'}$ is contained within another characterized by $\mathcal{P}$ by showing either of the following:

$$\mathcal{P} \cap \mathcal{P}^{'} = \mathcal{P}^{'} \; ; \tag{5.10a}$$

$$\mathcal{P} \cup \mathcal{P}^{'} = \mathcal{P} \; . \tag{5.10b}$$

In the above, remark that the *duality principle* of lattices holds: any equality formed using projectors, caps, and cups (for instance the first line) induces a second equality obtained by switching the caps and cups (the second line) which automatically holds. This principle greatly reduces the amount of proofs needed to show inclusion. In terms of projectors, the inclusion conditions will be concisely noted

$$\mathcal{P}^{'} \subseteq \mathcal{P} \; . \tag{5.11}$$

When comparing state structures, the inclusion of projectors shows inclusion (up to normalization) of a state structure within another since the other constraint, positivity, is common to all state structures. Moreover, because every operator system must at least contain the identity element and because an operator system cannot be bigger than the full space of self-adjoint operators, every projector in the algebra is contained between the depolarizing and identity projectors,

$$\mathcal{D} \subseteq \mathcal{P} \subseteq \mathcal{I}. \tag{5.12}$$

Conditions (5.10) thus define the partial order (5.11) quantifying whether a state structure is included into another one. Equation (5.12) states that $\mathcal{D}$ and $\mathcal{I}$ are respectively the least and greatest elements in the partial order, meaning that in a given space $\mathcal{L}\left(\mathcal{H}^{A}\right)$, the smallest state structure is the set $\{\frac{c_A}{d_A}\mathbb{1}\}$ and the largest is the set of self-adjoint operators. Hence, the projectors can be compared using the partial order, which corresponds to the embedding of an operator system into another one.

Nevertheless, the cap and the cup are new rules that cannot be expressed using the $\rightarrow$ connector. So, while it generalizes and simplifies the classification of higher-order transformations, the downside of the algebra is that it goes outside of the "typed-λ-calculus kind" of type-theoretic framework of references [10, 11] as the type system can no longer be expressed in terms of a single connector $\rightarrow$ from which all other connectors can be derived[7].

6: The proof that it is effectively a Boolean lattice is provided in Appendix D.1.1. Remark in passing that defined as such, the projector algebra is a finitely generated lattice with at most 2^{d^2-1} elements, see the comment in Appendix D.2.2.

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

7: Another type system can be defined as will be shown in Subsection 6.3.2.

5.1.2. The Boolean Algebra of Projectors

In a Boolean ring, it is always possible to define a unique *complement* operation, promoting it to a Boolean algebra. The *complement of a projector* in the algebra is noted $\overline{\cdot}$ and nicknamed 'negation'. It is defined by, $\forall \mathcal{P}$,

$$\mathcal{P} \cap \overline{\mathcal{P}} = \mathcal{D} \, ; \tag{5.13a}$$
$$\mathcal{P} \cup \overline{\mathcal{P}} = \mathcal{I} \, . \tag{5.13b}$$

The negation of a projector is none other than the operation appearing in Theorem 3.3.2,

$$\overline{\mathcal{P}} := \mathcal{I} - \mathcal{P} + \mathcal{D} \, , \tag{5.14}$$

and characterizing the dual state structure as in Definition 3.3.3. This operation $\overline{\cdot}$ effectively acts as the Boolean negation in the algebra of projectors since it makes $\overline{\mathcal{P}}$ complementary to $\mathcal{P}$, meaning they add up to the greatest element and since it is an involution, meaning applying it twice does nothing $\overline{\overline{\mathcal{P}}} = \mathcal{P}$. The algebra therefore simplifies the relation between a set of states and its set of unit effects into a simple application of the Boolean complement. Actually, for a given set of base state structures, the operations $\{\cup, \cap, \overline{\cdot}\}$ can be used to construct a full Boolean sublattice of the algebra by considering all possible combinations.

Proposition 5.1.1 (Boolean sublattice) *Let $\{\mathcal{P}_A^{(i)}\}_{i=1}^n$ be a set of n commuting projectors on operator systems in space $\mathcal{L}\left(\mathcal{H}^A\right)$. Then, this set spans*[8] *a sublattice of the projector algebra under the operations $\{\cup, \cap, \overline{\cdot}\}$.*

8: By its 'span', it is meant the set generated by all combinations of any number of elements from the base set mixed using the operations $\{\cup, \cap, \overline{\cdot}\}$.

Proof. The operations $\{\cup, \cap, \overline{\cdot}\}$ map projectors to projectors and preserve commutation are proven in the appendix (see Appendix D.2.1 and Appendix D.2.3). Since these operations define a Boolean lattice, with the least element being $\mathcal{D}_A$ (that can be obtained as $\mathcal{P}_A^{(i)} \cap \overline{\mathcal{P}_A^{(i)}}$ for any i) and the greatest being $\mathcal{I}_A$, it has to be a sublattice of the algebra of projectors. □

This sublattice contains all possible states and effects state structures that can be constructed from the base set. Therefore their comparison is straightforward using the partial order relation of the lattice ('is this state structure contained into this one?').

Moreover, a Boolean algebra obeys the De Morgan law:

$$\overline{\mathcal{P} \cap \mathcal{P}'} = \overline{\mathcal{P}} \cup \overline{\mathcal{P}'} \, , \tag{5.15}$$

which induces another duality principle[9]:

9: See Appendix D.2.3 for the proofs.

$$\mathcal{P}' \subseteq \mathcal{P} \iff \overline{\mathcal{P}} \subseteq \overline{\mathcal{P}'} \, . \tag{5.16}$$

These two rules greatly reduce the number of computations required to determine all the relations between the state structures in the lattice, as inclusions between functionals are dual to inclusions between the corresponding states. For example, in Equation (5.13), only one equation is to be proven and the other holds simply by the duality principle of the negation. The main properties of the Boolean lattice structure useful for doing computations on projectors are gathered in the following.

Proposition 5.1.2 (Properties of the cap and the cup) *The cap and the cup are associative, commutative, order-preserving, De Morgan dual to each other. Moreover, these two rules are distributive over each other. That is, the cap verifies*

$$\left(\mathcal{P}_A \cap \mathcal{P}'_A\right) \cap \mathcal{P}''_A = \mathcal{P}_A \cap \left(\mathcal{P}'_A \cap \mathcal{P}''_A\right); \tag{5.17a}$$

$$\mathcal{P}_A \cap \mathcal{P}'_A = \mathcal{P}'_A \cap \mathcal{P}_A \tag{5.17b}$$

$$\left.\begin{matrix} \mathcal{P}_A \subseteq \mathcal{P}'_A \\ \mathcal{P}''_A \subseteq \mathcal{P}'''_A \end{matrix}\right\} \Rightarrow \left(\mathcal{P}_A \cap \mathcal{P}''_A\right) \subseteq \left(\mathcal{P}'_A \cap \mathcal{P}'''_A\right); \tag{5.17c}$$

$$\overline{\mathcal{P}_A \cap \mathcal{P}'_A} = \overline{\mathcal{P}}_A \cup \overline{\mathcal{P}'_A}; \tag{5.17d}$$

and moreover,

$$\mathcal{P}_A \cap (\mathcal{P}'_A \cup \mathcal{P}''_A) = (\mathcal{P}_A \cap \mathcal{P}'_A) \cup (\mathcal{P}_A \cup \mathcal{P}''_A)\,. \tag{5.18}$$

Accordingly, the cup verifies the same five rules under the interchange $\cap \leftrightarrow \cup$.

Proof. See the appendix: in Appendix D.2.1, the lattice structure is discussed, proving the first three and the last equations; in Appendix D.2.3, the last one is proven. □

5.1.3. Type Theory and the No-Signaling Sublattice

The Boolean algebra of projectors sees each state structure as a global thing, characterized by a global projector $\mathcal{P}$. But how is the set of global projectors obtained? Actually, any projector splits as an expression built from the k projectors $\mathcal{P}_A, \mathcal{P}_B, \ldots, \mathcal{P}_K$ on the base state structures associated with each party. An example is the projector on the elementary state structure which reads $\mathcal{P}_{A\ldots K} = \mathcal{P}_A \otimes \ldots \otimes \mathcal{P}_K$.

Using the three rules under consideration in the type theory[10], $\{\overline{\cdot}, \otimes, \rightarrow\}$, the possible state structures for a given number of parties, and therefore the associated projectors, will be constructed by considering all possible combinations of these rules. The assumption that all such projectors commute with each other is then justified by the following lemma:

10: Which only needs the transformation connector $\rightarrow$ and the trivial system 1 to recover the other two.

Lemma 5.1.3 *The rules* $\{\overline{\cdot}, \otimes, \rightarrow\}$ *result in valid projectors and preserve commutation.*

This is provable by direct computation using the definitions; see Appendix D.2.3, Appendix D.3.1, and Appendix D.3.2. A direct corollary is that the sets they span are valid projector algebra.

Corollary 5.1.4 (The lattices of type theory) *Let* $\{\mathcal{P}_A^{(i)}\}_{i=1}^{n_A}$ *be a set of* n_A *commuting projectors on operator systems in space* $\mathcal{L}\left(\mathcal{H}^A\right)$. *Let* $\{\mathcal{P}_B^{(j)}\}_{j=1}^{n_B}$, $\{\mathcal{P}_C^{(k)}\}_{k=1}^{n_C}, \ldots$ *be a similarly defined sets in spaces* $\mathcal{L}\left(\mathcal{H}^B\right), \mathcal{L}\left(\mathcal{H}^C\right), \ldots$. *Then, the set of all expressions built from elements of these sets under the rules* $\{\overline{\cdot}, \otimes, \rightarrow\}$ *so that they project on* $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C \otimes \ldots\right)$ *spans a sublattice of the projector algebra on* $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C \otimes \ldots\right)$ *under operations* $\{\cup, \cap\}$.

As a simplification and for the sake of a meaningful comparison, these sets are often taken the same single-element sets:every party can only access to a single state space locally, and the state space of every party is the same. For example, in References [10, 11, 32, 88] it is assumed that every party has access only to quantum theory locally, so the set of base projectors for every party are reduced to the singletons made of the identity map, i.e. $\{\mathcal{P}_A^{(i)}\} = \{\mathcal{I}_A\}$. Using projective characterization, the comparison of higher-order transformations in the sense of Reference [11] is by consequent reduced to the study of lattices of commuting projectors on operator systems built from single-element base projectors. For k parties, any expression $\mathcal{P}$ built using $\{\overline{\cdot}, \otimes, \rightarrow\}$ is contained in a Boolean lattice closed under operations $\{\cap, \cup\}$ with smallest element $\mathcal{D}_A \otimes \ldots \otimes \mathcal{D}_K$ and biggest element $\mathcal{I}_A \otimes \ldots \otimes \mathcal{I}_K$. Of course, even when the base state structures are singletons, the number of possible combinations using the operations $\{\overline{\cdot}, \otimes, \rightarrow\}$ grows exponentially with the number of parties, so these lattices are huge. However, any projector in the lattice has natural subset and superset that are in general different than the tensor product of the depolarizing and of the identity projectors, meaning that projectors usually belong to a sublattice.

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.
[32]: Castro-Ruiz et al. (2018), *Dynamics of Quantum Causal Structures*.
[88]: Apadula et al. (2022), *No-signalling constrains quantum computation with indefinite causal structure*.

Definition 5.1.2 (No-signaling subset / Fully signaling superset) *Let $\mathcal{P}$ be a projector on operator system built using k base projectors $\mathcal{P}_A, \mathcal{P}_B \ldots \mathcal{P}_K$ composed together using operations $\{\overline{\cdot}, \cap, \cup, \otimes, \rightarrow\}$.*
The projector $\mathcal{P}_{NS}$ to its ***no signaling subset*** *is the largest projector embedded in $\mathcal{P}$ obtained as the tensor composition of single-partite projectors. By construction, the single-partite projector associated with party X can only be the base projector, its negation, or the depolarizing superoperator $\mathcal{D}_X$ so that $\mathcal{P}_{NS}$ reads*

$$\mathcal{P}_{NS} := \widetilde{\mathcal{P}}_A^{\mathcal{D}} \otimes \widetilde{\mathcal{P}}_B^{\mathcal{D}} \otimes \ldots \widetilde{\mathcal{P}}_K^{\mathcal{D}} \subseteq \mathcal{P} \,, \tag{5.19}$$

where the $\widetilde{\mathcal{P}}_X^{\mathcal{D}}$ notation means an element chosen among $\{\mathcal{P}_X, \overline{\mathcal{P}}_X, \mathcal{D}_X\}$ depending on the exact form of $\mathcal{P}$.
The projector $\mathcal{P}_{FS}$ to its ***fully signaling superset*** *is the smallest projector containing $\mathcal{P}$ obtained as the parr composition of single-partite projectors. By construction, the single-partite projector associated with party X can only be the base projector, its negation, or the identity superoperator $\mathcal{I}_X$ so that $\mathcal{P}_{FS}$ reads*

$$\mathcal{P} \subseteq \mathcal{P}_{FS} := \overline{\widetilde{\mathcal{P}}_A^{\mathcal{I}}} \rightarrow \left(\overline{\widetilde{\mathcal{P}}_B^{\mathcal{I}}} \rightarrow \ldots \left(\overline{\widetilde{\mathcal{P}}_J^{\mathcal{I}}} \rightarrow \widetilde{\mathcal{P}}_K^{\mathcal{I}}\right) \ldots\right) \,, \tag{5.20}$$

where the $\widetilde{\mathcal{P}}_X^{\mathcal{I}}$ notation means an element chosen among $\{\mathcal{P}_X, \overline{\mathcal{P}}_X, \mathcal{I}_X\}$ depending on the exact form of $\mathcal{P}$.

To see why these sets always exist, notice that the definition allows $\mathcal{P}_{NS}$ to be equal to $\mathcal{D}_A \otimes \ldots \otimes \mathcal{D}_K$ and $\mathcal{P}_{FS}$ to be $\mathcal{I}_A \otimes \ldots \otimes \mathcal{I}_K$ in the worst case scenario. The other thing to notice is that the cap and cup are defined respectively of which tensor factor they act on. This is because an intersection or union at the level of a single subsystem appearing in a single projector can be transferred as an intersection or union of two projectors over the full system. This is because the cap and the cup obey an interchange law with, respectively, the tensor and the parr[11],

11: Note that the converse is not true, the interchange of a cup with a tensor defines a subset, whereas the interchange of the cap with the parr defines a superset; see Equations (D.46) and (D.82) in the appendix.

$$\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) = (\mathcal{P}_A \otimes \mathcal{P}_B) \cap \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right) \ ; \tag{5.21a}$$
$$\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \parr \left(\mathcal{P}_B \cup \mathcal{P}_B^{'}\right) = (\mathcal{P}_A \otimes \mathcal{P}_B) \cup \left(\mathcal{P}_A^{'} \parr \mathcal{P}_B^{'}\right) \ . \tag{5.21b}$$

Using this law, the cap and cup connectors relative to the algebra over a tensor factor can always be lifted as caps and cups relative to the algebra on the full space by a rewriting like

$$\ldots \otimes (\mathcal{P}_J \cap \mathcal{P}_J^{'}) \otimes \ldots = \ldots \otimes \mathcal{P}_J \otimes \ldots \cap \ldots \otimes \mathcal{P}_J^{'} \otimes \ldots \ , \tag{5.22}$$

where on the left-hand side, the cap defined is with respect to the J subsystem, whereas on the right-hand side, it is with respect to the full space[12].

12: The exact procedure of making all the caps and cups global will be studied in details in Section 5.2 below.

In general, the no-signaling and fully signaling projectors define the least and greatest elements of a sublattice

$$\mathcal{D}_A \otimes \ldots \otimes \mathcal{D}_K \subseteq \mathcal{P}_{NS} \subseteq \mathcal{P} \subseteq \mathcal{P}_{FS} \subseteq \mathcal{I}_A \otimes \ldots \otimes \mathcal{I}_K \ , \tag{5.23}$$

around each projector $\mathcal{P}$. In Definition 5.1.2 one can use the parr instead of the transformation to define these sublattices, this is what is done in Reference [33]. Actually, it is hidden in the definition of the fully signaling superset:

[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

$$\mathcal{P}_{FS} \equiv \widetilde{\mathcal{P}}_A^{\mathcal{I}} \parr \widetilde{\mathcal{P}}_B^{\mathcal{I}} \parr \ldots \widetilde{\mathcal{P}}_K^{\mathcal{I}} \tag{5.24}$$

This makes sense in terms of the signaling interpretation: the parr compares better to the tensor since both elements in each side of the connective 'have the same nature', meaning that they are either both interpreted as inputs or outputs. In terms of the projector algebra, using the parr rather than using the transformation also makes more sense because, like the tensor, the parr has no directionality and is associative.

Proposition 5.1.5 (Properties of the tensor, the parr, and the transformation) *The tensor and the parr are associative, commutative*[13]*, order-preserving, De Morgan dual to each other. That is, the tensor verifies*

$$(\mathcal{P}_A \otimes \mathcal{P}_B) \otimes \mathcal{P}_C = \mathcal{P}_A \otimes (\mathcal{P}_B \otimes \mathcal{P}_C) \ ; \tag{5.25a}$$
$$\mathcal{P}_A \otimes \mathcal{P}_B \cong \mathcal{P}_B \otimes \mathcal{P}_A \ ; \tag{5.25b}$$
$$\left.\begin{array}{l}\mathcal{P}_A \subseteq \mathcal{P}_A^{'} \\ \mathcal{P}_B \subseteq \mathcal{P}_B^{'}\end{array}\right\} \Rightarrow (\mathcal{P}_A \otimes \mathcal{P}_B) \subseteq \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right) \ ; \tag{5.25c}$$
$$\overline{\mathcal{P}_A \otimes \mathcal{P}_B} = \overline{\mathcal{P}}_A \parr \overline{\mathcal{P}}_B . \tag{5.25d}$$

Accordingly, the parr verifies the same four rules under the interchange $\otimes \leftrightarrow \parr$. The transformation, however, is neither associative nor commutative, i.e., $\mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C) \neq (\mathcal{P}_A \to \mathcal{P}_B) \to \mathcal{P}_C$ and $\mathcal{P}_A \to \mathcal{P}_B \neq \mathcal{P}_A \leftarrow \mathcal{P}_B$. But it enjoys the following properties:

$$\mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C) = (\mathcal{P}_A \otimes \mathcal{P}_B) \to \mathcal{P}_C \ ; \tag{5.26a}$$
$$\mathcal{P}_A \to \mathcal{P}_B = \overline{\mathcal{P}}_A \leftarrow \overline{\mathcal{P}}_B \ . \tag{5.26b}$$

13: In the literature, and especially in the category theory literature (e.g., [124, 125]), the word 'symmetric' is often preferred over commutative, as the relation relies on an isomorphism of spaces, whence the use of $\cong$ instead of $=$ in (5.25b). At the level of algebraic relations, however, isomorphic spaces or maps are the same for all intents and purposes. Thus 'commutative' is used in a general, loose sense.

[124]: Roman (2017), *An Introduction to the Language of Category Theory*.
[125]: Heunen et al. (2019), *Categories for Quantum Theory: An Introduction*.

From the signaling interpretation of the last chapter, it should be clear that the parr compares to the tensor as a 'larger' composition because[14]

14: A one-line computation can prove these properties; see Appendix D.3.1 for the details.

$\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} \neq \mathcal{P}_A \otimes \mathcal{P}_B$ and

$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} = \mathcal{P}_A \parr \mathcal{P}_B. \tag{5.27}$$

Note that the inequality between these two projectors is fundamentally the reason why an indefinite causal order may arise. As argued in the biased quantum theory example, Subsection 3.6.2, the inequality must be verified in order for the two-way signaling space not to be trivial, i.e.

$$\mathcal{P}_A \otimes \mathcal{P}_B \neq \mathcal{P}_A \parr \mathcal{P}_B \Leftarrow \mathscr{A} \otimes \mathscr{B} \neq \mathscr{A} \parr \mathscr{B}\,. \tag{5.28}$$

Meaning that the support of the two-way signaling composition is strictly bigger than the one of the no-signaling composition, so there may be two-way signaling[15].

15: But, as was shown in the biased quantum state example, this is still not sufficient for ICO as one must then show that this support is also strictly bigger than the convex hull of the two one-way signaling compositions, so that it is not a mixture of signaling directions. And then to show that it allows for non-causal correlations. Only then can the process be said to have ICO.

When the parr is not equivalent to the tensor, the two of them allow refinement of the notion of a no-signaling/fully signaling subsets into a more compact form when $\mathcal{P}$ is built without intersection or unions.

Lemma 5.1.6 (No-signaling sublattice around a projector) *Let $\mathcal{P}$ be a projector on operator system built using k base projectors $\mathcal{P}_A, \mathcal{P}_B \ldots \mathcal{P}_K$ composed together using operations $\{\overline{\cdot}, \otimes, \rightarrow\}$. Then, it belongs to a sublattice closed under operations $\{\cap, \cup\}$ so that*

$$\widetilde{\mathcal{P}}_A \otimes \ldots \otimes \widetilde{\mathcal{P}}_K \subseteq \mathcal{P} \subseteq \widetilde{\mathcal{P}}_A \parr \ldots \parr \widetilde{\mathcal{P}}_K\,, \tag{5.29}$$

where the $\widetilde{\mathcal{P}}_X$ notation means an element chosen among $\{\mathcal{P}_X, \overline{\mathcal{P}}_X\}$ depending on $\mathcal{P}$. Importantly, the choice is the same on both sides of the equation.

This follows mainly from the property (5.25c) that $\mathcal{P}_A \subseteq \mathcal{P}'_A \subseteq \mathcal{P}''_A \Rightarrow \mathcal{P}_A \otimes \mathcal{P}_B \subseteq \mathcal{P}'_A \otimes \mathcal{P}_B \subseteq \mathcal{P}''_A \otimes \mathcal{P}_B \subseteq \overline{\overline{\mathcal{P}}''_A \otimes \overline{\mathcal{P}}_B}$, and from property (5.16)[16].

16: A proof is provided in the Appendix D.5.1 for completeness.

This fine-graining of the lattice is relevant for the study of indefinite causal order and signaling relations as it contains all scenarios similar in terms of the interpretation of base state structures either as inputs or outputs. For a state structure $\mathscr{A}$, the other state structures characterized by the no-signaling sublattice around its projector will be those in which each subsystem has the same interpretation, but with a different set of two-way signaling connections between any two subsystems. A simple example: given the state structure (i.e. the set of) of bipartite quantum channels, its no-signaling sublattice is composed of two elements: itself and the state structure of no-signaling channels. In the next section, this lattice will be further fine-grained by the addition of the prec connectors. Given a state structure, this further refinement is the sublattice of all state structures that are similar to the original one up to changes in signaling relations between subsystems. In the above example, the no-signaling sublattice around a bipartite channel will be enriched by two elements: the state structure of channels allowing signaling from Alice's side to Bob's and the one allowing signaling in the opposite direction.

The least and greatest projectors of Lemma 5.1.6 are found by repetitively using Equation (5.27). A rule of thumb for finding the no signaling projectors is to 'count the number of bars' above each base projector in a given expression $\mathcal{P}$. Since negation is an involution, an odd (respectively, even) amount will indicate that the projector is (not) negated in the no signaling subset. For example, to find the no signaling subset of $(\mathcal{P}_A \otimes (\mathcal{P}_B \to \mathcal{P}_C)) \to \mathcal{P}_D$, one first expresses it using negations and tensors products, $(\mathcal{P}_A \otimes (\mathcal{P}_B \to \mathcal{P}_C)) \to \mathcal{P}_D = \overline{\mathcal{P}_A \otimes \overline{\mathcal{P}_B \otimes \overline{\mathcal{P}}_C} \otimes \overline{\mathcal{P}}_D}$, then by inspection $\mathcal{P}_A$ and $\mathcal{P}_C$ have an odd number of negations above them (one and three, respectively) so they will be negated, whereas $\mathcal{P}_B$ and $\mathcal{P}_D$ have an even number (two and two) so they will not be. On that account, the no signaling subset of $(\mathcal{P}_A \otimes (\mathcal{P}_B \to \mathcal{P}_C)) \to \mathcal{P}_D$ is $\overline{\mathcal{P}}_A \otimes \mathcal{P}_B \otimes \overline{\mathcal{P}}_C \otimes \mathcal{P}_D$ and its fully signaling superset is $\overline{\mathcal{P}}_A \parr \mathcal{P}_B \parr \overline{\mathcal{P}}_C \parr \mathcal{P}_D = \overline{\mathcal{P}_A \otimes \overline{\mathcal{P}}_B \otimes \mathcal{P}_C \otimes \overline{\mathcal{P}}_D}$.

Thus, for a given $\mathcal{P}$, the subspace it defines is embedded between a no-signaling and a fully signaling subspace. What is more, the position of the negations in Equations (5.19) and (5.20) are the same because of Equation (5.27). The projectors to these subspaces are actually the least and greatest elements of a subalgebra stable under $\cap$ and $\cup$, i.e. a sublattice. This sublattice w.r.t a given $\mathcal{P}$ can be spanned as in Corollary 5.1.4 by defining each local projectors $\widetilde{\mathcal{P}}_X$ involved in its no-signaling subset as the set of base projectors, $\widetilde{\mathcal{P}}_X \mapsto \mathcal{P}_X$ (i.e., by redefining the base projectors so to incorporate the negations), and by restricting the choice of operations $\{\overline{\cdot}, \otimes, \to\}$ to a choice that preserves the same no-signaling subset, which, from of Equation (5.27), is $\{\otimes, \parr\}$.

Corollary 5.1.7 (Building a no-signaling sublattice.)
Let $\{\mathcal{P}_A, \mathcal{P}_B, \dots, \mathcal{P}_K\}$ be a set of k projectors on operator systems each associated with a specific Hilbert space $\mathcal{H}^A, \mathcal{H}^B \dots, \mathcal{H}^K$, i.e., a set of base projectors. Any expression $\mathcal{P}$ built using each element of this set once and under the rules $\{\otimes, \parr\}$ is a projector on an operator system over $\mathcal{L}\left(\mathcal{H}^A \otimes \dots \otimes \mathcal{H}^K\right)$ and belongs to the no-signaling lattice

$$\mathcal{P}_A \otimes \dots \otimes \mathcal{P}_K \subseteq \mathcal{P} \subseteq \mathcal{P}_A \parr \dots \parr \mathcal{P}_K\,, \tag{5.30}$$

spanned by $\{\cup, \cap, \otimes, \parr\}$.

This is the lattice mentioned above, the one obtained when studying a given state space of projector $\mathcal{P}$ with respect to modifications of the signaling structure while keeping the same base projectors. It is phrased as a corollary as it is a special case of the finer-grained 'signaling lattice' obtained under the same set of operations plus the prec connectors. This lattice is presented in Proposition 5.1.10 below.

Nonetheless, one should not be misled into thinking that the operations $\{\otimes, \parr\}$ are enough to span the sublattice. While it is trivially true for one party and also holds for two, with three parties and beyond this is no longer true as

$$\mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C \subsetneq ((\mathcal{P}_A \parr \mathcal{P}_B) \otimes \mathcal{P}_C) \cap (\mathcal{P}_A \otimes (\mathcal{P}_B \parr \mathcal{P}_C)) \subsetneq \mathcal{P}_A \otimes (\mathcal{P}_B \parr \mathcal{P}_C)\,; \tag{5.31a}$$

$$\mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C \subsetneq ((\mathcal{P}_A \parr \mathcal{P}_B) \otimes \mathcal{P}_C) \cap (\mathcal{P}_A \otimes (\mathcal{P}_B \parr \mathcal{P}_C)) \subsetneq (\mathcal{P}_A \parr \mathcal{P}_B) \otimes \mathcal{P}_C\,. \tag{5.31b}$$

So there is no expression involving only $\otimes$ and $\parr$ that is equivalent to $((\mathcal{P}_A \parr \mathcal{P}_B) \otimes \mathcal{P}_C) \cap (\mathcal{P}_A \otimes (\mathcal{P}_B \parr \mathcal{P}_C))$; this is a distinct element of the lattice that can only be expressed as an intersection. This issue happens because while the connectors are associative, they are neither associativite nor distributive with each other, i.e.,

$$\mathcal{P}_A \otimes (\mathcal{P}_B \parr \mathcal{P}_C) \subsetneq (\mathcal{P}_A \otimes \mathcal{P}_C) \parr \mathcal{P}_B \supsetneq (\mathcal{P}_A \parr \mathcal{P}_B) \otimes \mathcal{P}_C \,. \tag{5.32}$$

This is due to the richer substructures that can happen in the tripartite signaling scenarios. That $\mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C \neq ((\mathcal{P}_A \parr \mathcal{P}_B) \otimes \mathcal{P}_C) \cap (\mathcal{P}_A \otimes (\mathcal{P}_B \parr \mathcal{P}_C))$ happens because in the latter case there is still some 'controlled' or 'dynamic' way to pass a message: the first projector guarantees that Charlie can signal to neither Alice nor Bob and the second that Alice can signal to neither Bob nor Charlie, but there can still be signaling from Bob to Alice while Charlie is no-signaling to both or signaling from Bob to Charlie when Alice is no-signaling to both. This will be discussed in Chapter 6.

5.1.4. The No-Signaling Sublattice as (almost) Linear Logic

With the further additions of the tensor and of the parr operations, the Boolean algebra of projectors is lifted to another abstract lattice-like structure. This lattice is the one built from several Boolean algebras on different spaces $\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right), \ldots$ in the manner of Definition 5.1.2. It is by construction a sublattice of the Boolean lattice on $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \ldots\right)$ with the same least and greatest elements, $\mathcal{D}_A \otimes \mathcal{D}_B \otimes \ldots$ and $\mathcal{I}_A \otimes \mathcal{I}_B \otimes \ldots$, respectively.

Actually, the connectors $\otimes$ and $\parr$ behave like $\cap$ and $\cup$ under De Morgan duality. Compare Equations (5.17) with (5.25): In both cases it consists of a pair of associative and commutative connectors that obey a duality principle: any equality is valid under the interchange $\cap \leftrightarrow \cup$ and $\otimes \leftrightarrow \parr$; both preserve the partial order; and both obey a De Morgan law. And since the pair tensor/parr verifies a De Morgan law, it obeys the duality (5.16) which implies that

$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'} \iff \overline{\mathcal{P}}_A \parr \overline{\mathcal{P}}_B \supseteq \overline{\mathcal{P}_A^{'}} \parr \overline{\mathcal{P}_B^{'}} \,. \tag{5.33}$$

All these relations greatly reduce the number of computations needed to characterize a no-signaling sublattice. The only difference is their domains: the latter two take two expressions from the same space and form one still on the same space, i.e.[17]

17: Where the notation $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right)\right)$ is a shorthand for $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^A\right)\right)$; it indicates the space of superoperators on $\mathcal{H}^A$. That is, the space of operators defined over the Hilbert space $\mathcal{L}\left(\mathcal{H}^A\right) = \mathcal{L}\left(\mathcal{H}^A, \mathcal{H}^A\right)$.

$$\cap / \cup : \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \times \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \to \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \;, \tag{5.34}$$

whereas the former two take two expressions from different spaces and form one on the composite space, i.e.

$$\otimes \,/ \parr : \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right)\right) \times \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^B\right)\right) \to \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)\right) \;. \tag{5.35}$$

In linear logic terminology, the former ones are called additive connectors whereas the latter ones are called multiplicative connectors. The additive connectors are internal operations in the sense that they are kept within the same space whereas the multiplicative connectors are external in the sense that they 'glue' operations on two different spaces into an operation on the composite space. The lattice structure is defined with respect to the additive connectors as, while the multiplicative share many of their properties, they cannot satisfy certain lattice axioms like idempotency[18].

18: I.e., $\mathcal{P}_A \otimes \mathcal{P}_B$ is different than $\mathcal{P}_A$ even when $\mathcal{P}_A \cong \mathcal{P}_B$.

Still, this surprisingly regular behavior, while not defining a second lattice on top of the first, is a known structure. Just as the projector algebra is a Boolean lattice, which is a well-known model of classical logic, Boolean sublattice defined using $\{\overline{\cdot}, \cap, \cup, \otimes, \parr, \to\}$ is also close to a model of logic. Indeed, observe that the transformation between states is equivalent to the reverse transformation between effects,

$$\mathcal{P}_A \to \mathcal{P}_B = \overline{\mathcal{P}}_A \leftarrow \overline{\mathcal{P}}_B = \overline{\mathcal{P}_A \otimes \overline{\mathcal{P}}_B}\,. \tag{5.36}$$

This is a clear indication of the algebra under operations $\{\overline{\cdot}, \cap, \cup, \otimes, \parr, \to\}$ being an instance of logic: it follows the logic principle that if A implies B, then not B must imply not A. Interpreting the projectors as 'propositions' of a formal model of logic, the 'propositions' $\mathcal{P}$ consist of k 'sub-propositions' $\{\mathcal{P}_A, \dots \mathcal{P}_K\}$ that can be composed in two different manners: using $\otimes$ or $\parr$ and each (sub)proposition can be negated using $\overline{\cdot}$. To compare two propositions, the connectives $\cap$ and $\cup$ are used, and the comparison of two propositions is itself a valid proposition. The lattice is then the set of all propositions of a fixed length k. As such a model of logic, the projector algebra actually happens to form a degenerate[19] instance of *multiplicative additive linear logic* (MALL) [146] mainly because its connectives obey the De Morgan dualities:

19: See Appendix D.3.5 for the technicalities that prevent it from being an exact model of MALL.

[146]: Girard (1987), *Linear logic*.

$$\overline{\overline{\mathcal{P}}}_A = \mathcal{P}_A\,; \tag{5.37a}$$
$$\overline{\mathcal{P}_A \cap \mathcal{P}'_A} = \overline{\mathcal{P}}_A \cup \overline{\mathcal{P}'_A}\,; \tag{5.37b}$$
$$\overline{\mathcal{P}_A \otimes \mathcal{P}_B} = \overline{\mathcal{P}}_A \parr \overline{\mathcal{P}}_B\,. \tag{5.37c}$$

as well as the interchange laws (5.21). This extends an observation made in Reference [33] that the logic of higher-order quantum transformations form an instance of *multiplicative linear logic* (MLL)[20].

[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

20: While this is a connection between quantum physics and logic using projectors, it is different than one studied in the field of *quantum logic* [151]. In quantum logic, the propositions are projectors, and they form a non-distributive but orthomodular lattice. In the projective characterization, the propositions are superopertor projectors, and they form a distributive Boolean lattice.

[151]: Birkhoff et al. (1936), *The Logic of Quantum Mechanics*.

As is the case for the additive connectors, the rules obeyed by the no-signaling and two-way signaling composition are far from being arbitrary, and by exploiting their properties the characterization of signaling in higher-order theories can be made easier. An extra advantage of knowing that the logic of signaling is (a degenerate fragment of) Linear Logic is that this structure has been studied extensively in the context of automated proofs (see e.g., Reference [143]). For example, the package llprover is computer software that can be used to automate the search of equivalences in the fragment. What this suggests is that given a certain higher-order theory, it should be possible in principle to 1) write the projector associated with the states of the theory using the methods developed in this thesis and 2) use llprover or an automated proof program to find all equivalent projectors to the original projector. That way, any isomorphism can be found between the various signaling relations. A simple example is that in quantum theory the set of bipartite states is automatically no-signaling. This is computed by proving that $\mathcal{I}_A \parr \mathcal{I}_A = \mathcal{I}_A \otimes \mathcal{I}_B$. It could have been found automatically by 1) writing that a general bipartite quantum state is associated with $\mathcal{I}_A \parr \mathcal{I}_A$ and 2) letting an automated proof find that this formula is equivalent to $\mathcal{I}_A \otimes \mathcal{I}_B$. Of course, this example is trivial, but considering that the

[143]: Girard et al. (1989), *Proofs and Types*.

number of projectors in the lattice grows exponentially with the number of parties, automating the proofs may simplify the problem of finding such equivalences. This path was pioneered in References [33, 36], and a systematic treatment is left open for future research.

[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.
[36]: Simmons et al. (2022), *Higher-order causal theories are models of BV-logic*.

5.1.5. The Full Algebra of Projectors as the Signaling Lattice

With respect to the logic-like structure of the algebra, the addition of the tensor and parr promoted the Boolean algebra to a degenerate instance of Multiplicative Additive Linear Logic (MALL). With the further addition of the prec, the algebra can be seen as one of BV- [149, 152] or Pomset [147, 148] logic, as first noticed in Reference [36] (although their choice of additives connectives is different, see Subsection 5.1.4 for the comparison). Like with MALL, this fact alone already opens a path to automatic proofs for inferring the signaling structure in higher-order objects.

[149]: Guglielmi (2007), *A System of Interaction and Structure*.
[152]: Blute et al. (2012), *Deep Inference and Probabilistic Coherence Spaces*.
[147]: Rétoré (1993), *Réseaux et Séquents Ordonnés*.
[148]: Retoré (2021), *Pomset Logic*.

However automatic proofs give little information about the interpretation of the formulae, so this is not the direction followed here. Rather, this section presents the properties of the prec in order to conclude the investigation of the relations between the different connectors that appear in the projective characterization. With this, the sublattice spanned by negation, cap, cup, and the four compositions can be defined as a general notion, and conversely, a method of building such a lattice around any projector can also be defined. That way, information about the allowed signaling directions in a given state structure can be inferred from its neighbors in the lattice. As will be shown in the next section, the knowledge of how these lattices work can also be used to study signaling relations directly from reading the projector. In particular, a systematic way of decomposing a state space into causally ordered terms will be provided under the name normal form. Such a form is necessary to formalize the notions of non-fixed signaling structure, leading to an indefinite causal order, causal separability, and causal inequality for higher-order quantum processes.

Back to the properties of the prec. As is the case for the other compositions, the prec is a well-defined operation in the sense that it results in a valid projector in a manner that preserves commutation.

Lemma 5.1.8 *The rule $\prec$ results in a valid projector and preserves commutation.*

The proof is once again quite direct from definitions and relayed to Appendix D.3.6.

Like the tensor and the parr, it is an associative connector preserving partial order. However, like the transformation, the prec is not commutative.

Proposition 5.1.9 (Properties of the prec) *The prec is associative, order-*

preserving, and commutes with negation. That is, the prec verifies

$$(\mathcal{P}_A \prec \mathcal{P}_B) \prec \mathcal{P}_C = \mathcal{P}_A \prec (\mathcal{P}_B \prec \mathcal{P}_C) \ ; \tag{5.38a}$$

$$(\mathcal{P}_A \prec \mathcal{P}_B) \cap (\mathcal{P}_A \succ \mathcal{P}_B) = \mathcal{P}_A \otimes \mathcal{P}_B \ ; \tag{5.38b}$$

$$\left.\begin{array}{l} \mathcal{P}_A \subseteq \mathcal{P}_A' \\ \mathcal{P}_B \subseteq \mathcal{P}_B' \end{array}\right\} \Rightarrow (\mathcal{P}_A \prec \mathcal{P}_B) \subseteq \left(\mathcal{P}_A' \prec \mathcal{P}_B'\right) ; \tag{5.38c}$$

$$\overline{\mathcal{P}_A \prec \mathcal{P}_B} = \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B . \tag{5.38d}$$

Accordingly, the prec verifies the same four rules under the interchange $\prec \leftrightarrow \succ$, *albeit it is not commutative,* $\mathcal{P}_A \prec \mathcal{P}_B \neq \mathcal{P}_A \succ \mathcal{P}_B$.

The 'exotic' property of the prec compared to the other compositions is its commutation with the negation (5.38d). This feature will be important for defining a normal form of the algebra in the following.

In terms of distribution property with respect to the other connectives, the prec, the commutation with negation leads to another nice property compared to the other multiplicative connectors (compositions): it has an interchange law with both of the additives connectors (cap and cup; and consequently it is distributive over both),

$$\left(\mathcal{P}_A \cap \mathcal{P}_A'\right) \prec \left(\mathcal{P}_B \cap \mathcal{P}_B'\right) = \left(\mathcal{P}_A \prec \mathcal{P}_B\right) \cap \left(\mathcal{P}_A' \prec \mathcal{P}_B'\right) ; \tag{5.39a}$$

$$\left(\mathcal{P}_A \cup \mathcal{P}_A'\right) \prec \left(\mathcal{P}_B \cup \mathcal{P}_B'\right) = \left(\mathcal{P}_A \prec \mathcal{P}_B\right) \cup \left(\mathcal{P}_A' \prec \mathcal{P}_B'\right) . \tag{5.39b}$$

However, it is not distributive nor associative over the other multiplicative connectors. Nevertheless, there still exist some relations that can be inferred using the signaling interpretation of the last chapter (Definition 3.5.2 and Proposition 3.5.4): a state structure $\mathscr{A} \prec \mathscr{B}$ feature terms that can be signaling from A to B but none that can be signaling from B to A. From there, the following relation can be guessed:

$$\mathcal{P}_A \otimes \mathcal{P}_B = (\mathcal{P}_A \prec \mathcal{P}_B) \cap (\mathcal{P}_A \succ \mathcal{P}_B) \, , \tag{5.40a}$$

$$\mathcal{P}_A \parr \mathcal{P}_B = (\mathcal{P}_A \prec \mathcal{P}_B) \cup (\mathcal{P}_A \succ \mathcal{P}_B) . \tag{5.40b}$$

In plain words, the first line encodes the statement that the no-signaling composition is no-signaling from B to A (first term on the right-hand side) and (cap) from A to B (second term). Similarly, the second line encodes the statement that two-way signaling composition is no-signaling from B to A (first term on the r.h.s.) or (cup) from A to B (second term). These imply the following chains of inclusions,

$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \mathcal{P}_A \prec \mathcal{P}_B \subseteq \mathcal{P}_A \parr \mathcal{P}_B \, , \tag{5.41a}$$

$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \mathcal{P}_A \succ \mathcal{P}_B \subseteq \mathcal{P}_A \parr \mathcal{P}_B \, . \tag{5.41b}$$

These two partial order relations indicate that any projector whose expression features a prec has a lower bound obtained by replacing the prec by the tensor as well as an upper bound obtained by replacing the prec by the parr. As a consequence, while adding the prec connector to the set of operations $\{\overline{\cdot}, \cup, \cap, \otimes, \parr\}$ on projectors do increase the number of projectors in every sublattice closed under $\cap, \cup$ (like the no-signaling sublattice of Lemma 5.1.6), it does so in a manner that keeps lower and upper bounds intact. From the signaling interpretation this makes sense:

adding the prec is a fine-graining in which the intermediate situation of 'signaling in one direction but not the other' is slipped between the 'no-signaling' and 'two-way signaling' cases.

Any $\mathcal{P}$ can now be decomposed into base projectors $\mathcal{P}_A, \mathcal{P}_B, \ldots$ combined using any sequence of operations $\{\bar{\cdot}, \otimes, \prec, ⅋\}$ instead of $\{\bar{\cdot}, \otimes, ⅋\}$. The other way around, several commuting sets of projectors defined on different subsystems composed using the operations $\{\bar{\cdot}, \otimes, \prec, ⅋\}$ span a sublattice of the projector algebra on the full system. These are the lattices of projectors that the projective characterization of state structures defines.

Proposition 5.1.10 (The lattices of the projective characterization) *Let $\{\mathcal{P}_A^{(i)}\}_{i=1}^{n_A}$ be a set of n_A commuting projectors on operator systems in space $\mathcal{L}\left(\mathcal{H}^A\right)$. Let $\{\mathcal{P}_B^{(j)}\}_{j=1}^{n_B}$, $\{\mathcal{P}_C^{(k)}\}_{k=1}^{n_C}$, ... be a similarly defined sets in spaces $\mathcal{L}\left(\mathcal{H}^B\right), \mathcal{L}\left(\mathcal{H}^C\right), \ldots$. Then, the set of all expressions $\mathcal{P} : \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C \ldots\right) \to \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C \ldots\right)$ built using a combination of elements of these sets and the rules $\{\bar{\cdot}, \cup, \cap, \otimes, \prec, ⅋\}$ is a sublattice of the projector algebra on $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C \ldots\right)$.*

Proof. Collecting Propositions 5.1.1, 5.1.2, 5.1.5, and 5.1.9 as well as Lemmas 5.1.3 and 5.1.8, any $\mathcal{P}$ is a valid projector on operator system that preserve commutation.

The preservation of the commutation also relies on the fact that all caps and cups applied on subsystems can be turned into global caps and cups using the interchange laws for the tensor and the prec, Equations (5.21) and (5.39), as well as the De Morgan law (5.15) to interchange with the negation. (The interchange with the parr follows from the interchange with the tensor and De Morgan duality (5.37c).)

As cap and cup are valid operators in the construction of $\mathcal{P}$, the set is by definition a closed sublattice of the algebra of projectors. $\square$

When a global projector is given, one can construct the signaling lattice around this projector. As discussed between Lemma 5.1.6 and Corollary 5.1.7, this results in a lattice of projectors built from single base projectors –the ones appearing in the no-signaling subset of the projector– constructed under a restricted set of operations that 'preserve the number of negations over each base projectors'.

Proposition 5.1.11 (Building the signaling lattice) *Let $\mathcal{P}$ be a given element of a given lattice of projectors as in Proposition 5.1.10 and over a given space $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B \otimes \ldots \otimes \mathcal{H}^K\right)$ with k tensor factors. Let $\mathcal{P}_{NS} = \mathcal{P}_A \otimes \mathcal{P}_B \otimes \ldots \mathcal{P}_K$ be the projector on the no-signaling subspace of $\mathcal{P}$. Let $\{\mathcal{P}_A, \mathcal{P}_B, \ldots, \mathcal{P}_K\}$ be the set of k projectors on operator systems, each associated with a specific Hilbert space $\mathcal{H}^A, \mathcal{H}^B \ldots, \mathcal{H}^K$, that appear in $\mathcal{P}_{NS}$.*
*Then, $\mathcal{P}$ belongs to a sublattice, called **the signaling sublattice**, spanned by set $\{\mathcal{P}_A, \mathcal{P}_B, \ldots, \mathcal{P}_K\}$ under operations $\{\cap, \cup, \otimes, \prec, ⅋\}$.*

The proof, similar to the one of Lemma 5.1.6, is sketched in the appendix, Appendix D.5.2.

5.2. The Normal Form

With the prec and the signaling lattice, the signaling relations in state structures can now be analyzed. To do so systematically, a normal form will now be defined for projectors. As explained in the introduction, the non-associativity of the transformation connector is what makes the definition of order possible; in a tripartite formula like

$$\mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C) \neq (\overline{\mathcal{P}}_A \to \mathcal{P}_B) \to \mathcal{P}_C \,, \tag{5.42}$$

the left-hand side (l.h.s.) involves a transformation to a transformation, which is second order, whereas the right-hand side (r.h.s.) involves a nesting of two transformations, which is of first order. In such a case where the terms do not have the same order, deriving any conclusions regarding the two state spaces without any extra information is not feasible. Nonetheless, by articulating the formula purely through tensor and negation,

$$\overline{\mathcal{P}_A \otimes \mathcal{P}_B \otimes \overline{\mathcal{P}}_C} \neq \overline{\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} \otimes \overline{\mathcal{P}}_C} \,, \tag{5.43}$$

it can be inferred that these projectors share the same no-signaling subspace: $\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B \otimes \mathcal{P}_C$, so they must be comparable. The various refinements of the projector algebra make this comparison more and more clear. First, from what can be done using the type theory, the l.h.s. can be simplified using the uncurrying rule, $\mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C) = (\mathcal{P}_A \otimes \mathcal{P}_B) \to \mathcal{P}_C$, and using the parr, the formula becomes

$$(\mathcal{P}_A \otimes \mathcal{P}_B) \to \mathcal{P}_C \neq (\mathcal{P}_A \parr \mathcal{P}_B) \to \mathcal{P}_C \,. \tag{5.44}$$

Showing the inclusion of one into the other however cannot be done by some rewriting rules in the type theory and the early categorical treatment [10, 11, 33].

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.
[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

But using Equation (5.27), which is proven thanks to the cap and the cup, as well as De Morgan duality, it is possible to conclude that the r.h.s. is a subset of the l.h.s. by simple formulae manipulation once again:

$$\begin{aligned} &(\overline{\mathcal{P}}_A \parr \overline{\mathcal{P}}_B) \parr \mathcal{P}_C \supseteq (\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B) \parr \mathcal{P}_C \\ \iff &\overline{\mathcal{P}_A \otimes \mathcal{P}_B \otimes \overline{\mathcal{P}}_C} \supseteq \overline{(\mathcal{P}_A \parr \mathcal{P}_B) \otimes \overline{\mathcal{P}}_C} \\ \iff &\mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C) \supseteq (\overline{\mathcal{P}}_A \to \mathcal{P}_B) \to \mathcal{P}_C \,. \end{aligned} \tag{5.45}$$

One can then conclude that the inclusion is tight because of Equation (5.44). This is as far as the no-signaling sublattice goes. Adding the prec, i.e. looking at the expression with respect to the signaling lattice Proposition 5.1.11, provides a breakdown of the signaling relations between the parties. The relations (5.40) and (5.21) allow to write the expressions as:

$$\begin{gathered} (\overline{\mathcal{P}}_A \parr \overline{\mathcal{P}}_B) \parr \mathcal{P}_C \supseteq (\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B) \parr \mathcal{P}_C \iff \\ \big((\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B) \parr \mathcal{P}_C\big) \cup \big(\overline{\mathcal{P}}_A \succ \overline{\mathcal{P}}_B) \parr \mathcal{P}_C\big) \supseteq \big((\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B) \parr \mathcal{P}_C\big) \cap \big(\overline{\mathcal{P}}_A \succ \overline{\mathcal{P}}_B) \parr \mathcal{P}_C\big) \,. \end{gathered} \tag{5.46}$$

The two of them are pretty much the same object: the same two expressions are composed on both sides, with the notable difference that in the l.h.s., subsystem A can be used to signal (prec, $\prec$) to B OR (cup, $\cup$)

subsystem B can be used to signal to subsystem A (prec, $\succ$); whereas in the r.h.s, the signaling is from A to B AND (cap, $\cap$) from B to A. One concludes that there is two-way signaling in the former case, whereas there is no signaling in the latter. Since the rest of the expression is the same, the comparison is over.

Nevertheless, the analysis of the causal structure can be continued by developing these terms further. For example, in the r.h.s., relations (5.40) and (5.21) can be used again to turn the remaining parr into more unions[21],

21: The alphabetical ordering of the tensor factors is temporarily dropped for clarity.

$$\begin{aligned} &\mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C) = \\ &\qquad \big((\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B) \parr \mathcal{P}_C\big) \cup \big(\overline{\mathcal{P}}_A \succ \overline{\mathcal{P}}_B) \parr \mathcal{P}_C)\big) = \\ &\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B \prec \mathcal{P}_C \,\cup\, \mathcal{P}_C \prec \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B \,\cup\, \mathcal{P}_C \prec \overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A \,\cup\, \mathcal{P}_B \prec \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_C \,. \end{aligned} \tag{5.47}$$

This rewriting rule is due to the fact that the prec commutes with the negation, (5.38d), is associative, (5.38a), and at the same time, it can be interchanged with the additives connectors, (5.39). Compared to previous characterizations, it now becomes possible to analyze and compare state structures simply by algebraic manipulations of their projectors.

Each of the properties used for the rewriting above has an interpretation in terms of signaling: the prec commutes with negation because if a bipartite state is one-way signaling (say, characterized by $\mathcal{P}_A \prec \mathcal{P}_B$), corresponding to one half of the state (A) being prepared in the causal past of the other half (B), then a bipartite measurement on it ($\overline{\mathcal{P}_A \prec \mathcal{P}_B}$) must have at most the same causal structure ($\overline{\mathcal{P}_A \prec \mathcal{P}_B} = \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B$). Otherwise, if it was possible to deterministically signal from a measurement on the part in the causal future to the part in the causal past (i.e., by performing a measurement characterized by $\overline{\mathcal{P}_A \prec \mathcal{P}_B} \stackrel{?}{=} \overline{\mathcal{P}}_A \succ \overline{\mathcal{P}}_B$), it would result in a signaling loop. The state could be used to pass information from A to B, while at the same time, the measurement could be used to pass information from B to A. It suffices that parties A and B forward the information between the preparation and the measurement to achieve a loop. This is a remote consequence of the definition of measurement of state structure, Definition 3.2.4, imposing that the normalization of probabilities is always preserved. This condition shapes projector $\overline{\mathcal{P}}$ in order to avoid over-normalization of the trace, which in turn shapes the algebraic relations it must have with the multiplicative connectors so that this rule is respected. The logic rules followed by the negation (5.37) are but a translation of the signaling property that loops are forbidden. A more striking example is that the two-way signaling composition is dual to the no-signaling composition: this is so that closing a circuit (measuring a state) never results in a signaling loop.

Similarly, the logic rule that the prec is associative is also a consequence of a signaling property: transitivity. In an expression like $\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C$, the projector implicitly allows for signaling from A to C since[22]

22: This is proven by using the definition (5.3) then computing the intersection.

$$\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C \supseteq (\mathcal{P}_A \prec \mathcal{P}_C) \otimes \mathcal{P}_B \,, \tag{5.48}$$

meaning that prec chains contain terms that allow signaling to other parties down the chain irrespective of whether the intermediate term has received signaling. The r.h.s. indeed allow A to signal to C while B is no-signaling to both. This indicates that the signaling relations in a prec chain are transitive: A can signal not only to the next party in the chain (in that case, B), but to every following party in the chain (in that case C). $\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C$ is thus read as 'A can signal to B and to C, and B can signal to C'. Associativity,

$$(\mathcal{P}_A \prec \mathcal{P}_B) \prec \mathcal{P}_C = \mathcal{P}_A \prec (\mathcal{P}_B \prec \mathcal{P}_C)\,, \tag{5.49}$$

is read as 'A can signal to B, and the two of them can signal to C (l.h.s.) is equivalent to (=) A can signal to B and C, and B can signal to C (r.h.s.)'. Naturally, these two statements are equivalent to the one associated with $\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C$. The interpretation of formulae as logical statements about signaling will be discussed further in the next chapter Subsection 6.3.1.

Therefore, the amazing feature of writing the projector as 'unions of prec chains' is that the signaling structure of the state space can be directly read from its expression. Take decomposition (5.47) in the above example: it characterizes objects taking two 'inputs' (negations, $\overline{\cdot}$) from parties A and B, providing one 'output' to party C; this object allows for signaling from A to B and C and from B to C ($\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B \prec \mathcal{P}_C$) or ($\cup$) from C to B to A ($\mathcal{P}_C \prec \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B$) or ($\cup$) ... etc.

As is now shown, the above 'decomposition into prec chains' procedure can be conducted for any projectors as the tensor and the transformation split into precs, (5.40) and since the algebra obeys the De Morgan relations (5.37). Notice that any expression $\mathcal{P}$ involving $\otimes$ and $\rightarrow$ can be reduced to the union and intersection of expressions built using the prec alone. In the same way, all properties of the tensor and the parr (and the transformation) are derived from properties of the prec. For example, the interchange law (5.21) can be inferred from the interchange law of the prec, Equation (5.39). There is, moreover, a convenient aspect of expressing all multiplicative connectors in terms of prec: the expression would become associative 'prec chains', so the notion of order whatsoever is dropped in favor of union and intersections. Thus, on purely semantic grounds, any multiplicative connector other than the prec seems like 'syntactical sugar', whose only purpose is to shorten the expressions. Yet, these induce unnecessary complications. It is therefore reasonable to impose an unambiguous form to formulae, one using the one-way signaling composition and additives connectors only. Such a form will be called *normal*.

To obtain this normal form, remark that the cap and the cup distribute over each other in any Boolean algebra, Equation (5.18). They further distribute over the prec in the case of the projector algebra, Equation (5.39). Consequently, the additives connectors can always be put outside of expressions involving these three connectors so to make prec chains appear. Nevertheless, the cap and the cup can be interchanged, so there is still a choice of which normal form to choose in the Boolean algebra: either all formulae should be expressed as unions of intersections of prec chains; this is called a *disjunctive normal form*. Or they should be expressed as intersections of unions of prec chains; this is called a *conjunctive normal form*.

But splitting the projector expressions into prec chain was actually the goal of the algebra to begin with since it allows the interpretation of each chain as a term allowing a fixed signaling direction. From there, the signaling interpretation favors the disjunctive normal form. Indeed, the union can lead to more than one signaling direction at once, whereas the intersections reduce the length of the signaling chains. Formally[23],

$$(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_C \prec \mathcal{P}_A \prec \mathcal{P}_B) = (\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C \;; \tag{5.50a}$$
$$(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cup (\mathcal{P}_C \prec \mathcal{P}_A \prec \mathcal{P}_B) = (\mathcal{P}_A \prec \mathcal{P}_B) \parr \mathcal{P}_C \,. \tag{5.50b}$$

23: This is straightforward to prove: $(\mathcal{P}_A \prec \mathcal{P}_B)$ is a valid projector, so it obeys relations (5.40) with $\mathcal{P}_C$.

The first equation shows that the cap results in a shorter prec chain: the longest signaling chain that can be achieved is between two parties, compared to three for the terms on each side of the cap. The second equation, however, still has its longest signaling chain being three parties, but now it has two of them (either Charlie is the first or he is the last). Moreover, it is impossible that the intersection of two prec chains of length n results in the union of two prec chains of length $k \leq n$. Otherwise, this would contradict the interchange law, as the union can always be pulled outside of the chain[24]. As a consequence, once a term has a fixed signaling direction, the only way to have more is through the union with another term with a different signaling direction. Hence, by phrasing expressions as the union of intersections, it is direct to see when a projector defines a state structure with more than one signaling direction: its normal form features more than one prec chain.

24: Nonetheless, whilst the intersection cannot augment the number of signaling directions when merging two prec chains, it still has non-trivial rules with respect to signaling. For example, Equation (5.40a) will not apply in the tripartite case: the intersection of a prec chain with the reversed direction term will not yield the no-signaling projector, i.e. $(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_C \prec \mathcal{P}_B \prec \mathcal{P}_A) \neq (\mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C)$. This will be shown in Subsection 6.3.1

Definition 5.2.1 (Normal form) *Let $\mathcal{P}$ be a projector on operator system built from a set of k base projectors $\{\mathcal{P}_A, \mathcal{P}_B, \ldots, \mathcal{P}_K\}$ composed together under operations $\{\overline{\cdot}, \cap, \cup, \otimes, \prec, \rightarrow\}$ so that $\mathcal{P} \in \mathcal{L}\left(\mathcal{H}^A \otimes \ldots \otimes \mathcal{H}^K\right)$. Then, a* ***normal form*** *Γ of $\mathcal{P}$ is a projector equivalent to $\mathcal{P}$ obtained as unions of intersections of expressions built from operations $\{\overline{\cdot}, \prec\}$ alone. This means that a normal form has the following form:*

$$\Gamma := \bigcup_{i=1}^{x} \left(\bigcap_{j=1}^{y_i} \widetilde{\mathcal{P}}_{\sigma_{ij}(A)} \prec \widetilde{\mathcal{P}}_{\sigma_{ij}(B)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\sigma_{ij}(K)} \right) , \tag{5.51}$$

in which there are x unions of expressions labeled by index i, and each expression involves y_i intersections of sub-expressions labelled by index j. σ_{ij} is an element of the permutation group on k element, like e.g. $\sigma_{00}(A) = A, \sigma_{01}(A) = B, \ldots$, so that each $\widetilde{\mathcal{P}}_{\sigma_{ij}(A)}$ is a choice of a base projector which is potentially negated depending on indices i and j. Each sub-expression is thus a permutation of $\widetilde{\mathcal{P}}_A \prec \widetilde{\mathcal{P}}_B \prec \ldots \prec \widetilde{\mathcal{P}}_K$ where the position of the negations depends on i and j. (Note that the indices do not necessarily run over the full permutation group).

Theorem 5.2.1 *Any projector $\mathcal{P}$ as in Definition 5.2.1 has a normal form.*

Proof. By Proposition 5.1.10, it is a valid projector on operator system. By Equations (5.40), the projector can be put in a form involving operations $\{\overline{\cdot}, \cap, \cup, \prec\}$ only. The normal form can always be reached by first distributing negations over intersections and unions using De Morgan law (5.15) and over the prec using Equation (5.38d), then by distributing the prec over unions and intersections using (5.39), and finally by distributing the caps over the cups using (5.18). This procedure is explicitly proven in Appendix D.5.3. ☐

Definition 5.2.2 (Fixed Signaling Direction) *A state structure whose associated projector can be reduced in a normal form that only features intersections is said to have a* ***fixed signaling direction***[25].

25: Or *fixed causal order* colloquially.

This definition provides the most strict sufficient criterion for causal correlations as in Definition 1.2.3: a dual pair such that both state structures have a one-way signaling structure will always result in one-way signaling correlations between the same parties. Remark that the definition specifies that the normal form *can be reduced*, not *is*. This is due to the non-uniqueness of normal forms.

Indeed, because the normal form encodes the possible signaling directions, it is not unique for more than two parties. As mentioned above, the prec is associative because the possibility to signal is a transitive property: in an expression like $\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C$, A can signal to C so $\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C \cup (\mathcal{P}_A \prec \mathcal{P}_C) \otimes \mathcal{P}_B$ is a redundant formula, which induces the redundant normal form $\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C \cup ((\mathcal{P}_A \prec \mathcal{P}_C \prec \mathcal{P}_B) \cap (\mathcal{P}_A \prec \mathcal{P}_C \prec \mathcal{P}_B))$, which is equivalent to $\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C$. This implies that some formulae are redundant. For example, the following can be proven using the algebra[26]:

26: Remark that this is an example of a non-unique conjunctive normal form.

$$(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_C \prec \mathcal{P}_A \prec \mathcal{P}_B) = (\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_C \prec \mathcal{P}_A \prec \mathcal{P}_B) \cap (\mathcal{P}_A \prec \mathcal{P}_C \prec \mathcal{P}_B) \ . \quad (5.52)$$

This formula has an intuitive interpretation: if A cannot signal to B and C and vice-versa, this also includes the situation in which A receives signaling from B and signals to C. Therefore, the third term on the right-hand side of the above is redundant, and these two normal forms are equivalent. On top of that, the interchange law also presents a redundancy since the cap and the cup are commutative; see the discussion around Equation (D.96). Worse, the properties of the algebra make it so that sometimes exceptional isomorphism between formulae happens. Some of them are specific to quantum theory[27] and they will be discussed in the concluding example of this chapter. And some of them are due to the internal logic of the theory, as will be discussed in the next chapter, Subsection 6.3.1. Changing the definition to incorporate this redundancy to make the normal form unique is left open as a future research direction. Another concrete example of the use of the normal form, as well as of an instance of equivalent normal forms, will be used as the introductory example of the next chapter, Section 6.1.

27: All stem from the already mentioned fact that bipartite quantum states are automatically no-signaling; in categorical terms that the category of quantum processes is a compact closed subcategory of the category of higher-order processes, which is more generally $*-$autonomous [33, 137].

[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.
[137]: Abramsky et al. (2004), *A Categorical Semantics of Quantum Protocols*.

Contrariwise to this decomposition-driven approach, higher-order state spaces can also be devised by constructing their normal form through signaling requirements. Indeed, since expressions built using only the prec like $\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B \prec \mathcal{P}_C$ define the support of a theory with a fixed signaling order[28]. First, several prec chains are combined through the cap as a means to require no-signaling between some subsystems. Second, several such intersections are combined through the cup as a means to require two-way signaling. For example, to further impose no signaling between A and B in $\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B \prec \mathcal{P}_C$, one intersects it with $\left(\overline{\mathcal{P}}_A \succ \overline{\mathcal{P}}_B\right) \prec \mathcal{P}_C$ so that $\left(\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B \prec \mathcal{P}_C\right) \cap \left(\overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A \prec \mathcal{P}_C\right) = \left(\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B\right) \prec \mathcal{P}_C$. Then, to loosen the requirement that C can signal to neither A nor B, one takes the union of it with $\left(\mathcal{P}_C \prec \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B\right) \cap \left(\mathcal{P}_C \prec \overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A\right) =$

28: More precisely: certain operators M_{ABC} in $\overline{\mathscr{A}} \prec \overline{\mathscr{B}} \prec \mathscr{C}$ may allow A to signal *deterministically* to B and to C by suitably choosing the (*local deteministic*) state $V_A \in \mathscr{A}$ she will input in M_{ABC}. And the same way, B may signal to C by suitably choosing his state $V_B \in \mathscr{B}$, but he will never be able to signal to A, no matter his choice of V_B.

$(\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B) \succ \mathcal{P}_C$. The obtained projector has, by construction, the following normal form,

$$\begin{aligned}
&\big((\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B \prec \mathcal{P}_C) \cap (\overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A \prec \mathcal{P}_C)\big) \cup \big((\mathcal{P}_C \prec \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B) \cap (\mathcal{P}_C \prec \overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A)\big) \\
&\quad = \big((\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B) \parr \mathcal{P}_C) \cap (\overline{\mathcal{P}}_A \succ \overline{\mathcal{P}}_B) \parr \mathcal{P}_C)\big) \\
&\quad = (\overline{\mathcal{P}}_A \to \mathcal{P}_B) \to \mathcal{P}_C \,,
\end{aligned} \tag{5.53}$$

and the shortened versions written on the right side of the equality signs can be obtained using the rewrite rules of the algebra[29]. Surprise, this is the right-hand side term of the introductory example of this section, (5.42). From its normal form, it can be said that the set of objects transforming a transformation from an effect of A to a state of B into a state of C, i.e. $(\overline{\mathcal{P}}_A \to \mathcal{P}_B) \to \mathcal{P}_C$, is equivalently defined as an object which takes as inputs states from A and B and output a state of C. Its normal form shows furthermore that these tripartite objects are such that A and B cannot use it to signal to each other, but both can signal back and forth to the party at the output, C.

29: The first equality can more simply be obtained by directly interpreting the sentences telling how the normal form was constructed.

Compare it to the left-hand side of Equation (5.42), the set of objects transforming a state of A into a transformation from a state of B to one of C i.e. $\mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C)$. It was shown around Equation (5.45) that they are the same kind of objects: they take states from A and B and output one in C. Thus, they live in the same signaling lattice. Its normal form, derived at Equation (5.47), shows that the three parties can use these tripartite objects to achieve any direction of signaling. This is more directions than those in $(\overline{\mathcal{P}}_A \to \mathcal{P}_B) \to \mathcal{P}_C$ henceforth it is its superset. Actually, since it can also be put in a form involving the parr only, $\mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C) = \overline{\mathcal{P}}_A \parr \overline{\mathcal{P}}_B \parr \mathcal{P}_A$, its projector is the greatest element in their common signaling lattice.

With the normal form, any higher-order objects become comparable by using the signaling lattices of their state structures. Moreover, a specific object within a state structure can be shown to have less signaling directions than the maximum allowed by its state structure by demonstrating inclusion in a smaller element of the lattice. For instance, it can be proven that a bipartite channel is one-way signaling simply by applying the corresponding projector. The lattice structure and the normal form conclude the study of the projector algebra; they provide a set of tools to systematically infer the signaling relations in higher-order quantum theories as well as any theory built using state structures.

5.3. Example: Accidental Isomorphism for Quantum Theory

As a concluding example for this chapter, the theory of quantum networks [9, 13] is characterized using the projective methods. This example actually relies on a particularity of quantum theory already mentioned several times: that its no-signaling bipartite states are the same as its bipartite states. This is one of the accidental isomorphisms in the projector algebra, and, as will be shown, is the reason why quantum networks are causally ordered while at the same time interpretable as higher-order transformations. This is peculiar since higher-order transformations are

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks.*
[13]: Chiribella et al. (2008), *Quantum Circuit Architecture.*

built using the $\rightarrow$ connector which is related to the $\parr$ one and therefore should typically involve two-way signaling according to (5.40).

Avoiding the accidental isomorphism is the reason why the concluding example of the previous chapter was built on the exotic biased quantum theory. The presence of the isomorphism in quantum theory would have indeed tampered with the narrative of compositions conditioned by signaling requirement, as certain compositions would have been equivalent by accident. Note that an explanation of the latter result already exists within the framework of type theory [11], but the projector algebra recovers it in a more general fashion and it provides a nice illustration of the utilization of the normal form. The following example can also be interpreted as the comparison of two different ways of building higher-order objects using the developed formalism of projectors.

[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

Following reference [9], a **network** is defined as the causally ordered (i.e. one-way signaling) succession of 'nodes' of the same state structure. A '1-network of base $\mathscr{A}$' will be the set $\mathscr{A} \subset \mathcal{L}\left(\mathcal{H}^{A}\right)$ itself, thereafter noted with an index as $\mathscr{A}_0$ to distinguish between the multiple copies of the state structure $\mathscr{A}$ defined on an increasingly larger number of factors. The '2-network of base $\mathscr{A}$' will be the set $\mathscr{A}_0 \prec \mathscr{A}_1 \subset \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$, the '3-network' will be $\mathscr{A}_0 \prec \mathscr{A}_1 \prec \mathscr{A}_2$, etc. up to the '$n$-network' defined as $\mathscr{A}_0 \prec \mathscr{A}_1 \prec \ldots \prec \mathscr{A}_{n-1} \subset \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1} \otimes \ldots \mathcal{H}^{A_{n-1}}\right)$. A common occurrence of this structure in the literature is the network whose base is a quantum channel so that $\mathscr{A}$ is characterized by a projector $\mathcal{P}_{\mathscr{A}} = \mathcal{I}_{A_0} \rightarrow \mathcal{I}_{A_1}$; it is called a **quantum network** (notice that it will require twice as many systems since the base is defined as a state structure on two systems). This quantum network, here associated with some party that will be called Alice, represents the successive operations of that party. If Alice has a single node quantum network, it means that Alice acts once on subsystem A_0 with a quantum channel and outputs a quantum state in A_1. If she has a network with two nodes, she will act a first time on A_0 and output a first state at A_1, then a second time on A_2, now potentially using any size of ancillary qudit as a memory register she preserved from her first operation, and output a second state in A_3. And so on for all numbers of nodes, as defined recursively.

Another way of building a higher-order state structure is the **comb**, which consists of recursively transforming into a base type: the 1-comb of base $\mathscr{A}$ is again $\mathscr{A}_0$, then the $2-$comb is $\mathscr{A}_0 \rightarrow \mathscr{A}_1$, the $3-$comb is $(\mathscr{A}_0 \rightarrow \mathscr{A}_1) \rightarrow \mathscr{A}_2$, etc up to the $n-$comb defined as $(...(\mathscr{A}_0 \rightarrow \mathscr{A}_1) \rightarrow ...) \rightarrow \mathscr{A}_n \subset \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1} \otimes \ldots \otimes \mathcal{H}^{A_{n-1}}\right)$. As is the case for the network case, a common occurrence of this structure is the comb whose base is a quantum channel, called the **quantum comb**.

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.

When the two constructions were introduced in reference [9], it was proven that a quantum network *is* a quantum comb. When treated using the formalism developed here, there is a stark contradiction. All quantum combs are built using the transformation, $\rightarrow$, which permits two-way signaling. How could it be that they are all equivalent to networks which are built using the prec – that is, objects featuring a single direction of signaling? Besides, why is the 1-comb (built using the two-way signaling transformation) equivalent to the quantum channel (which is causal)? Why does the quantum 2-comb reduce to a two-node quantum network, i.e. a map acting on two quantum states, when by definition it should be a supermap, i.e. a map acting on a quantum channel? And why does it reduce to a succession of two operations that have a well-defined direction of signaling between parties, and not, as its projector suggests, a nesting of two two-way signaling compositions, resulting in four possible signaling directions?

To phrase this issue formally, let $\mathscr{A}_i^{state} \subset \mathcal{L}\left(\mathcal{H}^{A_i}\right)$ be the state structure of states on system A_i so that its projector is $\mathcal{I}_{A_i}$, and let $\mathscr{A}_{i,j}^{channel} \subset \mathcal{L}\left(\mathcal{H}^{A_i} \otimes \mathcal{H}^{A_j}\right)$ be the state structure of channels between systems A_i and A_j so that its projector is $\mathcal{I}_{A_i} \to \mathcal{I}_{A_j}$.
The state structure of quantum combs with n nodes reads $(\ldots(\mathscr{A}_{n-1,n}^{channel} \to \mathscr{A}_{n-2,n+1}^{channel})\ldots) \to \mathscr{A}_{0,2n-1}^{channel}$ which, in the language of [10, 11], carries the type $(\ldots((A_{n-1} \to A_n) \to (A_{n-2} \to A_{n+1})) \to \ldots) \to (A_0 \to A_{2n-1})$ where each A are the trivial type, and in terms of projector it is associated with:

$$\begin{aligned}\mathcal{P}_{\mathscr{A}_{\text{channel}}}^{\text{(n-comb)}} &:= (\ldots \\ ((\ldots((\mathcal{I}_{A_{n-1}} \to \mathcal{I}_{A_n}) &\to (\mathcal{I}_{A_{n-2}} \to \mathcal{I}_{A_{n+1}})) \to \ldots) \to (\mathcal{I}_{A_{n-1-j}} \to \mathcal{I}_{A_{n+j}})) \\ &\to \ldots) \to \left(\mathcal{I}_{A_0} \to \mathcal{I}_{A_{2n-1}}\right) .\end{aligned} \tag{5.54}$$

Note that the labeling has been chosen so as to coincide with the other state structures it will be compared to. The state structure of quantum networks with n nodes reads $\mathscr{A}_{0,1}^{channel} \prec \ldots \prec \ldots \prec \mathscr{A}_{2n-2,2n-1}^{channel}$; it carries the type $(A_0 \to A_1) \prec \ldots \prec (A_{2n-2} \to A_{2n-1})$; it is associated with the projector

$$\mathcal{P}_{\mathscr{A}_{\text{channel}}}^{\text{(n-network)}} := (\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \prec \ldots \prec \left(\mathcal{I}_{A_j} \to \mathcal{I}_{A_{j+1}}\right) \prec \ldots \prec \left(\mathcal{I}_{A_{2n-2}} \to \mathcal{I}_{A_{2n-1}}\right) . \tag{5.55}$$

The state structure of a comb based on states with $2n$ nodes reads $(\ldots((\mathscr{A}_0^{state} \to \mathscr{A}_1^{state}) \to \mathscr{A}_2^{state})\ldots) \to \mathscr{A}_{2n-1}^{state}$; it carries the type $(\ldots((A_0 \to A_1) \to A_2) \to \ldots) \to A_{2n-1}$; it is associated with the projector

$$\mathcal{P}_{\mathscr{A}_{\text{state}}}^{\text{(2n-comb)}} := (\ldots((\ldots((\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}) \to \mathcal{I}_{A_2}) \to \ldots) \to \mathcal{I}_{A_j}) \to \ldots) \to \mathcal{I}_{A_{2n-1}} . \tag{5.56}$$

It is known that 1) quantum channels are no signaling from output to input, and are equivalent to quantum 1-network and quantum 1-combs [13]; 2) that the first two state structures defined above are equivalent [9, Theorem 8]; 3) that the elements M of these state structures can be characterized in the CJ picture by the causality condition [9, Theorem 5] (see Equation (2.20)):

[13]: Chiribella et al. (2008), *Quantum Circuit Architecture*.
[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.
[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.

$$\begin{aligned}&\forall j \in 0, \ldots n, \text{ let } M^{(n)} := M,\ M^{(j)} := \frac{\mathrm{Tr}_{A_{2j}\ldots A_{2n-1}}[M]}{d_{A_{2j}} d_{A_{2j+2}} \ldots d_{A_{2n-2}}}, \\ &\text{then } \forall i \in 1, \ldots n, \quad \mathrm{Tr}_{A_{2i-1}}\left[M^{(i)}\right] = M^{(i-1)} \otimes \mathbb{1}_{A_{2i-2}} .\end{aligned} \tag{5.57}$$

4) that the last two state structures defined above are equivalent [11, Proposition 6].

[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

These results are now recovered and explained using the projective characterization. These four statements can indeed be proven simply by algebraic manipulations of the projectors. They amount to proving that

1) $\mathcal{I}_{A_0} \to \mathcal{I}_{A_1} = \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1}$;
2) Equations (5.54) and (5.55) are equivalent, i.e. $\mathcal{P}^{\text{(n-comb)}}_{\mathscr{A}_{\text{channel}}} = \mathcal{P}^{\text{(n-network)}}_{\mathscr{A}_{\text{channel}}}$;
3) Equation (5.57) is an equivalent way of defining the validity subspace of quantum networks, Eq. (5.55);
4) Equations (5.54) and (5.56) are equivalent, i.e. $\mathcal{P}^{\text{(n-comb)}}_{\mathscr{A}_{\text{channel}}} = \mathcal{P}^{\text{(2n-comb)}}_{\mathscr{A}_{\text{state}}}$.

5.3.1. The Notion of Causality in Quantum Process Theory

Every quantum channel $\mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right)$, when in CJ form, obeys Equation (5.57):

$$\mathrm{Tr}_B\left[M\right] = \mathbb{1}_A \,. \tag{5.58}$$

This is equivalent to the dual channel $\mathcal{M}^*$ being unital, or the channel being trace-preserving, and it is necessary for $\mathcal{M}$ to represent a valid quantum channel [52, 72, 90].

[52]: Nielsen et al. (2009), *Quantum Computation and Quantum Information*.
[72]: Kraus (1983), *States, Effects, and Operations: Fundamental Notions of Quantum Theory*.
[90]: Watrous (2018), *The Theory of Quantum Information*.

While it is necessary for the channel to be valid, it also makes the channel *causal* [78, 81, 130]. Being causal in this context is the condition described in Subsection 3.5.1 under the name no-signaling but applied from the output to the input. As explained in that section, no-signaling is first a theory-independent constraint on the probability distribution of the settings and outcomes of the parties at both sides of the channel. It imposes that the choice of setting (y) at the output (party B) cannot have probabilistic correlations with the outcome (a) at the input (party A). At the level of the Born rule given by

[78]: D'Ariano et al. (2017), *Quantum Theory from First Principles: An Informational Approach*.
[81]: Coecke et al. (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*.
[130]: Kissinger et al. (2017), *Equivalence of relativistic causal structure and process terminality*.

$$p(a,b|x,y) = \mathrm{Tr}\left[M_{AB} \cdot \left(V^A_{a|x} \otimes N^B_{b|y}\right)\right] \tag{5.59}$$

this bound imposes that there are no choices of deterministic operation $\sum_b N_{b|y} = N_{|y}$ at the output that can have an influence on the probabilistic operation $V_{a|x}$ at the input. Following the discussion in that chapter, this requirement was shown to imply Equation (3.96),

$$\mathrm{Tr}_B\left[M \cdot \left(\mathbb{1}_A \otimes N_{|y}\right)\right] = \mathrm{Tr}_B\left[M \cdot \left(\mathbb{1}_A \otimes c_{\overline{B}} \frac{\mathbb{1}_B}{d_B}\right)\right] \,. \tag{5.60}$$

This means that no choice of operation on Alice's side can distinguish the deterministic choice made by Bob from Bob maximally randomizing over his settings.

In the general framework of state structures, condition (5.58) follows from the equivalent conditions (3.57). Which is that a transformation $M \in \mathscr{A} \to \mathscr{B}$ is valid if it transforms a state of the input state structure $\mathscr{A}$ into one of the output $\mathscr{B}$, or equivalently if it transforms an effect of $\overline{\mathscr{B}}$ into an effect of $\overline{\mathscr{A}}$:

$$\forall N_B \in \overline{\mathscr{B}}, \quad N_B * M_{AB} \in \overline{\mathscr{A}} \,. \tag{3.57a}$$

This equivalence is due to Equation (5.36), i.e. that a map $M \in \mathscr{A} \to \mathscr{B}$ can be equivalently interpreted as one in $\overline{\mathscr{A}} \leftarrow \overline{\mathscr{B}}$; It recovers Equation (5.58) in the case of quantum theory: $\overline{\mathscr{A}}_{quant.} = \overline{\mathscr{B}}_{quant.} = \{\mathbb{1}\}$ so that $N_B = \mathbb{1}_B$ and

$$\mathbb{1}_B * M_{AB} \in \{\mathbb{1}_A\} \Rightarrow \mathrm{Tr}_B\left[M_{AB} \cdot \left(\mathbb{1}_A \otimes \mathbb{1}_B^T\right)\right] \in \{\mathbb{1}_A\} \Rightarrow \mathrm{Tr}_B\left[M_{AB}\right] = \mathbb{1}_A \,. \tag{5.61}$$

For a bipartite state structure to represent a valid channel (transformation) or a causal channel (no-signaling from output to input) are two different conditions: Equation (3.57a) is different than Equation (5.60). However, in the case of quantum channels, to be a valid channel, which implies verifying Equation (5.58), is sufficient to verify (5.60) as well because $\overline{\mathscr{B}} = \overline{\mathscr{B}}_{quant.} = \{\mathbb{1}\}$ so that (5.60) becomes tautological in that case. As it happens, the only choice of $N_{|y} \in \overline{\mathscr{B}}_{quant.}$ that can be made is $\mathbb{1}$, irrespective of y, so the condition becomes trivially true:

$$\mathrm{Tr}_B\left[M \cdot (\mathbb{1}_A \otimes \mathbb{1}_B)\right] = \mathrm{Tr}_B\left[M \cdot \left(\mathbb{1}_A \otimes \frac{d_B}{d_B}\mathbb{1}_B\right)\right] . \tag{5.62}$$

For this reason, in the literature of higher-order quantum processes, condition Equation (5.58) is taken as the causality condition, although it is the channel condition. This equivalence between a general transformation and a no-signaling one is only valid for the case of density matrices nonetheless (as proven explicitly in Lemma 5.3.1 below) and hides the subtle difference between the two definitions in the general case: being a valid transformation between state structures does not guarantee that the output cannot be used to signal to the input.

In the following, it will be shown that this overlap of definitions carries on at higher-order generalization of quantum channels, meaning that this tautology also applies to all of quantum comb formalism, and it is exactly the accidental isomorphism this section is about.

5.3.2. Transformations between Quantum States

First, in the case $n = 1$, equations (5.54), (5.55) and (5.56) all reduce to $\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}$, so items 2) and 4) hold. What remains to be proven are items 1) and 3) in the single-node case. The following lemma underlies the proofs.

Lemma 5.3.1 *The transformation operation on projectors, $\to$, simplifies into a prec operation when either of the projectors is identity. The prec operation on projectors, $\prec$, simplifies into a tensor operation when either of the projectors is depolarising. In equation,*

$$\mathcal{P}_A \to \mathcal{P}_B = \overline{\mathcal{P}}_A \prec \mathcal{P}_B \iff \mathcal{P}_A = \mathcal{I}_A \text{ or } \mathcal{P}_B = \mathcal{I}_B \,; \tag{5.63a}$$
$$\overline{\mathcal{P}}_A \prec \mathcal{P}_B = \mathcal{P}_A \otimes \mathcal{P}_B \iff \mathcal{P}_A = \mathcal{D}_A \text{ or } \mathcal{P}_B = \mathcal{D}_B \,. \tag{5.63b}$$

These relations are proven in detail in a dedicated section in Appendix D.4. There, these relations are depicted in terms of their support in Figure D.5 for the case where both A and B are quantum states. This should be compared with the general case of Figure D.4.

The first equation of this lemma implies that transformations between quantum states, characterized by $\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}$, are equivalent to a causal succession of a functional and a state, $\mathcal{D}_{A_0} \prec \mathcal{I}_{A_1}$. This proves statement 1), $\mathcal{I}_{A_0} \to \mathcal{I}_{A_1} = \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1}$. Remark these actually are two different ways of defining the quantum channel: as the most general transformation that takes a quantum state to a quantum state, or as a measurement coherently[30] followed by a preparation of a quantum state (this is also mentioned in the example of Subsection 4.1.3).

30: By coherent, it is meant 'as any element of the convex hull of measure-and-reprepare maps', so this set does not represent only entanglement breaking maps, i.e. of the form $\mathcal{M}(\bullet) = \rho\mathrm{Tr}\left[\sigma \cdot \bullet\right]$, but any linear map $\mathcal{M}(\bullet) = \sum_i \rho_i \mathrm{Tr}\left[\sigma_i \cdot \bullet\right]$). Recall that in this non-CJ representation, the CPTP condition comes as constraints on the ρ_i's and σ_i's.

More generally, the first equation (5.63a) states that any transformation from (or to) an arbitrary state structure to (or from) the state structure of quantum states will automatically be no signaling from output to input. The second equation states that any way of combining a state structure with the single element 'quantum measurement' state structure, i.e. $\{\mathbb{1}\}$, is automatically no signaling from and to it.

In contrast, Equation (5.63b) is for example the reason why the single partite process matrix reduces to an effect and a state in tensor product [5]: the 1-PM is a functional on channels, characterized by the negated projector $\overline{\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}}$. Using the first part of the lemma, $\overline{\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}} = \overline{\mathcal{D}_{A_0} \prec \mathcal{I}_{A_1}}$. The negation is distributed over a prec, $\overline{\mathcal{D}_{A_0} \prec \mathcal{I}_{A_1}} = \mathcal{I}_{A_0} \prec \mathcal{D}_{A_1}$, and finally the second part of the lemma is used, $\mathcal{I}_{A_0} \prec \mathcal{D}_{A_1} = \mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1}$. These algebraic manipulations on projectors quickly led to the conclusion that the functionals on quantum channels, characterized by $\overline{\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}}$, are equivalent to a quantum state on the input system followed by a measurement applied on its output, $\mathcal{I}_{A_0} \prec \mathcal{D}_{A_1}$, and that the state and measurement are causally disconnected, $\mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1}$.

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

Equation (5.63a) is the case explained in detail in Subsection 5.3.1 above, where the state structure definition of no-signaling, Equation (3.97), becomes redundant with the definition of transformation between state structures, Equation (5.58). The fact that these two equations coincide in this case is exactly statement 3) in the $n = 1$ case: $\forall M \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right),\ M \geq 0,\ \mathrm{Tr}\,[M] = c_A \ :\ \mathrm{Tr}_{A_1}[M] = c_A \frac{\mathbb{1}}{d_{A_0}} \iff (\mathcal{I}_{A_0} \to \mathcal{I}_{A_1})\{M\} = M$.

5.3.3. Equivalence of Quantum Combs and Networks

This result is now generalized for all n. Reinterpreting $\mathcal{I}_{A_0} \to \mathcal{I}_{A_1} = \mathcal{D}_{A_0} \prec \mathcal{I}_{A_1}$, the quantum channel can be seen as a sort of network in which the nodes alternate between an effect and a state, and it seems to imply (5.57) in the $n = 1$ case. Going to $n > 1$, a network of quantum channels is then an alternating network of effects and states as associativity can be used, and it indeed implies (5.57) in general.

Theorem 5.3.2 *The projectors to quantum combs* (5.54), *to a network of quantum channels* (5.55), *and to (twice) a network of quantum states* (5.56) *are all equivalent to the following projector to an alternating network of quantum effects and states:*

$$\mathcal{P}^{(2n)} = \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \ldots \prec \overline{\mathcal{I}}_{A_{2n-2}} \prec \mathcal{I}_{A_{2n-1}}\,. \tag{5.64}$$

Meaning that with suitable normalization, the state structure of networks of order n based on quantum channels is equivalent to the one of combs of order n based on quantum channels, and it is also equivalent to the one of combs of order $2n$ based on quantum states.
In addition, the elements M of the state structure defined by this projector obey equation (5.57).

The proof is presented in the appendix; see Appendix D.5.4. This theorem means that an operator M obeying the causality condition (5.57) can be interpreted as an element of four equivalent state structures: from Equation (5.54) as a valid quantum n-comb, i.e. (the CJ representation of) a map on $2n$ systems that transforms the recursively defined map on the $n-1$ nodes split between the $2n-2$ systems $A_1, \dots A_{2n-2}$ into a single-node map, i.e. a quantum channel from A_0 to A_{2n-1}. From Equation (5.55) as a valid quantum network of channels with n nodes, i.e. (the CJ representation of) a map on $2n$ systems $A_0, \dots A_{2n-1}$ that represents the n successive operations of a party on n nodes: at node j she applies a channel between systems A_{2j-2} and A_{2j-1} that can depend on (that is, use any size of ancillary memory from) the $j-1$ previous nodes. From Equation (5.56) as a valid comb of states on $2n$ nodes, i.e. (the CJ representation of) a map on $2n$ systems that transforms the recursively defined map on the $2n-1$ systems $A_0, \dots A_{2n-2}$ into a quantum state on the $2n-$th system A_{2n-1}. And from Equation (5.64) as a valid quantum network alternating between quantum measurements and states at each of the $2n$ nodes, i.e. (the CJ representation of) a map on $2n$ systems $A_0, \dots A_{2n-1}$ that represents the $2n$ successive operations of a party on $2n$ nodes: at odd nodes, she measures a quantum state, at even nodes, she prepares one. Each operation is potentially conditioned by the previous nodes but is independent of the future ones.

Although these four definitions appear quite different, notice that Equation (5.64) is actually the normal form of Equations (5.54), (5.55), and (5.56)! Hence, the equivalence between seemingly unrelated definitions of higher-order maps is inferred simply by working out their normal form. The fact that the two comb-like definitions result in structures with a single signaling direction is now directly apparent, as the structure features a single 'prec chain' in normal form. As mentioned above, this is a very peculiar behavior: From Equation (5.40b), one would have expected the normal forms to be the union of many prec chains with different signaling orderings. That is, the combs should typically have an unfixed signaling direction between their nodes. Lemma 5.3.1 auspiciously prevents that. The isomorphisms between $\prec$ and $\rightarrow$ in the case of composite state structures whose base structures are sets of quantum states explain this apparent counter-logical equivalence of a transformation (the network of states) with various higher-order transformations. Quantum theory is thus very tame in the sense that successive and/or comb-like-nested transformations[31] of quantum states or channels happen to have no signaling from output to input automatically.

31: In Section 6.1, it will be further shown that other ways of nesting based on quantum theory are also 'resistant' to multiple signaling directions: several layers of recursively defined supermaps are required before obtaining ICO.

On the contrary, nesting any other state structures in a comb-like construction will generally result in a non-fixed signaling structure. An example of this in the case of the MPM is provided in Chapter 6. It also features another example of the use of the normal form.

5.4. Conclusion and Outlooks for the Characterization

Defining and characterizing higher-order processes has been abstracted under the concept of a state structure in the Chapter 3. In particular, it was shown that any class of admissible higher-order processes is characterized only by the processes on top of which it is defined. In terms of state structures, this definition amounts to defining composite state structures out of base state structures, which in turn amounts to defining composition rules for the projector characterizing them. The properties of the various composition rules for projectors have been studied in depth in this chapter. The algebraic properties found in this chapter were shown to allow the characterization of the signaling structure of any higher-order process by algebraic manipulations on its projector only.

The key finding is that the various combinations of projectors built using the algebra and a given number of base state structures were shown to organize as a lattice, the *projector algebra*, Proposition 5.1.10. This lattice sorts the state structures it contains according to the possible signaling directions its states may show. Within it, a term is contained within another if it has 'more signaling' than another. For example, the partial order $\subseteq$ concludes that $\mathcal{P}_A \prec \mathcal{P}_B \supset \mathcal{P}_A \otimes \mathcal{P}_B$, meaning that the set of bipartite states that are one-way signaling from A to B have more signaling allowed than the ones that are no-signaling. Accordingly, the partial order cannot conclude anything for terms with different signaling directions, e.g., $\mathcal{P}_A \prec \mathcal{P}_B \neq \mathcal{P}_A \succ \mathcal{P}_B$.

In particular, the lattice can be restricted to the sub-lattice built without using the negation and using a single base state structure for each subsystem, the *signaling lattice*, Proposition 5.1.11. This signaling lattice characterizes all 'comparable state structures' inasmuch as it is the subset of state structures with the same local interpretation of their base state structures. If a party A considers her local share of a state belonging to a global state structure as something inputting a state from state structure $\mathscr{A}$, then she will also do so for every other state in every other state structures in the signaling lattice; she will not suddenly obtain a share of a state that is interpreted as $\mathscr{A}'$ or $\overline{\mathscr{A}}$. Within the signaling lattice, the no-signaling projector is contained within all elements: the smallest kind of global state that the parties can share is made of the ones that never allow for signaling from one party to another. Conversely, all elements of the lattice are contained in the fully signaling projector: the biggest kind of shared global state is made of those allowing signaling in every direction.

The second key result was introducing a normal form for the projectors, Definition 5.2.1, and showing that all of them admit at least one, Theorem 5.2.1. This normal form can be used to directly read the possible signaling directions a given lattice term may show. A caveat, nonetheless, is that the normal form is not unique. As shown by the example of the quantum comb formalism, in certain cases, a normal form can be reduced to another one that features fewer terms and, therefore, signaling directions. Because of that, one can be led to mistakenly interpret a state structure as permitting states with several possible signaling directions where they have only one, as is the case when naively putting the state structures of quantum combs in normal form.

This demonstrates that completely sorting out the signaling lattice for a given number of parties and set of base projectors is not as simple as it appears. As signaling defines the ordering of the lattice, it could have been that a term is 'above' another one every time its normal form has 'more unions' in its decomposition. But the isomorphic normal forms complicate such a simple approach. On the contrary, sorting these lattices will also help identify equivalent normal forms and, therefore, isomorphisms of state spaces. Such isomorphic state spaces are very interesting because they are characterized differently, while having the same signaling structure. Ergo, some classes of higher-order processes assumed to be fundamentally different can turn out to be the same. For example, in the case of the combs, it was shown in Theorem 5.3.2 that several ways[32] of defining the objects all led to the same state structure.

32: I.e., the combs can be defined as networks of channels, as networks of states, as transformations between channels.

Sorting a signaling lattice for two parties is still simple: these lattices are essentially characterized by (5.40). However, when there are more than two parties, the situation becomes more complex, especially since there can be projectors involving different multiplicative connectors like $\mathcal{P}_A \prec (\mathcal{P}_B \parr \mathcal{P}_C)$, and that these connectors do not distribute over each other, let alone verify a (weak) interchange law. Therefore, these non-distributive expressions lead to many more possibilities than the four equivalences classes defined by (5.40), and their decomposition into normal forms may lead to even more unsuspected isomorphisms of normal forms. In the outlook chapter, some preliminary results about the relations in the tripartite case will be presented in Subsection 6.3.1.

While it is a non-trivial matter, identifying the non-equivalent elements of a signaling lattice appears as a natural direction to continue the characterization: Identifying the non-equivalent points of the signaling lattice defines the equivalence classes of state structures. The knowledge of the normal form of each of these points, as well as a systematic way to turn a normal form into its other equivalent normal forms would complete the characterization fully. Indeed, if one knows how to turn a normal form of a given state structure until it becomes the normal form of one of the equivalence classes, they have precisely assessed the position of their state structure within the lattice, and, therefore, how much signaling the states in their state structure allows with respect to all the *comparable* state structures.

Nevertheless, as the quantum comb example demonstrated, the characterization tools developed in this chapter are already sufficient for a thorough analysis of the signaling in any family of higher-order processes. In the next chapter, the methods will be applied to nested quantum channels as a proof of concept. In addition, some preliminary considerations and insights about their use in the search for indefinite causal order will be presented.

Applications and Prospects | 6.

Nicely put, but as young Cerro said to King Vridank on their first date: “Does it have any practical uses?”

Zoltan Chivay to Geralt of Rivia in: Andrzej Sapkowski (1999), *The Lady of the Lake*

In Subsection 3.6.2, it was shown that defining a bipartite state or a channel between two state structures is enough to observe two-way signaling. In the channel case, it was demonstrated that specific admissible channels were not decomposable as a convex sum of two terms with opposite signaling directions. This ‘non-mixture of causal orders’ is colloquially referred to as an *indefinite causal order* (ICO) in the literature [4], and made precise under the theory-dependent notion of *causal non-separability* [5]. The biased quantum channel was moreover shown to beat the OCB game, a guess your neighbor’s input (GYNI) task providing a theory-independent inequality to certify that the process is *non-causal*, i.e. a *causal inequality*. Yet, in Section 5.3, it was shown that the higher-order quantum channel theory, known as the quantum comb formalism [9], never leads to two-way signaling, Theorem 5.3.2. The question is, ‘When does an indefinite causal order arise in higher-order quantum processes ?’.

[4]: Chiribella et al. (2013), *Quantum computations without definite causal structure*.

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.

With the tool of the normal form, Theorem 5.2.1, it was shown that the fixed signaling direction in the quantum combs is due to an accidental isomorphism, Lemma 5.3.1. As follows from Theorem 3.4.1 and Lemma 3.5.3, an admissible transformation is represented by a two-way signaling composition and should generally lead to two-way signaling. The answer provided by this thesis[1] is then *almost always, the quantum combs are an exception*.

In this thesis, a method for characterizing the signaling relations in higher-order processes has been developed. This last chapter before the conclusion presents some further consequences and future prospects implied by the methods.

This chapter begins with another example of the utilization of the formalism: nested quantum supermaps, Section 6.1. The aim is twofold: on the one hand, it will demonstrate how to use the characterization methods in another concrete case. On the other hand, it will show that non-fixed causal order, that is, multiple signaling directions, eventually arises when recursively defining admissible quantum transformations. In other words, this is done to confirm that the quantum combs are an exception to the rule; admissible quantum transformations lead to processes with non-fixed signaling direction, and possibly to indefinite causal order or non-causal processes.

1: Concordant conclusions were independently attained by Simmons and Kissinger [36] as well as Milz and Quintino [37].

[36]: Simmons et al. (2022), *Higher-order causal theories are models of BV-logic*.
[37]: Milz et al. (2023), *Transformations between arbitrary (quantum) objects and the emergence of indefinite causality*.

If it is the case that admissible quantum transformation lead to non-causal processes, one may wonder if it is possible to refine Perinotti and Bisio’s type system to define nested quantum transformations that do not. Preliminary insights in this direction are presented in Subsection 6.3.2. In this section are also sketched some preliminary results on the problem of characterizing the non-equivalent elements in the signaling lattice for the tripartite case, Subsection 6.3.1. As a whole, this section tries to utilize the lattice structure of projectors as a model of logic for discussing signaling: the signaling relations will be read directly from the projector formulae, and, as will be shown, the equivalence of formulae will be interpreted as logical equivalences. By this, it is meant that equivalences or inequivalences at the level of the algebra, computed using the algebraic rules, will be interpreted as logical equivalences, inferred using language and reasoning.

[41]: Vandenberghe et al. (1996), *Semidefinite Programming*.

[35]: Araújo et al. (2015), *Witnessing causal nonseparability*.

The last briefly mentioned outlook of this chapter is the connection of the characterization with Semi-Definite Programming [41], Section 6.2. Since the projective characterization yields projectors, hence linear constraints, these can be directly translated into linear programs for the numerical search of causally non-separable processes within a given state structure. Actually, this was the original purpose for introducing the projective characterization methods in Reference [35].

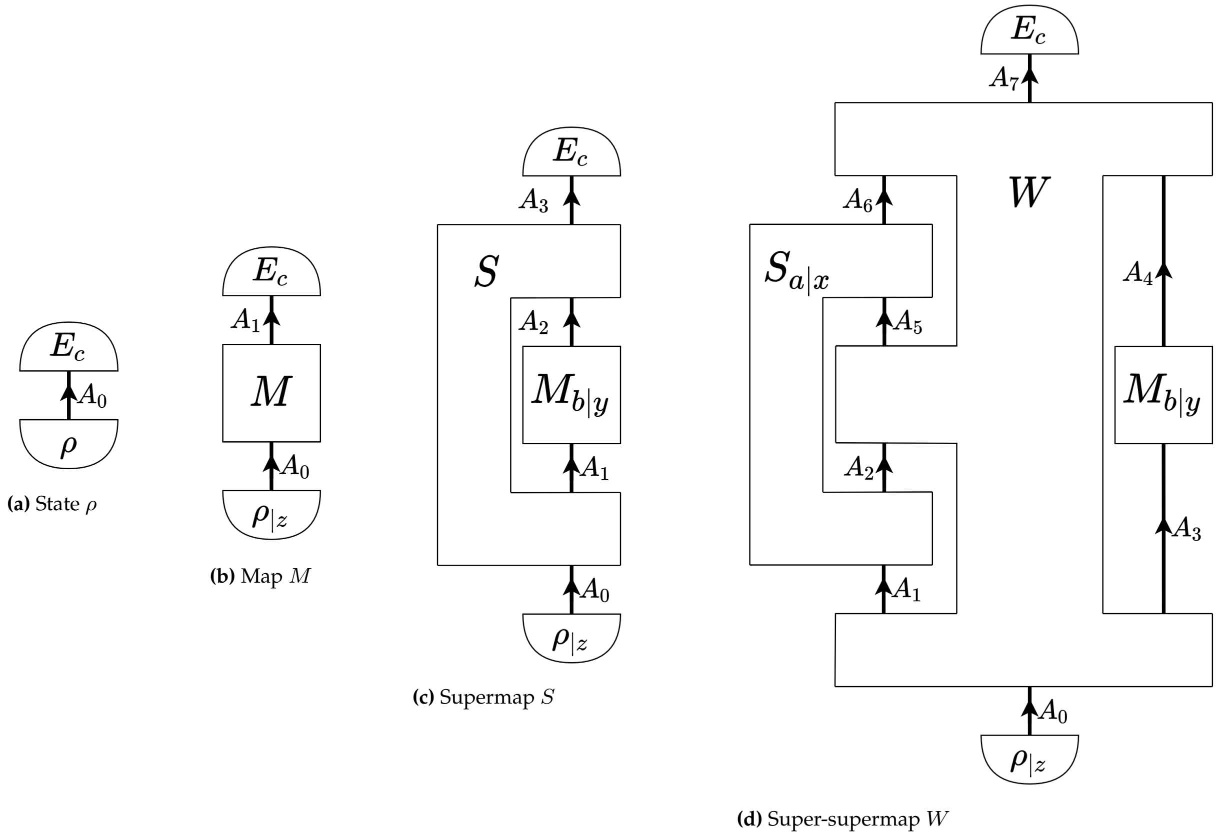
6.1. Example: Nesting Quantum Transformations until the Emergence of an Indefinite Causal Order

This example focuses on the case of higher-order transformations built upon quantum mechanics. The construction that lifts the POVM formalism into the quantum instrument formalism presented at the beginning of Chapter 3, revisited in Chapter 4 as an example for the state structure formalism as well as a motivation for studying the projector algebra in Chapter 5, is once again revisited in this section as an example of the use of the algebra. In a nutshell, it will be repeated on itself until an indefinite causal order (ICO) arises.

There are three reasons for choosing this example. Firstly, it demonstrates the application of characterization techniques in a concrete case, where all the objects that will be defined have already been studied in the literature. Secondly, and in accordance with the discussion of Section 5.3, it highlights how tame quantum theory is compared to other process theories characterized using projective means. Although the transformation operation $\rightarrow$ will be used repetitively, i.e. successive higher-order objects will be defined in a way that should allow for bidirectional signaling, it will not result in an object presenting ICO until the fourth order. This is in stark contrast with the example of the biased quantum theory presented in Section 3.6. There, one could observe signaling in both directions already at the level of bipartite states (bipartite and first-order type) as well as of channels (single partite and second-order type). Thirdly, this example recovers another previously studied tripartite process: the Multi-Round Process Matrix (MPM) [1].

[1]: Hoffreumon et al. (2021), *The Multiround Process Matrix*.

As is the case in Section 4.1 and Section 5.3, the objects under study are higher-order generalizations of quantum theory, so in this section, it is assumed that every base projector is the identity projector and that all base state structure is normalized to 1. However, the notion of order was shown to be somewhat misleading for quantum theory in Section 4.3.



(a) State ρ

(b) Map M

(c) Supermap S

(d) Super-supermap W

Figure 6.1.: The example considered in this section: from states to the super-supermap. Each new figure is defined by introducing a transformation of the previously introduced transformation: The channel M is a transformation on the state space of ρ, the supermap S is a transformation on the state space of the channels M, and the super-supermap is a transformation on the state space of the superchannels S. Note that, with respect to the graphical methods, these are not exactly the CJ representations of the maps as the 'wires are unbent' to increase readability; normally, in the CJ picture all these diagrams are half-circle acting on multiple subsystems.

Moreover, the normal form of Definition 5.2.1 makes it unnecessary. Consequently, this notion has to be formalized before 'constructing up to the fourth order'.

The basic idea is that an order is defined by a state and effect dual pair as in Definition 3.3.3. The base state structure determines the first order. To build the next order, an evolution (i.e. admissible mapping) from the states of the current order to themselves is postulated. The state structure of the evolutions is then declared as the 'states of order+1', which automatically fixes the 'effects of order+1' as its dual state structure, so this new pair forms the order+1. To give a process interpretation, the states at all orders will always be assumed to be an environment shared by the local effects of the order below. This is done for the sake of the argument: the introduced evolutions, which are bipartite states according to the order below, are interpreted as the global deterministic 'states' and everything else is interpreted as the local 'effects' controlled by some parties. Doing so will maximize the number of local parties acting probabilistically, which makes non-fixed signaling directions and ICO more obvious.

Therefore, given states in $\mathscr{A}_0 \subset \mathcal{L}\left(\mathcal{H}^{A_0}\right)$ and effects in $\overline{\mathscr{A}}_0$, the current order consists of pairs of objects in $\mathscr{A}_0 \times \overline{\mathscr{A}}_0$, and next order is constructed by defining $\mathscr{A}_0 \to \mathscr{A}_1 \subset \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_1}\right)$ as a shared state between party

A_0, acting on it with local effect $\mathscr{A}_0$ and party A_1, acting on it with $\overline{\mathscr{A}}_1 \cong \overline{\mathscr{A}}0$. Thus, the construction of the next order consists of the following redefinition of the dual pair:

$$\underbrace{\mathscr{A}_0}_{\text{State}} \times \underbrace{\overline{\mathscr{A}}_0}_{\text{Effect}} \mapsto \underbrace{\left(\mathscr{A}_0 \to \mathscr{A}_1\right)}_{\text{State}+1} \times \underbrace{\left(\overline{\mathscr{A}_0 \to \mathscr{A}_1}\right)}_{\text{Effect}+1} . \tag{6.1}$$

and the interpretation of the states as environments will promote the environment of order n to a local party at order $n+1$. In symbols, Environment $\mapsto$ Party B so that:

$$\underbrace{\mathscr{A}_0}_{\text{Environment}} \times \underbrace{\overline{\mathscr{A}}_0}_{\text{Party A}} \mapsto \left(\underbrace{\mathscr{A}_0 \to \mathscr{A}_1}_{\text{Environment}+1} \right) \times \left(\underbrace{\mathscr{A}_0}_{\text{Party B}} \otimes \underbrace{\mathscr{A}_1}_{\text{Party A}} \right) . \tag{6.2}$$

Remark that this interpretation is possible because the dual of a transformation is a no-signaling composition: $\overline{\mathscr{A}_0 \to \mathscr{A}_1} = \mathscr{A}_0 \otimes \overline{\mathscr{A}}_1$. Consequently, it can be split into two local parties without changing the support of the state structure.

6.1.1. The Quantum Channel is a 1-comb

This is yet again the introductory example of Section 3.1. The first order consists of a state and effect pair

$$(\rho, \mathbb{1}) \in \underbrace{\mathscr{A}_0}_{\text{Environment}} \times \underbrace{\overline{\mathscr{A}}_0}_{\text{David}} . \tag{6.3}$$

The resolutions of the effect are assumed to be under the control of party David, who records outcome[2] c with probability distribution

2: His settings are ignored since he performs a quantum measurement, which is always no-signaling; his outcome is labeled c instead of d to avoid confusion with the dimension.

$$p(c) = \mathrm{Tr}\left[E_c \cdot \rho\right] . \tag{6.4}$$

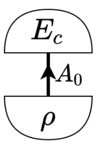


Figure 6.2.: Graphical representation of Equation (6.4)

To go to the second order, one postulates some evolution so that the state structure $\mathscr{A}_0$ is mapped to a similarly defined state structure $\mathscr{A}_1$ by a CPTP map $\mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ in a manner that preserves the Born rule,

$$1 = \left(\mathbb{1} \, , \, \rho\right)_{A_0} \mapsto \left(\mathbb{1} \, , \, \mathcal{M}^{A_0 \to A_1}\left(\rho\right)\right)_{A_1} = 1 . \tag{6.5}$$

According to Subsection 4.1.1, in the CJ picture, this results in a new state and effect dual pair[3]:

3: The numeral labeling of subsystems is changed after the introduction of each new order so to put the environment (state) of the previous order 'in between' the effects. The different operations remain assumed under the same party's control; here, Charlie is still preparing state $\rho_{|x}$ at A_0, but David is now measuring his effect E_c at A_1.

$$\left(\rho^{A_0} \otimes \mathbb{1}^{A_1}, M_{A_0 A_1}\right) \in \left(\mathscr{A}_0 \otimes \overline{\mathscr{A}_1}\right) \times \left(\mathscr{A}_0 \to \mathscr{A}_1\right) , \tag{6.6}$$

so that the normalization is preserved,

$$1 = \mathrm{Tr}\left[M \cdot (\rho \otimes \mathbb{1})\right] = \left(M \, , \, \rho \otimes \mathbb{1}\right)_{A_0 A_1} . \tag{6.7}$$

In this case, M is a quantum channel in CJ representation, i.e., a quantum 1-comb by Definition 2.3.2. As explained below Equation (4.6), this leads to the scenario of a communication through an 'environment' modeled by a channel M: the former environment, the quantum state ρ, is now a local effect modeling the action of a party Charlie. Letting Charlie choose his prepared quantum state according to setting[4] z while David is still

4: As will be shown, Charlie will always end up in the global past, so his intervention is assumed deterministic since no probabilistic course of action would allow his outcome to be correlated with any of the other parties's settings.

applying a POVM with outcome c, the distribution is

$$p(c|z) = \mathrm{Tr}\left[M \cdot \left(\rho_{|z} \otimes E_c^T\right)\right] \ . \tag{6.8}$$

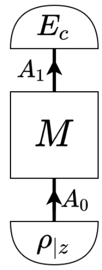


Figure 6.3.: Diagrammatic representation of Equation (6.8)

In this example, completing the second-order theory requires the set of states to be extended to all functionals normalized on the effects. That is, to all W in $\overline{\mathscr{A}_0 \to \mathscr{A}_1}$ so that the normalization of probabilities becomes

$$1 = (W\,,\, M)\ , \tag{6.9}$$

where W is a single partite PM and M is a 1-comb, so that both are second-order objects.

Nonetheless, it was shown in Subsection 4.1.4 that the single partite PM trivially decomposes into quantum states and measurements, which means that the states W of the second order always 'redescend to the first order': $\overline{\mathscr{A}_0 \to \mathscr{A}_1} = \mathscr{A}_0 \otimes \overline{\mathscr{A}}_1$. The explanation in terms of the projector algebra is direct from the definition $\mathcal{P}_{A_0} \to \mathcal{P}_{A_1} := \overline{\mathcal{P}_{A_0} \otimes \overline{\mathcal{P}}_{A_1}}$. In terms of signaling structure, because the transformation is two-way signaling, $\mathcal{P}_{A_0} \to \mathcal{P}_{A_1} = \left(\overline{\mathcal{P}}_{A_0} \prec \mathcal{P}_{A_1}\right) \cup \left(\overline{\mathcal{P}}_{A_0} \succ \mathcal{P}_{A_1}\right)$, the functional on transformations have to be no-signaling by De Morgan duality (5.37c): $\overline{\mathcal{P}_{A_0} \to \mathcal{P}_{A_1}} = \overline{\overline{\mathcal{P}}_{A_0} \prec \mathcal{P}_{A_1}} \cap \overline{\mathcal{P}_{A_0} \succ \mathcal{P}_{A_1}} = \left(\mathcal{P}_{A_0} \prec \overline{\mathcal{P}}_{A_1}\right) \cap \left(\mathcal{P}_{A_0} \succ \overline{\mathcal{P}}_{A_1}\right) \overset{(5.40a)}{=} \mathcal{P}_{A_0} \otimes \overline{\mathcal{P}}_{A_1}$. Therefore, the single partite process matrix for any base state structure always reduces to state and effect pairs. While the state structure redescending to the state and effect pair is not a specificity of quantum theory, the fact that all single parties process matrices factorizes, i.e. that W can always be written as a pure tensor product like $W = \rho \otimes \mathbb{1}$ is one. This is due to the fact that the state structure of quantum measurement is a singleton $\overline{\mathcal{P}}_{A_1} = \{\mathbb{1}\}$.

Another difference noticed in Subsection 4.1.1 is that the 1-comb is one-way signaling despite being built using the transformation connector. Now, this fact has been explained in Section 5.3: the 1-comb is characterized by $\mathcal{I}_{A_0} \to \mathcal{I}_{A_1}$ which, by Lemma 5.3.1, is equivalent to the normal form

$$\mathcal{I}_{A_0} \to \mathcal{I}_{A_1} \overset{(5.63a)}{=} \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1}\ . \tag{6.10}$$

Hence, it is guaranteed that the 1-combs have a fixed signaling direction, so they cannot show indefinite causal order. If the base state structure were anything other than quantum states, this would not have been the case; this is a specificity of quantum theory. Interestingly, while the effects lose a signaling direction, the states remain no-signaling,

$$\overline{\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1}} = \overline{\mathcal{D}}_{A_0} \prec \mathcal{D}_{A_1} \overset{(5.63b)}{=} \mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1}\ , \tag{6.11}$$

the normal form $\overline{\mathcal{D}}_{A_0} \prec \mathcal{D}_{A_1}$ is indeed subject to one of the accidental isomorphism of Lemma 5.3.1. Therefore, in the second order, the dual pair, which was to feature up to two directions of signaling, only features one,

$$\left(\overline{\mathscr{A}_0 \to \mathscr{A}_1}\right) \times \left(\mathscr{A}_0 \to \mathscr{A}_1\right) = \left(\mathscr{A}_0 \otimes \overline{\mathscr{A}}_1\right) \times \left(\overline{\mathscr{A}}_0 \prec \mathscr{A}_1\right)\ . \tag{6.12}$$

The only signaling direction is from the input of the effect (i.e., the quantum channel) to its output; this scenario is causally ordered in both its states and effects.

6.1.2. The Quantum Supermap is a 2-comb

[8]: Chiribella et al. (2008), *Transforming quantum operations: Quantum supermaps.*

The third order is obtained by assuming the existence of an evolution of the current evolution. This transformation of transformation was defined in the literature as the quantum supermap [8]. In the same way operators defined on operators can be nicknamed 'superoperators', the 'supermap' $\mathcal{S}$ is a linear map between two maps in the same state structure. Let the quantum channel $\mathcal{M}$ be defined between subsystems A_1 and A_2. Then, $\mathcal{S}$ as a completely CPTP-preserving supermap that send $\mathcal{M}$ to a similar map $\widetilde{\mathcal{M}}$ between subsystems A_0 and A_3, $\mathcal{S}\left(\mathcal{M}^{A_1 \to A_2}\right) = \widetilde{\mathcal{M}}^{A_0 \to A_3}$, so that

$$1 = \left(\mathbb{1}\,,\, \mathcal{M}\left(\rho\right)\right)_{A_2} \mapsto \left(\mathbb{1}\,,\, \left[\mathcal{S}\left(\mathcal{M}\right)\right]\left(\rho\right)\right)_{A_3} = 1\,. \tag{6.13}$$

5: Normally, $M = \mathfrak{C}(\mathcal{M})$ should be transposed since it went from the l.h.s. to the r.h.s. of the inner product. This transposition is assumed to be hidden in a redefinition like $M \mapsto M^T$ to lessen clutter. Omitting it is unimportant for the discussion since operator systems are closed under the transposition in any case.

Going to the CJ picture[5], the following deterministic probability rule is obtained:

$$1 = \text{Tr}\left[S \cdot \left(\rho \otimes M \otimes \mathbb{1}\right)\right] = \left(S\,,\, \rho \otimes M \otimes \mathbb{1}\right)\,. \tag{6.14}$$

The quasi-orthogonal pair is of the form

$$(S, \rho \otimes M \otimes \mathbb{1}) \in \underbrace{\left(\left(\mathscr{A}_1 \to \mathscr{A}_2\right) \to \left(\mathscr{A}_0 \to \mathscr{A}_3\right)\right)}_{\text{Environment}} \times \big(\underbrace{\mathscr{A}_0}_{\text{Charlie}} \otimes \underbrace{\left(\mathscr{A}_1 \to \mathscr{A}_2\right)}_{\text{Bob}} \otimes \underbrace{\overline{\mathscr{A}_3}}_{\text{David}}\big)\,. \tag{6.15}$$

The channel $M \in \mathscr{A}_1 \to \mathscr{A}_2$ is now interpreted as a local effect. It is assumed under the control of party Bob, who is resolving it into a collection $\{M_{b|y}\}$ according to his setting y and outcome b. Letting Charlie and David resolve their respective quantum states ρ and unit effect $\mathbb{1}$, their outcome distribution without the supermap is now a scenario in which Charlie prepares a state, sends it to Bob, who applies a quantum instrument on it then forwards the output to David, who finally applies a destructive measurement on it:

$$p(b, c|y, z) = \text{Tr}\left[M_{b|y} \cdot \left(\rho_{|z} \otimes E_c^T\right)\right]\,. \tag{6.16}$$

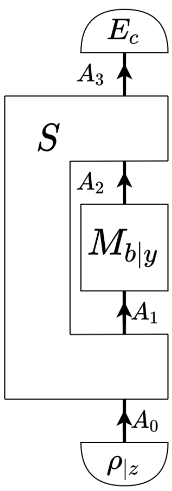


Figure 6.4.: Diagrammatic representation of Equation (6.17)

With the supermap as an environment between their operations, the probability becomes

$$p(b, c|y, z) = \text{Tr}\left[S \cdot \left(\rho_{|z} \otimes M_{b|y} \otimes E_c^T\right)\right]\,. \tag{6.17}$$

Defined as such, S is a 2-comb, an object that transforms Bob's 1-comb (a quantum channel) into the 1-comb seen by Charlie and David. In the previous subsection, channels themselves were shown to be one-way signaling objects, and, moreover, in Section 5.3 it was demonstrated that any comb is one-way signaling. But without using Theorem 5.3.2, this could also have been found from the normal form of its projector. This is now done for completeness. Some manipulations yield

$$\begin{aligned}(\mathcal{I}_{A_1} \to \mathcal{I}_{A_2}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_3}) &\overset{(5.63a)}{=} \left(\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2}\right) \to \left(\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_3}\right) \\ &\overset{(5.40b)}{=} \left(\overline{\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2}}\right) \prec \left(\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_3}\right) \cup \left(\overline{\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2}}\right) \succ \left(\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_3}\right) \\ &= \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_3} \cup \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_3} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \,. \end{aligned} \tag{6.18}$$

At this point, it may be concluded that the most general 2-comb is a superposition of two possible quantum networks: the left-hand side is the one in which the channel M is measured, $\mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} = \overline{\mathcal{I}_{A_1} \to \mathcal{I}_{A_2}}$, then reprepared in its causal future ($\prec$) as $\tilde{M}$, $\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_3}$; whereas the right-hand side is the one in which $\tilde{M}$ is first prepared then M is measured.

By Definition 5.2.2, one may be led to conclude that the supermap has a non-fixed signaling structure. Yet, there is again the accidental isomorphism at play: using Equation (5.63b),

$$(\mathcal{I}_{A_1} \to \mathcal{I}_{A_2}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_3}) = \left((\overline{\mathcal{I}}_{A_0} \otimes \mathcal{I}_{A_1} \otimes \overline{\mathcal{I}}_{A_2}) \prec \mathcal{I}_{A_3}\right) \cup \left(\overline{\mathcal{I}}_{A_0} \prec (\mathcal{I}_{A_1} \otimes \overline{\mathcal{I}}_{A_2} \otimes \mathcal{I}_{A_3}\right) \,, \tag{6.19}$$

but both sides of the union are actually equivalent. This is a particular instance of an uncurrying-like rule for the prec[6]:

6: But the prec does not obey the uncurrying rule in general; this is discussed in Subsection 6.3.2.

$$\left(\overline{\mathcal{I}}_{A_0} \otimes \left(\mathcal{I}_{A_1} \otimes \overline{\mathcal{I}}_{A_2}\right)\right) \prec \mathcal{I}_{A_3} = \overline{\mathcal{I}}_{A_0} \prec \left(\left(\mathcal{I}_{A_1} \otimes \overline{\mathcal{I}}_{A_2}\right) \otimes \mathcal{I}_{A_3}\right) = \overline{\mathcal{I}}_{A_0} \prec \left(\mathcal{I}_{A_1} \otimes \overline{\mathcal{I}}_{A_2}\right) \prec \mathcal{I}_{A_3} \,. \tag{6.20}$$

To get to the normal form, one normally uses Equation (5.40b) on the right-hand side of the above so to get

$$(\mathcal{I}_{A_1} \to \mathcal{I}_{A_2}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_3}) = \left(\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_3}\right) \cap \left(\overline{\mathcal{I}}_{A_0} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_1} \prec \mathcal{I}_{A_3}\right) \,. \tag{6.21}$$

However, Equation (5.63b) makes it a redundant normal form. Indeed, $\mathcal{I}_{A_1} \otimes \overline{\mathcal{I}}_{A_2} = \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2}$, so using (5.40b) in

$$\mathcal{I}_{A_1} \otimes \overline{\mathcal{I}}_{A_2} = (\mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2}) \cap (\mathcal{I}_{A_1} \succ \overline{\mathcal{I}}_{A_2}) \tag{6.22}$$

added a new, unnecessary $\mathcal{I}_{A_1} \succ \overline{\mathcal{I}}_{A_2}$ term, which resulted in an extra prec chain in the normal form (6.21) (i.e., the one on the right-hand side of the cap). Simplifying the normal form results in a different normal form featuring a single prec chain this time:

$$(\mathcal{I}_{A_1} \to \mathcal{I}_{A_2}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_3}) = \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_3} \,. \tag{6.23}$$

Equation (6.21) is a first example of a non-trivially redundant normal form for (6.23)[7]: the normal form can be shortened because of an accidental isomorphism that is specific to the base projectors that were used to construct it.

7: By trivially redundant normal form it is meant one where terms are repeated like $\mathcal{P} = \mathcal{P} \cap \mathcal{P}$, the latter expression is trivially redundant. In the case of (6.21), it is non-trivial as the equivalence relies on a non-trivial inclusion relation between different projectors like $\mathcal{P} = \mathcal{P} \cap \mathcal{P}'$.

This specificity of quantum theory, which reduces the number of terms in the normal form, means that the third-order supermaps are objects with a fixed signaling direction. This is in accordance to Theorem 5.3.2[8]: Equation (6.23) shows that the 2−combs are equivalent to quantum networks with two nodes. This makes the states of the third-order (the

8: The proof of this theorem provides an alternative way to show (6.23) using the uncurrying for the transformation (D.68):

$$\begin{aligned}&(\mathcal{I}_{A_1} \to \mathcal{I}_{A_2}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_3}) \\ &\overset{(D.68)}{=} (\mathcal{I}_{A_0} \otimes (\mathcal{I}_{A_1} \to \mathcal{I}_{A_2})) \to \mathcal{I}_{A_3} \\ &\overset{(5.63a)}{=} \overline{(\mathcal{I}_{A_0} \otimes (\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2}))} \prec \mathcal{I}_{A_3} \\ &\overset{(5.63b)}{=} \overline{(\mathcal{I}_{A_0} \prec (\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2}))} \prec \mathcal{I}_{A_3} \\ &= \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_3} \,. \end{aligned} \tag{6.24}$$

supermaps) descend to a causally ordered succession of two second-order states (which are quantum channels):

$$(\mathscr{A}_1 \to \mathscr{A}_2) \to (\mathscr{A}_0 \to \mathscr{A}_3) = \left(\overline{\mathscr{A}}_0 \prec \mathscr{A}_1\right) \prec \left(\overline{\mathscr{A}}_2 \prec \mathscr{A}_3\right) . \tag{6.25}$$

As the second-order transformations have been shown to have a single signaling direction, so is the third-order.

Using Lemma 5.3.1 on the normal form, the third-order can be further descended to a composition of first-order objects since the projector becomes

$$\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_3} = \overline{\mathcal{I}}_{A_0} \prec \left(\mathcal{I}_{A_1} \otimes \overline{\mathcal{I}}_{A_2}\right) \prec \mathcal{I}_{A_3} , \tag{6.26}$$

so the third-order states have the state structure

$$(\mathscr{A}_1 \to \mathscr{A}_2) \to (\mathscr{A}_0 \to \mathscr{A}_3) = \overline{\mathscr{A}}_0 \prec \left(\mathscr{A}_1 \otimes \overline{\mathscr{A}}_2\right) \prec \mathscr{A}_3 . \tag{6.27}$$

Thus, an ordered succession of four first-order objects that allow for signaling from A_0 to all the nodes, and from A_1 and A_2 to A_3. This is very surprising as it is the dual of the tripartite no-signaling composition of the effects,

$$\mathscr{A}_0 \otimes \left(\overline{\mathscr{A}}_1 \prec \mathscr{A}_2\right) \otimes \overline{\mathscr{A}}_3 = \overline{(\mathscr{A}_1 \to \mathscr{A}_2) \to (\mathscr{A}_0 \to \mathscr{A}_3)} , \tag{6.28}$$

so the state structure of the environments (states) before simplification was $\overline{\mathscr{A}_0 \otimes \left(\overline{\mathscr{A}}_1 \prec \mathscr{A}_2\right) \otimes \overline{\mathscr{A}}_3} = \overline{\mathscr{A}}_0 \parr \left(\mathscr{A}_1 \prec \overline{\mathscr{A}}_2\right) \parr \mathscr{A}_3$ which featured two-way signaling connectors. This means that the two two-way signaling compositions ($\parr$) were reduced to a one-way signaling ($\prec$) and the one-way signaling ($\prec$) to a no-signaling ($\otimes$) one. Without the isomorphism, one may have expected third-order states to be a fully signaling tripartite state structure (i.e., composed using the $\parr$ only, and treating $\mathscr{A}_1 \prec \overline{\mathscr{A}}_2$ as a single state structure associated with Bob), which, in that case, would have featured up to six different one-way signaling terms in the decomposition. (Indeed, with two-way signaling between three projectors, the normal form was expected to feature the union of 3! different prec chains.) But instead, the quantum supermap reduces to an object with, at most, a single signaling direction and characterized by Equation (6.27).

Moreover, it is not even signaling along the four subsystems: in a supermap, it is impossible to signal from the first output at $\mathscr{A}_1$ to the second input at $\mathscr{A}_2$. Note that it is this property that makes the quantum supermap interpretable as a quantum bipartite channel reduced to a one-way signaling subset [64] (also called a semicausal box [61, 62, 65]). Without it, the channel causality condition (5.58) could not be verified when treating the two subsystems A_0 and A_2 as the input of a bipartite channel outputting subsystems A_1 and A_3.

[64]: Piani et al. (2006), *Properties of quantum nonsignaling boxes*.
[61]: Beckman et al. (2001), *Causal and localizable quantum operations*.
[62]: Eggeling et al. (2002), *Semicausal operations are semilocalizable*.
[65]: D'Ariano et al. (2011), *No Signaling, Entanglement Breaking, and Localizability in Bipartite Channels*.

6.1.3. The Quantum Super-supermap has Non-Fixed Causal Order

To go to the fourth order, the same procedure is applied again: the 'super-supermaps' $\mathcal{W}$ are introduced as the admissible mappings from supermaps like $\mathcal{S}$ to themselves, $\mathcal{W}(\mathcal{S}) = \widetilde{\mathcal{S}}$. In this scenario, the relabeling of subsystems is such that $\mathcal{S} \in \mathcal{L}\left(\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2} \otimes \mathcal{H}^{A_5} \otimes \mathcal{H}^{A_6}\right)$,

$\widetilde{\mathcal{S}} \in \mathcal{L}\left(\mathcal{H}^{A_0} \otimes \mathcal{H}^{A_3} \otimes \mathcal{H}^{A_4} \otimes \mathcal{H}^{A_7}\right)$, so that the Born rule is mapped to

$$1 = (\mathbb{1}\,,\ [\mathcal{S}\,(\mathcal{M})]\,(\rho))_{A_3} \mapsto (\mathbb{1}\,,\ [[\mathcal{W}\,(\mathcal{S})]\,(\mathcal{M})]\,(\rho))_{A_7} = 1\,. \tag{6.29}$$

In the CJ representation[9], the normalization of probabilities reads

$$1 = \mathrm{Tr}\left[W \cdot (\rho \otimes S \otimes M \otimes \mathbb{1})\right]\ . \tag{6.30}$$

9: Like the channel (see note 5), the supermap S should be transposed in (6.30); it is implicitly assumed that it has been redefined like $S \mapsto S^T$.

Accordingly, the valid states and effects are taken from a dual pair of state structures defining the fourth order as

$$\begin{aligned}&(W, \rho \otimes S \otimes M \otimes \mathbb{1}) \in \\ &\underbrace{([(\mathscr{A}_2 \to \mathscr{A}_5) \to (\mathscr{A}_1 \to \mathscr{A}_6)] \to [(\mathscr{A}_3 \to \mathscr{A}_4) \to (\mathscr{A}_0 \to \mathscr{A}_7)])}_{\text{Environment}} \\ &\times (\underbrace{\mathscr{A}_0}_{\text{Charlie}} \otimes \underbrace{(\mathscr{A}_2 \to \mathscr{A}_5) \to (\mathscr{A}_1 \to \mathscr{A}_6)}_{\text{Alice}} \otimes \underbrace{(\mathscr{A}_3 \to \mathscr{A}_4)}_{\text{Bob}} \otimes \underbrace{\overline{\mathscr{A}_7}}_{\text{David}})\,.\end{aligned} \tag{6.31}$$

The 2-comb S is promoted as a local effect representing the intervention of a party Alice. She prepares it according to her setting x and obtains outcome a when resolving it probabilistically.

$$p(a,b,c|x,y,z) = \mathrm{Tr}\left[W \cdot \left(\rho_{|z} \otimes S_{a|b} \otimes M_{b|y} \otimes E_c^T\right)\right]\ . \tag{6.32}$$

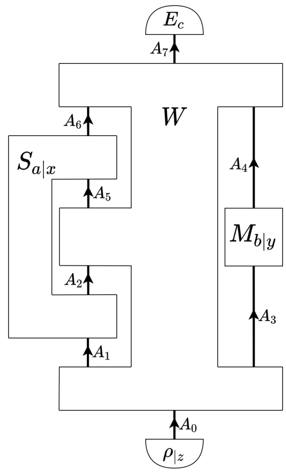


Figure 6.5.: Diagrammatic representation of Equation (6.32)

The local effects consist of Charlie preparing a state, Alice applying a probabilistic supermap, Bob applying a quantum instrument, and David measuring a POVM. Using the results of the previous section, the signaling allowed in the local interventions is drastically reduced,

$$\begin{aligned}&\overline{[(\mathscr{A}_2 \to \mathscr{A}_5) \to (\mathscr{A}_1 \to \mathscr{A}_6)] \to [(\mathscr{A}_3 \to \mathscr{A}_4) \to (\mathscr{A}_0 \to \mathscr{A}_7)]} \\ &\quad = \mathscr{A}_0 \otimes \left(\overline{\mathscr{A}}_1 \prec \left(\mathscr{A}_2 \otimes \overline{\mathscr{A}}_5\right) \prec \mathscr{A}_6\right) \otimes \left(\overline{\mathscr{A}}_3 \prec \mathscr{A}_4\right) \otimes \overline{\mathscr{A}}_7\,.\end{aligned} \tag{6.33}$$

and the operation of Alice –applying the resolution of a supermap– can be restated as a network with two nodes: Alice does a first intervention modeled by a quantum instrument acting between subsystems A_1 and A_2, then she does a second intervention also modeled by a quantum instrument but between subsystems A_5 and A_6 this time. Between her two interventions, she can keep an ancillary memory represented by a side channel.

The projector characterizing the state structure of the environment W is put in normal form to extract its signaling structure. First, successive applications of the uncurrying rule (D.68) yield

$$\begin{aligned}&[(\mathcal{I}_{A_2} \to \mathcal{I}_{A_5}) \to (\mathcal{I}_{A_1} \to \mathcal{I}_{A_6})] \to [(\mathcal{I}_{A_3} \to \mathcal{I}_{A_4}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_7})] \\ &\quad = [(\mathcal{I}_{A_3} \to \mathcal{I}_{A_4}) \otimes ((\mathcal{I}_{A_2} \to \mathcal{I}_{A_5}) \to (\mathcal{I}_{A_1} \to \mathcal{I}_{A_6}))] \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_7}) \\ &\quad = \left[\mathcal{I}_{A_0} \otimes \left((\mathcal{I}_{A_3} \to \mathcal{I}_{A_4}) \otimes ((\mathcal{I}_{A_2} \to \mathcal{I}_{A_5}) \to (\mathcal{I}_{A_1} \to \mathcal{I}_{A_6}))\right)\right] \to \mathcal{I}_{A_7}\,.\end{aligned} \tag{6.34}$$

Next, Equations (5.63) as well as (6.23) are used alternatively:

$$
\begin{aligned}
&= \big[\mathcal{I}_{A_0} \otimes \big((\mathcal{I}_{A_3} \to \mathcal{I}_{A_4}) \otimes ((\mathcal{I}_{A_2} \to \mathcal{I}_{A_5}) \to (\mathcal{I}_{A_1} \to \mathcal{I}_{A_6})) \big)\big] \to \mathcal{I}_{A_7} \\
&= \big[\mathcal{I}_{A_0} \otimes \big(\big(\overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4}\big) \otimes \big(\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6}\big)\big)\big] \to \mathcal{I}_{A_7} \\
&\overset{(5.63b)}{=} \big[\mathcal{I}_{A_0} \prec \big(\big(\overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4}\big) \otimes \big(\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6}\big)\big)\big] \to \mathcal{I}_{A_7} \\
&\overset{(5.63a)}{=} \overline{\mathcal{I}_{A_0} \prec \big(\big(\overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4}\big) \otimes \big(\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6}\big)\big)} \prec \mathcal{I}_{A_7} \\
&= \overline{\mathcal{I}}_{A_0} \prec \overline{\big(\big(\overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4}\big) \otimes \big(\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6}\big)\big)} \prec \mathcal{I}_{A_7} \ ,
\end{aligned}
\tag{6.35}
$$

where the negation got distributed over the prec in the last step.

Simplifying the negation at the center of the formula provides another example of a non-unique normal form. Using Equation (5.40a):

$$
\begin{aligned}
&\overline{\mathcal{I}}_{A_0} \prec \overline{\big(\big(\overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4}\big) \otimes \big(\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6}\big)\big)} \prec \mathcal{I}_{A_7} \\
&= \overline{\mathcal{I}}_{A_0} \prec \overline{\big(\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6} \prec \overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4}\big) \cap \big(\overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4} \prec \overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6}\big)} \prec \mathcal{I}_{A_7} \\
&\overset{(5.15)}{=} \overline{\mathcal{I}}_{A_0} \prec \Big(\overline{\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6} \prec \overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4}} \cup \overline{\overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4} \prec \overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6}}\Big) \prec \mathcal{I}_{A_7} \\
&= \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_3} \prec \overline{\mathcal{I}}_{A_4} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_5} \prec \overline{\mathcal{I}}_{A_6} \prec \mathcal{I}_{A_7} \cup \\
&\qquad \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_5} \prec \overline{\mathcal{I}}_{A_6} \prec \mathcal{I}_{A_3} \prec \overline{\mathcal{I}}_{A_4} \prec \mathcal{I}_{A_7} \ .
\end{aligned}
\tag{6.36}
$$

Therefore,

$$
\begin{gathered}
[(\mathcal{I}_{A_2} \to \mathcal{I}_{A_5}) \to (\mathcal{I}_{A_1} \to \mathcal{I}_{A_6})] \to [(\mathcal{I}_{A_3} \to \mathcal{I}_{A_4}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_7})] \\
= \\
\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_3} \prec \overline{\mathcal{I}}_{A_4} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_5} \prec \overline{\mathcal{I}}_{A_6} \prec \mathcal{I}_{A_7} \cup \\
\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_5} \prec \overline{\mathcal{I}}_{A_6} \prec \mathcal{I}_{A_3} \prec \overline{\mathcal{I}}_{A_4} \prec \mathcal{I}_{A_7} \ .
\end{gathered}
\tag{6.37}
$$

From this normal form, it can be inferred that the super-supermaps W can feature an indefinite causal order as it features the union of two different prec chains and, compared to Equation (6.23), no accidental isomorphism can make one term disappear. Actually, the indefiniteness is between Alice and Bob. To make it more clear, the state structure associated the unit effects of each local party is gathered into a single symbol, $(\mathscr{A}_0\otimes(\mathscr{A}_2 \to \mathscr{A}_5) \to (\mathscr{A}_1 \to \mathscr{A}_6)\otimes(\mathscr{A}_3 \to \mathscr{A}_4)\otimes\overline{\mathscr{A}_7}) \mapsto (\mathscr{C}\otimes\mathscr{A}\otimes\mathscr{B}\otimes\mathscr{D})$, so that the local state structures are redefined as the base state structures. The associated projectors are accordingly defined; let

$$
\begin{aligned}
\mathcal{P}_A &:= \overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6} \ ; \\
\mathcal{P}_B &:= \overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4} \ ; \\
\mathcal{P}_C &:= \mathcal{I}_{A_0} \ ; \\
\mathcal{P}_D &:= \overline{\mathcal{I}}_{A_7} \ ;
\end{aligned}
\tag{6.38}
$$

be associated with the (normal form of the) state structures of, respectively, the deterministic interventions $S, M, \rho, \mathbb{1}$ (i.e., unit effects) of Alice, Bob, Charlie, and David.

The state structure of the shared environment W is then

$$[(\mathcal{I}_{A_2} \to \mathcal{I}_{A_5}) \to (\mathcal{I}_{A_1} \to \mathcal{I}_{A_6})] \to [(\mathcal{I}_{A_3} \to \mathcal{I}_{A_4}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_7})] \\ = \\ \overline{\mathcal{P}}_C \prec \overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_D \cup \overline{\mathcal{P}}_C \prec \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_D \,. \quad (6.39)$$

Interpreting the above as the projector associated with a state structure in which states are shared between the four parties, it is straightforward to see that Charlie's intervention is in the global past whereas David's is in the global future. These two parties are thus in a fixed causal structure; in the language of process matrix formalism, any four-partite process W belonging to this state structure is at most "3-causal" [153] meaning that the processes will decompose into terms in which at least three parties are fully ordered. Indeed, since Charlie has to be in the past and David has to be in the future, it is certain that Alice and Bob will be in between, so at least three parties will have a fixed ordering in signaling relations.

[153]: Abbott et al. (2017), *Genuinely multipartite noncausality*.

Finally, the dual pair at the fourth order is

$$(W, S \otimes M \otimes \rho \otimes \mathbb{1}) \in ((\overline{\mathscr{C}} \prec \overline{\mathscr{B}} \prec \overline{\mathscr{A}} \prec \overline{\mathscr{D}}) \cup (\overline{\mathscr{C}} \prec \overline{\mathscr{A}} \prec \overline{\mathscr{B}} \prec \overline{\mathscr{D}})) \times (\mathscr{A} \otimes \mathscr{B} \otimes \mathscr{C} \otimes \mathscr{D}) \,. \quad (6.40)$$

Once again, according to the effect state structure of effects, the states should have belonged to the fully signaling four-partite state structure $\overline{\mathscr{A}} \parr \overline{\mathscr{B}} \parr \overline{\mathscr{C}} \parr \overline{\mathscr{D}}$ and therefore featured a union of up to $4! = 24$ different fixed signaling directions (i.e., prec chains). Yet, because of the recurring accidental isomorphism due to the base state structure being the quantum states, there are only 2 of them. Still, because there are two of them, and because they are not contained in one another, the fourth order is the first order in this hierarchy of nested maps to have a non-fixed signaling direction. Because of that, it is additionally the first level in the hierarchy that does not descend into the first order; it is *genuinely* a higher-order transformation as not all states W can be decomposed as some succession of objects of lower orders. This is due to the multiple terms appearing in the normal form: nothing prevents operators from belonging to the affine hull of the union. I.e., operators whose description features terms supported on the two subspaces defined by the two prec chains of different signaling directions.

The follow-up question is whether the affine hull is convex. That is, whether the states W have a convex decomposition like

$$W = q \left(\overline{\mathcal{P}}_C \prec \overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_D\right) W + (1-q) \left(\overline{\mathcal{P}}_C \prec \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_D\right) W, \quad (6.41)$$

for some $q \in [0,1]$. If this is the case, one can conclude that the super-supermaps W have a non-fixed signaling direction but one that is just a classical mixture of two directions. If so, then the process is automatically causally separable and by consequent causal as in Definition 1.2.3.

6.1.4. The Quantum Super-supermap is a Multi-round Process Matrix

To see how the set of quantum super-supermaps admits processes that can have ICO and, even more, be non-causal, a small reinterpretation of them must be done. Because of all the isomorphisms at the lower levels, it so happens that Alice, Bob, Charlie, and David's interventions are each a quantum network as in Definition 2.3.1. Alice has two nodes; Bob, Charlie, and David have one. In addition, the nodes of Charlie and David are special: Charlie has trivial input, and David has trivial output. Hence, the set of valid super-supermaps W can also be seen as a special kind of process functionals from the quantum networks to probabilities. This class of processes was previously studied in the literature under the name Multi-round Process Matrix (MPM) according to the definition of Oreshkov and myself [1, 34].

[1]: Hoffreumon et al. (2021), *The Multi-round Process Matrix*.
[34]: Hoffreumon (2019), *Processes with indefinite causal structure in quantum theory: The Multi-Round Process Matrix*.

To fit the definition of an MPM, the signaling structure of W will therefore be treated with respect to parties seen as nodes of quantum networks. By treating the state structure of W as a functional normalized on networks, Alice's two interventions are effectively treated as if they were independent parties. Whence the natural question of the causal ordering of Alice's nodes with respect to the others. To treat this question, Alice's state structure is split: $\mathscr{A} = \mathscr{A}^{(1)} \prec \mathscr{A}^{(2)}$, her projectors is split accordingly $\mathcal{P}_A = \mathcal{P}_A^{(1)} \prec \mathcal{P}_A^{(2)}$, so that

$$\begin{aligned} \mathcal{P}_A^{(1)} &:= \overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \,; \\ \mathcal{P}_A^{(2)} &:= \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6} \,. \end{aligned} \tag{6.42}$$

This refinement of the projectors actually induces two normal forms for the projector of W. This is due to the same kind of redundancy as the example of Equation (5.52):

$$\begin{aligned} &(\overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A) \cap (\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B) = \\ &\qquad \left(\overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A^{(1)} \prec \overline{\mathcal{P}}_A^{(2)}\right) \cap \left(\overline{\mathcal{P}}_A^{(1)} \prec \overline{\mathcal{P}}_A^{(2)} \prec \overline{\mathcal{P}}_B\right) = \\ &\left(\overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A^{(1)} \prec \overline{\mathcal{P}}_A^{(2)}\right) \cap \left(\overline{\mathcal{P}}_A^{(1)} \prec \overline{\mathcal{P}}_A^{(2)} \prec \overline{\mathcal{P}}_B\right) \cap \left(\overline{\mathcal{P}}_A^{(1)} \prec \overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A^{(2)}\right) \,, \end{aligned} \tag{6.43}$$

and, using

$$\left(\overline{\mathcal{P}}_B \prec \overline{\mathcal{P}}_A^{(1)} \prec \overline{\mathcal{P}}_A^{(2)}\right) \cap \left(\overline{\mathcal{P}}_A^{(1)} \prec \overline{\mathcal{P}}_A^{(2)} \prec \overline{\mathcal{P}}_B\right) = \overline{\left((\overline{\mathcal{I}}_{A_3} \prec \mathcal{I}_{A_4}) \otimes (\overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \overline{\mathcal{I}}_{A_5} \prec \mathcal{I}_{A_6})\right)} \,, \tag{6.44}$$

the longer form can be reinjected at (6.36) to derive the following equivalent normal form to (6.37):

$$\begin{aligned} &[(\mathcal{I}_{A_2} \to \mathcal{I}_{A_5}) \to (\mathcal{I}_{A_1} \to \mathcal{I}_{A_6})] \to [(\mathcal{I}_{A_3} \to \mathcal{I}_{A_4}) \to (\mathcal{I}_{A_0} \to \mathcal{I}_{A_7})] \\ &= \\ &\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_3} \prec \overline{\mathcal{I}}_{A_4} \prec \mathcal{I}_{A_5} \prec \overline{\mathcal{I}}_{A_6} \prec \mathcal{I}_{A_7} \cup \\ &\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_3} \prec \overline{\mathcal{I}}_{A_4} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_5} \prec \overline{\mathcal{I}}_{A_6} \prec \mathcal{I}_{A_7} \cup \\ &\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_5} \prec \overline{\mathcal{I}}_{A_6} \prec \mathcal{I}_{A_3} \prec \overline{\mathcal{I}}_{A_4} \prec \mathcal{I}_{A_7} \,. \end{aligned} \tag{6.45}$$

Ergo, the normal form is multiply defined in that case because of redundancy in the description of the signaling structure. This general equivalence of two normal forms is interesting because it is a redundancy independent of the choice of base state structure. Moreover, this redundancy in Equation (6.43) can be inferred from logical reasoning alone: the state structure characterized by this projector allows for signaling from the nodes of Alice to the one of Bob and vice-versa. But, if Bob's node is allowed to be both before or after the Alice's nodes, he can also be in the middle of the two. The issue with these logically equivalent normal forms is that they may hide some simplifications when decomposing a process into pieces with fixed signaling directions, and it will be studied in greater detail in Subsection 6.3.1.

However, finding the minimal decomposition is not the point here. What is interesting for proving that there is an effective ICO is the structure of signaling between the nodes. The situation can be simplified by supposing that either Charlie and David acted already or with trivial systems. By doing so, the state and effect pair becomes

$$(W, S \otimes M) \in ((\overline{\mathscr{A}^{(1)}} \succ \overline{\mathscr{A}^{(2)}}) \succ \overline{\mathscr{B}}) \cup (\overline{\mathscr{A}^{(1)}} \prec \overline{\mathscr{A}^{(2)}}) \prec \overline{\mathscr{B}}) \times ((\mathscr{A}^{(1)} \prec \mathscr{A}^{(2)}) \otimes \mathscr{B}) \tag{6.46}$$

using the first normal form (6.37), and where each base state structure in the above is the state structure of quantum channels. This state and effect pair is exactly the one of the example in Reference [1]. Such MPMs where Alice has two rounds and Bob one can not only be causally separable, but they can also be *causally activated*, and they can violate the same causal inequality as the OCB process matrix. The reader is invited to refer to this paper for the details. Nevertheless, this roundabout way proves that the quantum super-supermaps are a special kind of MPM that can be non-causal and, as a consequence, that the super-supermaps allow for an indefinite causal order (ICO).

To summarize, the conclusion that can be drawn from this example is that higher-order quantum theory stands out compared to theories based on different state structures because it forbids two-way signaling in many simple cases. It was tried to obtain ICO by recursively nesting quantum transformations. But compared to the biased quantum theory example of Section 3.6, where ICO appears as soon as transformations are defined, the nesting depth of quantum transformations had to be threefold before obtaining a genuinely higher-order process that features an indefinite causal ordering. Still, this indefinite causal order is not genuinely multipartite [153]: it is only between party Alice and Bob that the effect manifests itself. Charlie and David are always well-localized in, respectively, the global past and future.

[153]: Abbott et al. (2017), *Genuinely multipartite noncausality*.

6.2. Fixed Causal Order in State Structures: Towards Causal Witnesses

Thus far, the notion of signaling has been applied to sets of transformations as a whole. Given a dual pair featuring more than one local intervention (pictured as multipartite effects that factor into a pure tensor product), the states are said to allow for more than one signaling direction as soon as the normal form featured the union of more than one non-redundant terms. But this is a too naive notion for indefinite causal order (ICO): having more than one signaling direction is only a necessary condition to achieve it. In particular, while a state structure may feature the union of two signaling directions, nothing guarantees that the terms in this union will not factorize into a convex mixture of the two signaling directions. This happens in the bipartite biased quantum theory presented in Subsection 3.6.1. These bipartite states may allow signaling from one side to the other, and they may even allow for a convex combination of it, but they cannot outperform a classical protocol as these bipartite states are never 'in superposition' of two causal orders; they are only mixed.

However, the tools of the projective characterization can also be adapted for the study of a given process. While the general problem of proving that a process will be non-causal as in Definition 1.2.3 is not linear, the projective methods can still detect certain cases where it is causal. Indeed, besides algebraic properties, using the disjunctive normal form is also motivated by the sufficient condition for causal separability it induces. Recall that causal separability is a decomposition of operators that guarantees its inability to violate a causal inequality and, thus, that it is causal. It was shown in a previous work on the Multi-round Process Matrix that the general question of defining causal separability for higher-order objects is actually non-trivial due to the parties' ability to use lower-order states as shared ancillary resources. This general issue is left open for future work. Nonetheless, the normal form induces a 'strong' notion of causal separability. This notion is the ability to decompose the operator into a convex sum of causally ordered terms. This is, therefore, a restricted notion insomuch as it precludes dynamical causal orders, which are typically causal but not caught by this notion.

To see how the normal form always implies such a decomposition, remark that a 'prec chain' has a fixed signaling direction. In addition to that, the intersection of several prec chains can never achieve two-way signaling[10]. These two properties mean that when a normal form is expressed as

$$\Gamma = \bigcup_{i=1}^{n} \Gamma_i \ , \tag{6.47}$$

each Γ_i is a valid projector that projects in a subspace whose signaling structure is, at most, a totally ordered chain. Therefore, each Γ_i defines a subspace of valid state structures with a fixed signaling direction. Following the original idea of Reference [35], these projectors can be used to decompose any state W into causally ordered pieces. There are two required properties for this decomposition to be possible: that the projectors are positive- and trace-preserving. Assuming that the base state structure is quantum theory, for which this is the case[11], the decomposition only amounts to applying each projector and summing the reduced operators to obtain the original one. Suppose the projector characterizing a state structure $\mathscr{A}$ has a normal form with n terms. Then, any operator W in the structure, i.e. for which $\Gamma(W) = W$, can potentially be decomposed as the following weighted sum:

$$W = \sum_{i}^{n} q_i \Gamma_i(W) \ , \tag{6.48}$$

where $\Gamma_i(W) \in \mathscr{A}$ for all i and $\sum q_i = 1$. In this expression, the weights q_i compensate for all the double-counting of terms that belong to the image of several Γ_i. Of course, such a decomposition is not guaranteed to exist, but if it does and with convex weights, then the operator has been

10: Suppose it does, then there exists $\mathcal{P} = \mathcal{P}_K \prec \mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_L$ and $\mathcal{P}' = \mathcal{P}_R \prec \mathcal{P}_B \prec \mathcal{P}_A \prec \mathcal{P}_S$ (with $\mathcal{H}^K \otimes \mathcal{H}^L \cong \mathcal{H}^R \otimes \mathcal{H}^S$, but not assuming that $\mathcal{H}^K \cong \mathcal{H}^R$), so that $\mathcal{P} \cap \mathcal{P}' = \dots(\mathcal{P}_A \invamp \mathcal{P}_B)\dots$. If the situation is different than $\mathcal{P}_A = \mathcal{P}_B = \mathcal{I}$, then this would contradict Equation 5.40 since the subsystems K, R, L, W are arbitrary and so can be taken as 1-dimensional systems.

11: The projectors are built from the CPTP math $\mathcal{I}$, which keep these properties under the tensor and the operations of the algebra as all of them result into valid projectors. However, as discussed in Appendix C.1.4, certain projectors on state structures, like the one on the biased quantum theory, are not CP. Consequently, this discussion –like the rest of the chapter– is restricted to the case of higher-order quantum processes.

decomposed into a convex sum of causally ordered terms. And therefore, the operator is causally separable.

Hence, the problem of the decomposition into chains with fixed signaling direction can also be phrased as an optimization problem under linear constraints, which can be solved using Semi-Definite Programming [41]. In the case of Equation (6.48), the optimization problem is to find the vector of encoding the $q_i's$ such that each $q_i \geq 0$ and the condition $W \geq 0$ is verified. As shown in [35], this method can be refined into an analog to the concept of an entanglement witness (see e.g. [98]), but which witnesses this strict notion of causal separability instead of entanglement. In general, this kind of witness is formulated as a Hermitian operator S, which lives in the state structure dual to the normal form because

[41]: Vandenberghe et al. (1996), *Semidefinite Programming*.

[98]: Horodecki et al. (2009), *Quantum entanglement*.

$$\begin{aligned}(S\,,\,W) &= \left(S\,,\,\bigcup_{i=1}^{n}\Gamma_i(W)\right)\\ &= \left(\overline{\bigcup_{i=1}^{n}\Gamma_i(S)}\,,\,W\right)\,.\end{aligned} \tag{6.49}$$

The problem then consists in minimizing $(S\,,\,W)$ under the constraint that S is a valid element of the dual, $\overline{\bigcup_{i=1}^{n}\Gamma_i(S)}$. This optimization problem can also be automatized by SDP, and if it returns a negative value, it witnesses that at least one of the weights is negative.

The utilization of this SDP problem in concrete cases, as well as its refinement, are left open as a future research direction, although abundant literature already exists on the matter (see [35, 40, 43–46, 91–93] for instance). Compared to these works, the original result of this thesis is to provide a systematic way to derive the linear constraints to be put into the SDP problem: these are none other than the projective methods. It should be mentioned that a concording approach was presented in [37] around the same time I presented mine [2]. In this work, they specifically study the projector algebra with the goal of formulating signaling and causal separability as general classes of SDP problems. While not going into as many details in the algebra, this complements the results of this thesis nicely as they concretely demonstrate how to use the projector algebra to devise SDP characterization methods.

[35]: Araújo et al. (2015), *Witnessing causal nonseparability*.

[40]: Abbott et al. (2016), *Multipartite causal correlations: Polytopes and inequalities*.

[43]: Feix et al. (2016), *Causally nonseparable processes admitting a causal model*.

[44]: Chiribella et al. (2016), *Optimal quantum networks and one-shot entropies*.

[45]: Bavaresco et al. (2019), *Semi-device-independent certification of indefinite causal order*.

[46]: Milz et al. (2022), *Resource theory of causal connection*.

[91]: Quintino et al. (2019), *Probabilistic exact universal quantum circuits for transforming unitary operations*.

[92]: Bavaresco et al. (2021), *Strict Hierarchy between Parallel, Sequential, and Indefinite-Causal-Order Strategies for Channel Discrimination*.

[93]: Bavaresco et al. (2022), *Unitary channel discrimination beyond group structures: Advantages of sequential and indefinite-causal-order strategies*.

[37]: Milz et al. (2023), *Transformations between arbitrary (quantum) objects and the emergence of indefinite causality*.

[2]: Hoffreumon et al. (2022), *Projective characterization of higher-order quantum transformations*.

6.3. Towards the Logic of No-Signalling

The projective characterization is a new method for analyzing the signaling relations in higher-order processes. The tools that have been developed in the previous chapter can be used to analyze the signaling structure of a wide class of interventions and processes. The first way to do so is at the level of a whole theory: given the state structures of the state and effect pair and their partitioning into parties, the possible signaling directions of the objects in this theory can be inferred. To do so, one considers the decomposition of the projectors into their normal form and searches to reduce it so as to extract the minimal number of fixed signaling directions. This approach can then answer specific questions using projector algebra, such as whether a given signaling scenario can be obtained and which operators should be used. It can also answer general questions using the lattice structure, like if a given state structure is the one allowing the most possible signaling directions, whether it is equivalent to another, and which are the closely related state spaces. The second way, which was briefly considered in the previous section, is to apply the tools to specific processes, since the projectors give linear constraints that can be formulated as SDP problems. For example, one can certify that a process has a fixed signaling direction or find the optimal process to beat a given causal inequality induced by a given game. These developments are the expected utilization of the formalism in the yet-to-come follow-ups on this thesis.

Yet, these prospective applications rely on processes or communication protocols known a priori that would benefit from the tools for optimization. But these prosaic implementations do not make the most of the logic of the projector algebra. Instead, the logic itself could be explored to find new interesting processes. As shown by Theorem 5.3.2, the fact that quantum networks and quantum combs are the same is purely accidental. One can then wonder how many more such accidental isomorphisms exist and if, like was the case for the switch and the OCB compared to the network formalism, there are interesting processes to be uncovered at the boundary between the exception and the rule. This is a very promising way to use the formalism, especially since this search can be automatized (for example, by using llprover as mentioned in Subsection 5.1.4).

In this section, two preliminary results on the logic of the algebra are sketched. These will hopefully pave the way for continuation works.

6.3.1. Isomorphisms in Tripartite Projectors and Interpretation of the Formulae as Logic

Isomorphic state structures complexify the use of the normal form. The issue is that terms that are written as a union of prec chains can sometimes reduce to fewer prec chains. To avoid drawing erroneous conclusions, it is important to discover the general ways such isomorphisms may happen. Another reason for searching these isomorphisms is to optimize the SDP search: a brute force algorithm to certify fixed causal order would test over the $n!$ possible combinations. However, if the goal is to test for multiple signaling directions, the optimization does not have to run over all possibilities. Proving that at least two terms are non-zero is sufficient, but how can such proof be practically realized from projectors? Beyond these aspects that are specific to the normal form, more general questions may be asked: Isomorphisms of state spaces offer a promising lead for identifying interesting processes, but are there that many? On the other hand, why not just classify all combinations of signaling relations for a given number of parties into equivalence classes, as was discussed in the conclusion of Chapter 5?

Some elements of answers are presented in this final part. As will be shown, as soon as more than two parties are involved, the situation becomes much more complicated: the normal forms can be non-unique in many different ways, something which makes the characterization and classification of the terms in the projector algebra and in its signaling sublattices harder. But on the other hand, it shows that there are many potentially interesting isomorphisms in the lattice.

Starting with a single party, the signaling lattice is simply a lattice of commuting projectors built from the various base state structures. Using the negation, intersection, and union completes the lattice of commuting projectors. The normal form is unique up to trivial redundancy since any single-partite projector is in normal form by definition, and the only isomorphisms that may happen is that the dual of a base state structure is equivalent to another base state structure.

The first non-trivially equivalent normal forms arrive in the bipartite case: the presence of more than one base projector induces a redundancy by permutation involving unions and intersections of the different bases. This is because the cap commutes with the prec, and the two of them distribute over the prec. As explained in the appendix, this can lead to artificially multiplying the number of normal forms through permutations over the same subsystem, as in the following (D.96):

$$\begin{aligned}
\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \prec \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) &= \left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \prec \left(\mathcal{P}_B^{'} \cap \mathcal{P}_B\right) \\
&= \left(\mathcal{P}_A^{'} \cap \mathcal{P}_A\right) \prec \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) \\
&= \left(\mathcal{P}_A \prec \mathcal{P}_B^{'}\right) \cap \left(\mathcal{P}_A^{'} \prec \mathcal{P}_B\right) .
\end{aligned} \tag{6.50}$$

This problem is specific to the study of state spaces involving potentially different base state structures, like the quantum to classical channels for example. This problem is set aside in the following by assuming that the projector algebra (as in Proposition 5.1.10) is built with a single base projector associated with each party.

In such a case, the lattice for two parties is quick to characterize: it only consists of the four terms in Equation (5.40), corresponding to the four possible ways of signaling. As the equation gives their relations in terms of interconnected unions and intersections, any other term in the lattice falls into one of these four equivalence classes of projectors. That is, any term is either equivalent to $\mathcal{P}_A \otimes \mathcal{P}_B$, $\mathcal{P}_A \prec \mathcal{P}_B$, $\mathcal{P}_A \succ \mathcal{P}_B$, or $\mathcal{P}_A \parr \mathcal{P}_B$, up to negation.

Inspired by this solution, one may try to generalize this to the multipartite case: once the no-signaling projector has been fixed, so that the position of the negations is known, the lattice splits according to the number of signaling relations. At the bottom of the lattice is the no-signaling projector, then the terms allowing for one-way signaling, then the two-way signaling ones, etc. Nonetheless, the situation is not that easy. Already in the tripartite case, this intuition proves wrong. Consider three parties A, B, and C, and the projectors that characterize their shared state such that there is only one direction of signaling between only two parties in total. This is the kind of scenario in which the states may allow signaling from Alice to Bob, but none from or to Charlie for instance. These states are characterized by projectors like $(\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C$ $((\mathcal{P}_A \succ \mathcal{P}_B) \otimes \mathcal{P}_C)$, $((\mathcal{P}_A \succ \mathcal{P}_C) \otimes \mathcal{P}_B)$, etc.

If these projectors encode a single two-partite signaling direction, their intersections should lead to no-signaling projectors as a generalization of Equation (5.40b). As it turns out, there is an oddity. Computing the various cases up to relabelling[12] leads to:

12: These formulae are proven from the definition, Equation (5.3), and using the algebra.

$$((\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C) \cap ((\mathcal{P}_A \succ \mathcal{P}_B) \otimes \mathcal{P}_C) = \mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C\,, \quad (6.51a)$$
$$((\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C) \cap (\mathcal{P}_A \otimes (\mathcal{P}_B \prec \mathcal{P}_C)) = \mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C\,, \quad (6.51b)$$
$$((\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C) \cap ((\mathcal{P}_A \succ \mathcal{P}_C) \otimes \mathcal{P}_B) = \mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C\,, \quad (6.51c)$$
$$((\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C) \cap (\mathcal{P}_A \otimes (\mathcal{P}_B \succ \mathcal{P}_C)) = \mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C\,, \quad (6.51d)$$
$$((\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C) \cap ((\mathcal{P}_A \prec \mathcal{P}_C) \otimes \mathcal{P}_B) \neq \mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C \quad (6.51e)$$

It appears that Alice being able to signal to Bob but not to Charlie while being able to signal to Charlie but not to Bob does not forbid her to signal. It is not the situation in which she can signal to Bob and Charlie, since

$$((\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C) \cap ((\mathcal{P}_A \prec \mathcal{P}_C) \otimes \mathcal{P}_B) \subsetneq \mathcal{P}_A \prec (\mathcal{P}_B \otimes \mathcal{P}_C)\,. \quad (6.52)$$

Moreover, it appears that this is a scenario for which the direction of signaling is important since:

$$\mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C \subsetneq ((\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C) \cap ((\mathcal{P}_A \prec \mathcal{P}_C) \otimes \mathcal{P}_B)\,; \quad (6.53a)$$
$$\mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C = ((\mathcal{P}_A \succ \mathcal{P}_B) \otimes \mathcal{P}_C) \cap ((\mathcal{P}_A \succ \mathcal{P}_C) \otimes \mathcal{P}_B)\,. \quad (6.53b)$$

To understand this situation in terms of signaling structure, it is easier to consider the De Morgan dual:

$$\mathcal{P}_A \parr \mathcal{P}_B \parr \mathcal{P}_C \supsetneq ((\mathcal{P}_A \prec \mathcal{P}_B) \parr \mathcal{P}_C) \cup ((\mathcal{P}_A \prec \mathcal{P}_C) \parr \mathcal{P}_B)\,; \quad (6.54a)$$
$$\mathcal{P}_A \parr \mathcal{P}_B \parr \mathcal{P}_C = ((\mathcal{P}_A \succ \mathcal{P}_B) \parr \mathcal{P}_C) \cup ((\mathcal{P}_A \succ \mathcal{P}_C) \parr \mathcal{P}_B)\,. \quad (6.54b)$$

In that case, there is a purely semantic explanation. Thus far, it has been said that an expression like $\mathcal{P}_A \prec \mathcal{P}_B$ 'allows for signaling from A to B'. But the statement of Lemma 3.5.3 is less ambiguously interpreted as 'forbids signaling from B to A'. This is a stricter way to phrase the condition, as allowed to signal does not necessarily mean that the party will signal, whereas forbidden to signal necessarily means that the party will not signal; A state characterized by $\mathcal{P}_A \prec \mathcal{P}_B$ *may* allow for signaling from A to B but it *never* allows for signaling from B to A. Accordingly, the reason for working with the De Morgan dual of these formulae is that these will be shorter, as they encode only a small number of interdictions. For example, $\mathcal{P}_A \parr \mathcal{P}_B \parr \mathcal{P}_C$ is 'no forbidden directions in the signaling structure' and $((\mathcal{P}_A \prec \mathcal{P}_B) \parr \mathcal{P}_C)$ is 'a single forbidden direction in the signaling structure: Bob can never signal to Alice'. To the contrary, $\mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C$ encodes six such interdictions. Whence, to interpret the formulae as reasonably short sentences, it is best to use De Morgan duality to go as high as possible in the signaling lattice, even if it means interpreting their dual instead.

In the above, the second line, Equation (6.54b), then reads 'no restriction on the signaling structure (the fully signaling projector, $\mathcal{P}_A \parr \mathcal{P}_B \parr \mathcal{P}_C$) is equal to forbidding Alice to signal to Bob ($(\mathcal{P}_A \succ \mathcal{P}_B) \parr \mathcal{P}_C$) or ($\cup$) to forbidding Alice to signal to Charlie ($(\mathcal{P}_A \succ \mathcal{P}_C) \parr \mathcal{P}_B$)'. Since she can always use the other situation to bypass the interdiction, there are effectively no restrictions on Alice's signaling. Proving the equivalence between the two formulae using the algebra amounts to this purely logical reasoning. On the other hand, Equation (6.54a) reads (from right to left) 'forbidding Charlie to signal to Alice ($(\mathcal{P}_A \prec \mathcal{P}_C) \parr \mathcal{P}_B$) or ($\cup$) forbidding Bob to signal to Alice ($(\mathcal{P}_A \prec \mathcal{P}_B) \parr \mathcal{P}_C$) is more restrictive ($\subsetneq$) than imposing no restriction'. This is indeed the case since if Bob or Charlie cannot signal to Alice, it precludes the situation in which Bob and Charlie are both signaling to Alice simultaneously. Hence, there is a non-trivial condition compared to (6.54b), but it is weaker than forbidding the two of them to signal to Alice ($\mathcal{P}_A \prec (\mathcal{P}_B \parr \mathcal{P}_C)$).

Interestingly, when changing the intersection to a union in Equation (6.53), the unexpected case happens in the other signaling direction[13]:

$$\mathcal{P}_A \prec (\mathcal{P}_B \otimes \mathcal{P}_C) = ((\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C) \cup ((\mathcal{P}_A \prec \mathcal{P}_C) \otimes \mathcal{P}_B)\ ; \tag{6.55a}$$

$$\mathcal{P}_A \succ (\mathcal{P}_B \otimes \mathcal{P}_C) \supsetneq ((\mathcal{P}_A \succ \mathcal{P}_B) \otimes \mathcal{P}_C) \cup ((\mathcal{P}_A \succ \mathcal{P}_C) \otimes \mathcal{P}_B)\ . \tag{6.55b}$$

13: It is also interesting to notice that the 'two-to-one' kind of signaling of the r.h.s. of Equation (6.55b) is the kind of signaling on which relies the so-called 'Lugano' process [77, 154].

[77]: Baumeler et al. (2016), *The space of logically consistent classical processes without causal order*.

[154]: Baumeler et al. (2014), *Perfect signaling among three parties violating predefined causal order*.

The interpretation is again easier from the dual:

$$\mathcal{P}_A \prec (\mathcal{P}_B \parr \mathcal{P}_C) = ((\mathcal{P}_A \prec \mathcal{P}_B) \parr \mathcal{P}_C) \cap ((\mathcal{P}_A \prec \mathcal{P}_C) \parr \mathcal{P}_B)\ ; \tag{6.56a}$$

$$\mathcal{P}_A \succ (\mathcal{P}_B \otimes \mathcal{P}_C) \subsetneq ((\mathcal{P}_A \succ \mathcal{P}_B) \parr \mathcal{P}_C) \cap ((\mathcal{P}_A \succ \mathcal{P}_C) \parr \mathcal{P}_B)\ . \tag{6.56b}$$

The first line states that 'Bob and Charlie being no-signaling to Alice is equivalent to Bob being no-signaling to Alice and Charlie being no-signaling to Alice'. The second line states that 'Alice being no-signaling to Bob and Charlie is a stronger condition than her being no-signaling to Bob and her being no-signaling to Charlie'. Indeed, in that latter case, nothing is said about the scenarios where Alice signals to both simultaneously.

This shows that there are two different intermediate situations between the one-way and no-signaling projectors. The interpretation of which depends on the signaling direction, yet both rely on the idea of simultaneous signaling. This hints that for more parties, more intermediate situations will arise as not only the bipartite coincidences must be considered but also the tripartite, quadripartite, etc.

The fact that Equation (6.55b) is a 'little more' than the union of two one-way signaling connections is actually necessary for the decomposition of the prec chains:

$$\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C \supsetneq (\mathcal{P}_A \prec (\mathcal{P}_B \otimes \mathcal{P}_C)) \cup (\mathcal{P}_A \otimes (\mathcal{P}_B \prec \mathcal{P}_C))\ ; \tag{6.57a}$$

$$\mathcal{P}_A \succ \mathcal{P}_B \succ \mathcal{P}_C = (\mathcal{P}_A \succ (\mathcal{P}_B \otimes \mathcal{P}_C)) \cup (\mathcal{P}_A \otimes (\mathcal{P}_B \succ \mathcal{P}_C))\ . \tag{6.57b}$$

The right-hand term of the first line, which can be further decomposed into the union of three one-way signaling bipartite compositions using Equation (6.55a), shows that a prec chain is more than that. This means that the decomposition of prec chains into smaller prec chains is also impacted by these "coincidental" isomorphisms. The second line of the above is in line with the intuition: like the two previous examples, (the De Morgan dual of) Equation (6.57b) states that 'Alice being no-signaling to Bob and Charlie while Bob is no-signaling to Charlie ($\mathcal{P}_A \succ \mathcal{P}_B \succ \mathcal{P}_C$) can be decomposed into Alice being no-signaling to Bob and Charlie ($\mathcal{P}_A \succ (\mathcal{P}_B \otimes \mathcal{P}_C)$) and into Bob being no-signaling to Charlie independently of Alice ($\mathcal{P}_A \otimes (\mathcal{P}_B \succ \mathcal{P}_C)$)'. By contrast, the first line does not decompose as such: (the dual of) Equation (6.57a) states that 'Charlie being no-signaling to Bob and Alice while Bob is no-signaling to Alice ($\mathcal{P}_A \succ \mathcal{P}_B \succ \mathcal{P}_C$) is a weaker constraint that Bob and Charlie being no-signaling to Alice ($\mathcal{P}_A \prec (\mathcal{P}_B \otimes \mathcal{P}_C)$) and Charlie being no-signaling to Bob independently of Alice ($\mathcal{P}_A \otimes (\mathcal{P}_B \prec \mathcal{P}_C)$)'.

The further consequence is that one cannot obtain a no-signaling channel by simply combining two chains with opposite signaling directions. This is again an oddity in the expected behavior when combining prec chains (all these equations but the last are proven by associativity and using Equation (5.40)):

$$
\begin{aligned}
(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_B \prec \mathcal{P}_A \prec \mathcal{P}_C) &= (\mathcal{P}_A \otimes \mathcal{P}_B) \prec \mathcal{P}_C\,, && \text{(6.58a)}\\
(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_A \prec \mathcal{P}_C \prec \mathcal{P}_B) &= \mathcal{P}_A \prec (\mathcal{P}_B \otimes \mathcal{P}_C)\,, && \text{(6.58b)}\\
(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_B \prec \mathcal{P}_C \prec \mathcal{P}_A) &= \mathcal{P}_A \otimes (\mathcal{P}_B \prec \mathcal{P}_C)\,, && \text{(6.58c)}\\
(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_C \prec \mathcal{P}_A \prec \mathcal{P}_B) &= (\mathcal{P}_A \prec \mathcal{P}_B) \otimes \mathcal{P}_C\,; && \text{(6.58d)}\\
(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_C \prec \mathcal{P}_B \prec \mathcal{P}_A) &\neq \mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{P}_C && \text{(6.58e)}
\end{aligned}
$$

Using the decomposition (6.57b) and distributing over the unions, this latter equation reduces to

$$(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_C \prec \mathcal{P}_B \prec \mathcal{P}_A) = ((\mathcal{P}_A \otimes \mathcal{P}_B) \prec \mathcal{P}_C) \cap (\mathcal{P}_A \succ (\mathcal{P}_B \otimes \mathcal{P}_C))\,, \tag{6.59}$$

which can be further simplified[14] into the first 'coincidental' example (6.53a) (up to the interchange $A \leftrightarrow B$),

14: Proven by direct application of Equation (5.3).

$$(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cap (\mathcal{P}_C \prec \mathcal{P}_B \prec \mathcal{P}_A) = ((\mathcal{P}_A \succ \mathcal{P}_B) \otimes \mathcal{P}_C) \cap (\mathcal{P}_A \otimes (\mathcal{P}_B \prec \mathcal{P}_C))\,. \tag{6.60}$$

This means that the states allowing signaling from B to A and C simultaneously belong to two oppositely directed prec chains in which they are in the middle. On the other hand, the fact that the intersection is not the no-signaling subspace means that

$$(\mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C) \cup (\mathcal{P}_C \prec \mathcal{P}_B \prec \mathcal{P}_A) \subsetneq \mathcal{P}_A \parr \mathcal{P}_B \parr \mathcal{P}_C\,. \tag{6.61}$$

So, these two prec chains do not exhaust all the signaling directions in the fully signaling subspace: at least three prec chains should be used to give a normal form to $\mathcal{P}_A \parr \mathcal{P}_B \parr \mathcal{P}_C$. Hence, at least three signaling directions must be tested before concluding that any operator in this state structure has a single signaling direction.

To conclude, this observation can be phrased as part of a bigger problem: given a projector Γ, what is the minimal number of prec chains that can be used to decompose it as a union? And what are those chains? Knowing the answer to that question would be very useful to provide the minimal amount of terms used to decompose any operators to test for a single signaling direction. Whether this problem has a general answer and whether this answer would be useful to formulate a more efficient SDP test of non-fixed signaling structure are questions left open for future work.

6.3.2. A Type System Based on the Prec and BV-Logic

According to the discussion in Section 4.3, the projectors under the transformation connector $\rightarrow$ form a type system, first studied by Bisio and Perinotti [10, 11]. As was mentioned in Subsection 5.1.4, this type system corresponds to a fragment of a logic system called multiplicative linear logic (MLL) [33, 146]. Yet, according to the discussion in Section 5.2, the $\to$ connector is an operation derived from the one-way signaling connector $\prec$; it can be removed from the equations using a combination of the prec with the single-projector operations $\cap, \cup, \overline{\cdot}$. By doing so, the underlying logic is promoted to (almost[15]) BV-logic [36, 149, 152], a model of multiplicative additive linear logic enriched by a non-commutative connector $\prec$ [147].

Considering these two points, one can wonder whether the $\prec$ can be used as the constructor of a new type system. The ensuing type system is a second direction for possible continuation works concerning the study of the logic of signaling connections in state structures. Here are briefly presented some preliminary considerations about this possible type system.

Starting from a set of base projectors $\mathcal{P}_A, \mathcal{P}_B$,... the type theory is built from a connector relating an output type with an input type, built by negating the input of the $\prec$. Following References [147, 149], this connector is called the *seq* (for *sequent*) and noted as $\triangleleft$:

$$\mathcal{P}_A \triangleleft \mathcal{P}_B := \overline{\mathcal{P}}_A \prec \mathcal{P}_B \,. \tag{6.62}$$

Consequently, compared to Perinotti and Bisio's type system, this constructor is related to the CJ representation of the set of one-way signaling maps rather than the set of all admissible maps. Because of Lemma 5.3.1, the linear type system collapses into the type system built on the prec as soon as the base projectors are the identity. Also because of this lemma, remark that when the base projectors are the depolarizing ones, both type systems collapse to a third one, based on the no-signaling composition.

The $\triangleleft$ types obey the same basic rules on single projectors as the $\to$ types. That is, every type is equivalent to a seq from the trivial system:

$$\mathcal{P}_A = 1 \triangleleft \mathcal{P}_A \,, \tag{6.63}$$

and the negation is recovered as the seq towards the trivial system:

$$\overline{\mathcal{P}}_A = \mathcal{P}_A \triangleleft 1 \,. \tag{6.64}$$

However, as soon as multipartite formulae are involved, the properties diverge. While the seq is also non-associative,

$$(\mathcal{P}_A \triangleleft \mathcal{P}_B) \triangleleft \mathcal{P}_C \neq \mathcal{P}_A \triangleleft (\mathcal{P}_B \triangleleft \mathcal{P}_C) \,, \tag{6.65}$$

it does not obey the uncurrying rule (D.68):

$$\mathcal{P}_A \triangleleft (\mathcal{P}_B \triangleleft \mathcal{P}_C) \neq (\mathcal{P}_A \otimes \mathcal{P}_B) \triangleleft \mathcal{P}_C \,. \tag{6.66}$$

This is because

$$\mathcal{P}_A \triangleleft (\mathcal{P}_B \triangleleft \mathcal{P}_C) = \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B \prec \mathcal{P}_C = (\mathcal{P}_A \prec \mathcal{P}_B) \triangleleft \mathcal{P}_C \,, \tag{6.67}$$

so the equivalence is only true when $\mathcal{P}_B = \mathcal{I}_B$ or when $\mathcal{P}_A = \mathcal{D}_A$. Therefore, to be in a situation where the two types systems are equivalent. For the same reason, the tensor product cannot be recovered from the $\triangleleft$

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.
[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.
[146]: Girard (1987), *Linear logic*.

15: As explained in Appendix D.3.5, the projector algebra is not exactly a model of logic, but still it is very close to BV-logic [149], which is a sublogic of pomset logic [147] in which the $\prec$ is self-dual. This property is indeed verified by the projector algebra, Equation (5.38d). Remark that the projector algebra can become a model of logic if the additive connectors are changed [36].

[36]: Simmons et al. (2022), *Higher-order causal theories are models of BV-logic*.
[149]: Guglielmi (2007), *A System of Interaction and Structure*.
[152]: Blute et al. (2012), *Deep Inference and Probabilistic Coherence Spaces*.
[147]: Rétoré (1993), *Réseaux et Séquents Ordonnés*.

connector alone, except when equivalent to quantum theory. From these preliminary considerations, the type system enriched by adding the $\cap$ connective looks more promising as it recovers all the other connectors in the algebra without reducing to the type system of transformations.

Briefly summarized, the main observation is that this type system is similar to Bisio and Perinotti's linear type theory for single systems but significantly departs from it in the multipartite case when the base theory is not quantum theory. The difference between the introductory example of this chapter and Theorem 5.3.2 can be interpreted using this fact: the type system of quantum combs reduces to the type system based on the seq, whereas the general type system is based on the $\rightarrow$, and, eventually, the two constructions must diverge. This divergence is embodied by the super-supermap being an MPM instead of being a comb.

Conclusion

> Look into my heart. I know that everything you need is in there. It has to be. I never sold my soul to anyone! It's mine, it's human! You take from me what it is I want... it just can't be that I would want something bad! Damn it all, I can't think of anything, except those words of his... 'HAPPINESS FOR EVERYBODY, FREE, AND NO ONE WILL GO AWAY UNSATISFIED!'
>
> Arkady and Boris Strugatsky (1972), Roadside Picnic*

Higher-order quantum processes are a rich theoretical landscape of which this thesis only scratched the surface. Starting from the observation that nested quantum transformations lead to processes with non-fixed signaling structures, this thesis has tried to present a comprehensive characterization method for assessing the signaling structure of general processes. With the tools developed in this thesis, it is now possible to devise and characterize any kind of higher-order quantum process, either by requiring a specific signaling structure or by requiring a specific input and output. This thesis also put forward the idea that higher-order processes are characterized by the intrinsic logic of their signaling relations: the processes are defined by their state space, which is defined by a projector, which in turn are defined as the elements of a Boolean lattice. These nested relations summarize the main concepts in this thesis: the admissible *processes*, Definition 1.1.1, are defined by their *state structures*, Definition 3.2.2, which correspond to *projectors on operator systems*, Definition 3.2.7, which are elements of the projector algebra, Definition 5.1.1. The contribution of this thesis lies in bridging the gap between them: the signaling structure of processes *is* the logic of how projectors are composed.

More specifically, in order to achieve this goal, some elements presenting the process formalism were gathered into Chapter 1. The *process* was presented as a collection of probabilistic assignments on every possible intervention made by a party during the exchange of a system with her environment. While trying to be faithful to the original work [5], the presentation tried to articulate the formalism as a generalized probabilistic theory [50, 79], i.e. without assuming a too specific form of the intervention. The aim was to convey the main idea of the process formalism –that the interventions of local parties are the only constraint on their vicinity– under a probabilistic approach. Using this probabilistic approach, the notion of signaling, Definition 1.2.1, as well as of causal correlations, Definition 1.2.3, can be introduced independently of the theory that rules the local interventions. In the remainder of the chapter, the formalism was applied to local quantum theory. The single-partite process scenario was modeled as a quantum operation, representing the party's intervention, and the process functional, representing the process, meaning the party's global environment.

This scenario was then extended to the multipartite case in Chapter 2. An original point sustained in this chapter is that if the *admissible* environments and interventions are everything compatible with the probabilistic interpretation of the process, then quantum interventions may and will generally be higher-order if the environment lets them do so. To represent all these higher-order interventions on the same footing, the tool of the Choi-Jamiołkowski correspondence [85, 86, 89] was introduced as a means to substitute all linear maps by operators. To close this section, some landmarks in the development of the theory of higher-order quantum transformations were reviewed: the quantum combs as the first formalism based on the idea of admissible quantum transformations, the quantum switch as the first example of a higher-order process with indefinite causal ordering, and the process matrix as the first formalism leading to non-causal correlations.

At that point, the research question is formulated as 'What difference in the definition of these processes leads to the difference in their signaling relations?' and 'How to characterize any class of higher-order processes, and when does a non-fixed signaling direction arise?'.

* Quoting the 1977 translation by Antonina W. Bouis; this novel is better known under the name of its movie adaptation by Andrei Tarkovsky, *Stalker*.

This led to the technical core of the thesis, Chapters 3 and 5. Firstly, in Chapter 3, the main concepts of a state structure, of a resolution, and of an admissible higher-order transformation were defined as the abstract modelization of, respectively, a deterministic intervention, a probabilistic one, and an evolution; see Definitions 3.2.2, 3.2.4, and 3.4.3. Then, they were used to characterize the processes of measurement and evolution as state structures related to the state structure they are acting on, Theorems 3.3.2 and 3.4.1. This led to two general considerations: first, that all classes of higher-order processes correspond to a state structure and that higher-order transformations correspond to composite state structures. Second, the projectors on operator systems (Definition 3.2.7) involved in the characterization of state structures encode most of the information about it. Whence, the characterization can be shifted to the study of these projectors, and the characterization of higher-order processes to the study of how these projectors are composed. Doing so, the notion of signaling was used as a guiding principle to define the various ways of composting two projectors, and, by extension, to compose state structures; see Lemma 3.5.3. This led to the realization that there are essentially four such compositions, $\otimes, \prec, \succ, \parr$ (see Definition 3.5.2), corresponding to the four possible signaling directions that a bipartite object shared by Alice and Bob may allow: none, A to B, B to A, both. In addition, the measurement of state structures was also defined as a new state structure obtained by applying an operation on the original one, called the negation $\overline{\cdot}$. Using the negation, it was made clear that the transformation is a bipartite state structure composed using a mix of the two-way signaling composition $\parr$ and the negation. As a proof of concept, the toy model of *biased quantum theory* was used to demonstrate the use of these connectors. As expected, the biased quantum channels allowed two-way signaling. On top of that, they even featured ICO.

This led to the natural question of ‘Why do the quantum channels have one-way signaling then?’. This question was motivated by several examples presented in Chapter 4. Then, Chapter 5 delved into the structure of the algebra of projectors defined under the set of operations $\{\overline{\cdot}, \otimes, \prec, \parr\}$ to provide an answer. Two new connectives, the intersection $\cap$ and the union $\cup$, were added so their utilization induces an order relation on the set of projectors. The algebra of projectors is a lattice under these operations, and the partial order amounts to gauging whether the state structure represented by a projector allows for more signaling directions than another, Proposition 5.1.10. The related concept of a signaling lattice was defined as the sublattice that encompasses all projectors ‘comparable to a given projector’, Proposition 5.1.11. (Here, by ‘comparable’, it is meant the projectors that characterize the state structure of processes in which inputs and outputs are of the same nature.) It was also noted that, as a specific kind of Boolean lattice, the projector algebra was very close to a model of logic called BV-logic [149], corroborating an observation made by another group working on the matter [36]. A final technical tool, the normal form Definition 5.2.1, was introduced at that stage. This normal form allows to write all projectors so that the allowed signaling structures of the set of higher-order processes they characterize can be read directly from it. As a proof of concept, the proof that quantum combs have a fixed signaling direction was rederived in Theorem 5.3.2 (it was first derived in Reference [11]). Since it is the kind of higher-order transformation whose simplest instances are the quantum channels, this answered the question.

Finally, Chapter 6 presented another example of nested quantum transformations to insist on the idea that quantum theory is tame in that certain classes of supermaps avoid having non-fixed signaling direction because of some accidental isomorphisms. However, the point is that the very concept of an admissible higher-order transformation is based on the *two-way signaling* composition which, apart from the special case of quantum combs, would necessarily lead to indefinite causal order in any class of processes at a high enough order. This corroborates an observation made by yet another group [37]. In this chapter were also mentioned some future prospects for the formalism developed through this thesis.

The first of these prospects is the adaptation of numerical methods to automatize the search for new interesting higher-order processes. The first aspect of this search will consist of adapting the already existing semi-definite programming methods [37] to the broader context of the projective characterization. The first necessary task towards this goal will be to adapt the definition of causal separability to higher-order processes. Convex decomposition into causally ordered pieces is indeed a too strong condition to impose as a definition of causal separability since it cannot take into account the processes with dynamical causal order (i.e., the kind of processes in which the ordering of the parties classically depends on the actions of the other parties). In its own respect, the definition of causal separability for higher-order processes is an interesting endeavor since it is the only way to talk about an *indefinite causal order* meaningfully. This faces the risk of becoming an order-dependent concept: nothing prevents a priori that a process is causally separable provided that the parties are not allowed to perform interventions of a higher order. For example, certain MPMs are causally separable so long that the parties cannot use quantum channels as side resources but only quantum states.

Devising SDP to develop causal witness and possibly to bound the set of correlations attainable by a process is one way that numerical methods may be useful for higher-order theories. Another way, mentioned in Subsection 5.1.4, is to use the linear-logic-like structure of the algebra of projectors. It is reasonable to consider an adaptation of automated provers like llprover for the task of finding interesting isomorphisms of signaling structure, as was first considered in [33, 36]. This would provide an ad hoc complement to the SDP algorithms: the linear logic part can automatize the search of classes of processes that may be interesting, and the SDP part can automatize the search for causally non-separable processes within this class.

The other main direction mentioned in Chapter 6 is to study the algebra of projectors as a logic model. One can wonder how often two differently defined structures with a common no-signaling subset are equivalent. And how often do the structures based on the transformation connector reduce into a normal form to structures based on a prec/seq connectors? The question of classifying the isomorphic constructions is also left open. A systematic way to obtain results in that direction would be to characterize the points in the signaling lattice for a given number of parties. I.e., to characterize the equivalence classes of formulae. As was shown in Subsection 6.3.1, this task is already non-trivial for three parties. However, the use of automated provers can also help in that case.

Other aspects have also not been addressed concerning the algebra of projectors. The first of which is the non-uniqueness of the normal form. Some criterion must always exist to favor a certain normal form over an equivalent one; there must be a possible refinement of the definition to make it unique. Remark that changing the definition to incorporate the redundancy of normal form cannot be done without a prior identification of all the possible ways that two normal forms can be equivalent, so this task is intrinsically linked with the study of the logic of the algebra.

Another issue that was overlooked is the one of the composition of types in the sense of References [11, 88]: knowing the projector characterizing a transformation is sufficient to know what will be the projector characterizing its set of outputs. However, if the input state structure is now restricted to a subset with a different projector, is there a rule to apply on the projector of the transformation to get the projector on the possibly restricted output state structure? For example, an n-comb takes $(n-1)$-combs as inputs and outputs a 1-comb. One can ask what happens when the inputs are restricted to the no signaling subset: when $(n-1)$ 1-combs are plugged into an $n-$comb instead, is the output set still the full set of 1-combs?

When setting up the concepts necessary to define the notion of a resolution, a frame function, and an admissible transformation, a certain amount of a priori hypothesis has been imposed. In particular, while these hypotheses can be taken as definitions, it is still worth discussing their physical justification.

The assumption that any element of any resolution corresponds to an actual physical realization could be justified by devising a procedure to realize them concretely in a lab, as can be done with POVMs and quantum instruments [76]. But this issue is part of the more general issue of realizability: given a state structure, is it possible to realize each of its elements in a lab experiment? Is there a systematic way to relate these abstract mathematical objects to a circuit realization, as is the case for quantum combs [9] and for time-delocalized subsystems [28, 30]? One may expect that the normal decomposition of general state structures into the union of several one-way signaling structures will prove useful for answering this question.

On the other hand, it is not clear how the frame functions can get rid of the hypothesis of Gleason-kind non-contextuality that was required for their definition, but a concrete first step would be to define a notion of higher-order purification so to make the connection with previous works on POVM contextuality [58, 59, 115, 116, 155]. Remark that this notion of purification may also help with the realization of the resolutions, in the same way that Naimark and Stinespring dilations are involved in the proof of realizability for POVM and instruments (see e.g., [52, 90, 156]).

Regarding the definition of admissible transformation, there is a case to be made against the fact that they allow two-way signaling albeit quantum channels are not two-way signaling transformations. But this is not the concern at hand, another less fundamental hypothesis was still slipped in the definition: that local parties between which the admissible transformation can be realized should in principle be able to share any no-signaling composite state, including the entangled ones. As mentioned in the text, this hypothesis relies on the difference between no-signaling and localizable bipartite processes [61] (which is a concept that is yet to be defined in general for higher-order processes). Because certain no-signaling channels require communication to be obtained (e.g., [65]), does it make sense to speak about *local* parties in a process? The derivation of the theorems without assuming the parties to have access to the full affine hull of the set of no-signaling states would close that loophole. See in particular Reference [36] where this issue is discussed in much more depth than in this thesis.

Another enjoyable aspect of the broad definition of state structure that was under-considered in the thesis is their ability to represent other higher-order theories than higher-order quantum theory. For instance, the biased quantum theory is a new class of theories that can be considered by taking a different base projector than the identity. Note that in the biased quantum theory examples already there were interesting connections still to be explored see the conclusion of Section 3.6. However, a task of interest that was not considered at all in the thesis is to devise classical/quantum mixed theories. The investigation of the state structure of quantum to classical channels and related higher-order construction is a standalone path that can also be interesting for future works on decoherence or on classical and quantum control of processes as in Reference [157].

Finally, the operational/general probabilistic theory [78, 79] presentation of the background is also a whole topic that has not been studied yet. The interpretation of the signaling lattice as a class of GPT seems like a good starting point. Another more general approach that was not considered in this thesis albeit lurking in the margins is the categorical theoretic treatment. Many of the results presented in this thesis were independently attained by another group [33, 36] using the formalism of category theory. Providing a Rosetta stone between the two approaches, as well as phrasing the projective characterization as a categorical construction looks a very promising path for future research.

Appendix

Appendices to Chapter 1 A.

They were funny-looking pictures. And I did think consciously: Wouldn't it be funny if this turns out to be useful and the Physical Review would be all full of these funny looking pictures. It would be very amusing.

R. Feynman on Feynman diagrams*

The ultimate goal of life, the universe, and everything is of course to replace horrible symbolic manipulation by diagrams.

Coecke and Kissinger (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning* [81]

Boy, that escalated quickly.

Ron Burgundy (Will Farrel) in: Anchorman: The Legend of Ron Burgundy (2004) (dir. Adam McKay, DreamWorks Pictures)

A.1. Mathematical Methods

In this section, some mathematical methods are reviewed with the particular goal of presenting the notation in more detail. These are only a few scattered facts about the theory of Hilbert-Schmidt spaces and about the theory of probabilities that may be useful to be reminded of. Besides the one cited, the main sources used for this section are References [96, 107, 124, 159–161].

[96]: Roman (2008), *Advanced Linear Algebra*.
[107]: Hiai et al. (2014), *Introduction to Matrix Analysis and Applications*.
[124]: Roman (2017), *An Introduction to the Language of Category Theory*.
[159]: Hall (2013), *Quantum Theory for Mathematicians*.
[160]: Pearl (2009), *Causality*.
[161]: Wackerly et al. (2008), *Mathematical Statistics with Applications*.

A.1.1. Operators on a Hilbert Space

Operators on a Hilbert space, which are linear maps between isomorphic input and output spaces identified together, play a special role in this thesis. In addition to the properties of linear maps, some extra concepts can be defined on them, like the positivity or the trace. This section briefly reviews these two notions and related concepts for reference.

Definition A.1.1 *An operator $V \in \mathcal{L}(\mathcal{H})$ is* ***Positive SemiDefinite*** *(PSD) if, for all vectors $|\psi\rangle \in \mathcal{H}$, the condition*

$$\langle\psi|V|\psi\rangle \geq 0 \tag{A.1}$$

* Quoted from Lancaster and Blundell (2014), *Quantum Field Theory for the Gifted Amateur* [158], chapter 19, p. 175.

*holds. If the inequality is strict, $\langle\psi|V|\psi\rangle > 0$, then it is **positive definite**.*

For simplicity, 'positive' will be used to say 'positive semi-definite' and so will the shorthand notation $V \geq 0$ to indicate that an operator is positive. Actually, positivity, as well as most of the properties of operators, are all encoded in their *spectrum*.

Proposition A.1.1 (Classification of operators.) *An operator $V \in \mathcal{L}(\mathcal{H})$ is **normal** if and only if it commutes with its adjoint,*

$$V^\dagger V = V V^\dagger \, . \tag{A.2}$$

*A normal operator can be **diagonalized**,*

$$V = \sum_{i=1}^{d} \lambda_i \, |e_i\rangle\!\langle e_i| \ , \tag{A.3}$$

*where $\{|e_i\rangle\}$ is an orthonormal basis (ONB) of $\mathcal{H}$ and the set of coefficients $\lambda_i \in \mathbb{C}$ is called the **spectrum** of V, and the coefficients its **eigenvalues**. Then,*

- *V is **self-adjoint** if and only if its spectrum is real, $\forall i: \ \lambda_i \in \mathbb{R}$;*
- *V is **unitary** if and only if its spectrum lies on the unit circle, $\forall i$: $|\lambda_i| = 1$;*
- *V is **positive** if and only if its spectrum is (real and) positive, $\forall i$: $\lambda_i \geq 0$;*
- *V is a **projector** if and only if all its eigenvalues are equal to either 1 or 0 or, equivalently, if and only if it verifies*

$$V^2 = V \, ; \tag{A.4}$$

- *V is the **identity (operator)**, noted $\mathbb{1}$, if and only if all its eigenvalues are equal to 1.*

When operators are defined on tensor-composite Hilbert spaces like $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$, a weaker notion of positivity which depends on the factorization can also be defined. It is called *positivity on pure tensors* [123].

[123]: Barnum et al. (2005), *Influence-free states on compound quantum systems*.

Definition A.1.2 (Positive on Pure Tensors) *An operator $W \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ is **positive on pure tensors** (POPT) if and only if, for all vectors $|\psi\rangle \in \mathcal{H}^A$ and $|\phi\rangle \in \mathcal{H}^B$, the condition*

$$\left(\langle\psi| \otimes \langle\phi|\right) W \left(|\psi\rangle \otimes |\phi\rangle\right) \geq 0 \tag{A.5}$$

holds.

A related concept is the *trace* of an operator.

Definition A.1.3 (Trace.) *Let $M \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ be a bipartite operator, let $\{|e_i\rangle\}$ be a basis of $\mathcal{H}^A$ and $\{|f_j\rangle\}$ be one of $\mathcal{H}^B$. The linear map*

$\mathrm{Tr} : \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right) \to \mathbb{C}$ *defined as*

$$Tr\left[M\right] := \sum_i \sum_j \langle e_i \otimes f_j | M | e_i \otimes f_j \rangle \tag{A.6}$$

is called the ***Trace*** *of* M.
In addition, the linear maps $\mathrm{Tr}_A : \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right) \to \mathcal{L}\left(\mathcal{H}^B\right)$ *and* $\mathrm{Tr}_B : \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right) \to \mathcal{L}\left(\mathcal{H}^A\right)$ *defined as*

$$Tr_A\left[M\right] := \sum_i \langle e_i | M | e_i \rangle \ , \tag{A.7a}$$

$$Tr_B\left[M\right] := \sum_j \langle f_j | M | f_j \rangle \ , \tag{A.7b}$$

are respectively called the ***partial traces*** *over subsystems* A *and* B.

Proposition A.1.2 (Properties of the Trace.)

1. *The trace is basis-independent ;*
2. *The trace of a normal operator is equal to the sum of its eigenvalues ;*
3. *The trace can be used to define a norm on the space of operators, called the* ***trace norm****:*
$$||V||_1 := Tr\left[\sqrt{VV^\dagger}\right] \ ; \tag{A.8}$$
When restricted on the set of positive operators this norm is the trace itself:
$$V \geq 0 \Rightarrow ||V||_1 = Tr\left[V\right] \ ; \tag{A.9}$$
4. *The partial traces commute with each other*
$$Tr_A\left[Tr_B\left[V\right]\right] = Tr_B\left[Tr_A\left[V\right]\right] = Tr_{AB}\left[V\right] := Tr_{AB}\left[V\right] \ ; \tag{A.10}$$
5. *The partial traces commute with the tensor product*
$$Tr_{AB}\left[V \otimes N\right] = Tr_A\left[V\right] Tr_B\left[N\right] \ . \tag{A.11}$$

A.1.2. Functionals and the Dual Space

A *functional* is a linear map from a Hilbert space to its base field, $f : \mathcal{H} \to \mathbb{C}$. The (algebraic) dual space of a Hilbert space, noted $\mathcal{H}^*$ (or, alternatively, $\mathcal{L}\left(\mathcal{H}, \mathbb{C}\right)$ using the notation for spaces of linear maps of this thesis), is the vector space of all linear functionals on it. Note that to talk about a Hilbert space with respect to its dual, the terms *direct* or *base* Hilbert space are used. By a well-known theorem of Riesz and Fréchet, and in the case of finite-dimensional Hilbert space, the dual space is also a Hilbert space which moreover is isometrically anti-isomorphic to the direct[1].

1: In the infinite-dimensional case, it is also true for the *continuous* dual space, i.e. the space of all continuous linear functionals on the direct space. In the finite-dimensional case, these two notions happen to coincide, so there is no need for disambiguation when mentioning the dual space.

Theorem A.1.3 (Riesz Representation Theorem) *Let* $\mathcal{H}$ *be a Hilbert space with inner product* $\langle \cdot\,, \cdot \rangle_{\mathcal{H}}$. *Let* $\{e_i\}_{i=0}^{d-1}$ *be a basis of* $\mathcal{H}$. *Let* $\mathcal{H}^*$ *be the space of all (continuous) linear functionals on* $\mathcal{H}$.

Then $\mathcal{H}^*$ is a Hilbert space under the inner product

$$\langle f\,,\,g\rangle_{\mathcal{H}^*} = \sum_i \overline{f(e_i)}\,g(e_i)\;, \tag{A.12}$$

where $f, g \in \mathcal{H}^*$.
Moreover, this Hilbert space has a representation in $\mathcal{H}$ called the ***Riesz representation****: for every (continuous) linear functional $f \in \mathcal{H}^*$, there exists a unique vector $\varphi_f \in \mathcal{H}$ such that*

$$f(\psi) = \langle \varphi_f\,,\,\psi\rangle\;. \tag{A.13}$$

The Riesz representation can indeed be obtained through an antilinear isomorphism of spaces.

Corollary A.1.4 *The space $\mathcal{H}$ is anti-isomorphic to its dual $\mathcal{H}^*$.*
Therefore, the identification

$$\mathcal{H} \simeq \mathcal{H}^* \tag{A.14}$$

can be realized by an invertible linear map $\mathfrak{R}$ sending the inner product in $\mathcal{H}^*$ to its conjugate in $\mathcal{H}$,

$$\begin{gathered}\exists\mathfrak{R} : \mathcal{H}^* \to \mathcal{H} \;:\quad \forall f, g \in \mathcal{H}^*,\; \exists\varphi_f, \vartheta_g \in \mathcal{H}\;:\\ \mathfrak{R}\,(f) = \varphi_f;\\ \mathfrak{R}\,(g) = \vartheta_g;\\ \langle f\,,\,g\rangle_{\mathcal{H}^*} = \langle \vartheta_g\,,\,\varphi_f\rangle_{\mathcal{H}}\;.\end{gathered} \tag{A.15}$$

The antilinear nature of the correspondence makes it basis-dependent when expressed using a linear map like $\mathfrak{R}$ since a basis has to be fixed in order to define the complex conjugation operation. However, a complex conjugation is an involution, so applying the Riesz representation twice gets rid of the basis-dependency. This means that a functional on a functional, like $\gamma_\psi : \mathcal{H}^* \to \mathbb{C}$ such that $\gamma_\psi(f) = f(\psi)$, is naturally isomorphic to a vector in the direct, $\gamma_\psi \cong \psi$. This is the content of the following well-known corollary.

Corollary A.1.5 *The spaces $\mathcal{H}$ and its double dual $(\mathcal{H}^*)^*$ are canonically isomorphic.*
Therefore, the identification

$$\mathcal{H}^A \cong \left(\left(\mathcal{H}^A\right)^*\right)^* \tag{A.16}$$

can always be realized by an invertible linear map in a basis-independent way.

The bra-ket notation makes good use of this theorem. The identification of a functional f with a vector φ_f is made apparent using the 'bra' notation $f = \langle\varphi_f|$. The functional is a vector of the dual, thus put in the antilinear part of the inner product, the bra, so to be combined with a vector of the direct ψ, put into the linear part, the ket $|\psi\rangle$.

The symbol $\dagger$ is then used to denote the antilinear identification of a vector with its dual, $|\psi\rangle^\dagger = \langle\psi|$, using Corollary A.1.4. This identification can be made loosely since it is antilinear and therefore basis-independent. However, as stated in the theorem, trying to make this identification in a linear manner leads to a linear map $\mathfrak{R}$ that depends on a choice of basis. When identifying the kets with vectors of $\mathbb{C}^{d_A \times 1}$, the usual way of obtaining a linear identification is the transposition with respect to a basis $\{e_i\}_{i=0}^{d-1}$,

$$\mathfrak{R}\left(\langle\varphi_f|\right) = \left(\langle\varphi_f|\right)^T := \sum_i \langle\varphi_f|e_i\rangle\,|e_i\rangle\ . \tag{A.17}$$

The dagger then corresponds to the conjugate transpose,

$$\left(\langle\varphi_f|\right)^\dagger := \sum_i \overline{\langle\varphi_f|e_i\rangle}\,|e_i\rangle = \sum_i \langle e_i|\varphi_f\rangle\,|e_i\rangle\ , \tag{A.18}$$

which is effectively basis-independent[2]. These seemingly pedantic precisions about the bra-ket notation will come in handy in the following sections when nested linear maps are considered.

2: Which is quick to prove using the completeness relation $\sum_i |e_i\rangle\!\langle e_i| = \mathbb{1}$:

$$\begin{aligned}\sum_i \langle e_i|\varphi_f\rangle\,|e_i\rangle &= \sum_i |e_i\rangle\,\langle e_i|\varphi_f\rangle \\ &= \sum_i (|e_i\rangle\,\langle e_i|)\,|\varphi_f\rangle = \mathbb{1}\,|\varphi_f\rangle \\ &= |\varphi_f\rangle\ .\end{aligned} \tag{A.19}$$

Since the completeness relation holds for all bases, the choice of a particular base $\{|e_i\rangle\}$ has no influence on the definition of the $\dagger$.

A.1.3. Hilbert-Schmidt Spaces of Operators

The inner product on a Hilbert space $\mathcal{H}$ can be used to define an inner product between operators $N, V \in \mathcal{L}(\mathcal{H})$ as

$$(N\,,\,V) := \sum_i \langle N(e_i)\,,\,V(e_i)\rangle\ . \tag{A.20}$$

where $N, V \in \mathcal{L}(\mathcal{H})$ and the set of vectors $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$.

Definition A.1.4 (Hilbert-Schmidt Inner Product) *Let $\mathcal{H}^A$ and $\mathcal{H}^B$ be Hilbert spaces with inner product $\langle\,,\,\rangle_A$ and $\langle\,,\,\rangle_B$ respectively. Let $\{e_i\}$ be a basis of space $\mathcal{H}^A$. Let $M, N \in \mathcal{L}\left(\mathcal{H}^A, \mathcal{H}^B\right)$ be a two linear maps.*
Then, the sesquilinear form defined by

$$(M\,,\,N) := \sum_i \langle M(e_i)\,,\,N(e_i)\rangle_A\ , \tag{A.21}$$

*is called the **Hilbert-Schmidt inner product**.*

It can be checked that this induced sesquilinear form is indeed an inner product on $\mathcal{L}(\mathcal{H})$. Moving the terms around a little,

$$(N\,,\,V) = \sum_i \langle N(e_i)\,,\,V(e_i)\rangle = \sum_i \left\langle e_i\,,\,(N^\dagger\cdot V)(e_i)\right\rangle\ , \tag{A.22}$$

it can be expressed as a trace:

$$(N\,,\,V) \equiv \mathrm{Tr}\left[N^\dagger \cdot V\right]\ . \tag{A.23}$$

This hints that this is indeed a basis-independent construction as it can be shown. The Hilbert-Schmidt inner product actually makes the space of linear maps a Hilbert space with respect to it as well.

Therefore, a Hilbert space $(\mathcal{H}, \langle\cdot\,,\,\cdot\rangle)$ always induces another Hilbert space of operators on itself $(\mathcal{L}(\mathcal{H}), (\cdot\,,\,\cdot))$ called the Hilbert-Schmidt [74, 162] or Liouville [156, 163] space[3]. In the finite-dimensional case, this construction is also well-defined for linear maps between spaces

[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.
[162]: Farenick (2000), *Algebras of Linear Transformations*.
[156]: Heinosaari et al. (2011), *The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement*.
[163]: Audretsch (2007), *Entangled Systems: New Directions in Quantum Physics*.

3: In infinite dimension, this also holds but the space of operators must be restricted to the space of bounded and compact operators that have finite Hilbert-Schmidt norm.

of different dimensions, in which case it is often called the Frobenius inner product instead [164]. As will be shown in the following, this self-repeating property is the mathematical reason why the theory of higher-order processes, which mathematically is nothing short of defining linear maps between linear maps recursively, is so well-behaved and always presents the same kind of structure.

[164]: Horn et al. (2013), *Matrix Analysis*.

Remark that the theory of categories is best suited to abstract this repeating pattern behavior between a Hilbert space and the operators defined on it. In categorical language, the property that the (Hilbert-Schmidt) operators on a Hilbert space are also a Hilbert space is called *closure*. And that the Hilbert-Schmidt space is isomorphic to the tensor product between the base Hilbert space and its dual is called *monoidal closure*. See References [81, 125] for an introduction in the context of quantum information theory.

[81]: Coecke et al. (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*.

[125]: Heunen et al. (2019), *Categories for Quantum Theory: An Introduction*.

When the input and output spaces are isomorphic, the Hilbert-Schmidt space of operators is naturally isomorphic to the tensor product of the base Hilbert space with its (algebraic) dual $\mathcal{H}^*$,

$$\mathcal{L}\left(\mathcal{H}\right) \cong \mathcal{H} \otimes \mathcal{H}^* \,. \tag{A.24}$$

This fact is what justifies Dirac bra-ket notation as dyads (or ket-bras) such as $|\phi\rangle\langle\psi|$ can be interpreted both as an operator or as the tensor product between a vector and a dual vector $|\phi\rangle\langle\psi| \cong |\phi\rangle \otimes \langle\psi|$. This fact is also what underlies the Choi-Jamiołkowski isomorphism, hence, with Choi's theorem, these are the crucial ingredients for the representation of higher-order processes.

Adjoint operators are defined on the Hilbert-Schmidt space as well; to avoid confusion these will be noted by a $*$ instead of a $\dagger$. The adjoint of a linear map $\mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right)$ is given by the unique map $\mathcal{M}^* \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^B\right), \mathcal{L}\left(\mathcal{H}^A\right)\right)$ satisfying

$$\left(N\,,\, \mathcal{M}\left(V\right)\right)_B = \left(\mathcal{M}^*(N)\,,\, V\right)_A \,, \tag{A.25}$$

for all $V \in \mathcal{L}\left(\mathcal{H}^A\right)$ and $N \in \mathcal{L}\left(\mathcal{H}^B\right)$. That is the only $\mathcal{M}^*$ such that

$$\mathrm{Tr}\left[N^\dagger \cdot \mathcal{M}(V)\right] = \mathrm{Tr}\left[\mathcal{M}^*(N)^\dagger \cdot V\right] \,. \tag{A.26}$$

As is the case with linear maps, if the input and output spaces are the same, $\mathcal{L}\left(\mathcal{H}^A\right) \cong \mathcal{L}\left(\mathcal{H}^B\right)$, the map are called operators on $\mathcal{L}\left(\mathcal{H}^A\right)$. To distinguish the sets of operators on a Hilbert space $\mathcal{L}\left(\mathcal{H}^A\right)$, from the set of operators on the set of operators on a Hilbert space $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right)\right)$, the elements of the later are called *superoperators* [8].

[8]: Chiribella et al. (2008), *Transforming quantum operations: Quantum supermaps*.

A.1.4. Probabilities

Let a be a realization of a random variable $\hat{a}$, and let b be one of $\hat{b}$. In this work, all random variables are assumed to be discrete and finite. That is, realizations of $\hat{a}$ are assumed to take values a from a countable set of events Ω_a whose cardinality is finite, $|\Omega_a| = n_a < \infty$. Therefore the values $\hat{a}$ can take range over the set $\Omega_a = \{a = 1, a = 2, \ldots, a = n_a\}$. And the same holds for $\hat{b}$.

Definition A.1.5 *Let $P : \Omega_a \to \mathbb{R}$ be a probability measure on Ω_a, the* ***probability distribution***[4] *of $\hat{a}$, or* ***distribution*** *in short, is the function $p : \mathbb{R} \to [0, 1]$ defined by*

$$p(a) := P(\hat{a} = a) \,. \tag{A.27}$$

4: Also called **probability mass function** or **discrete probability density function**.

A distribution summed over all possible realizations of $\hat{a}$, i.e. all the possible values the random variable can take, gives a probability of 1:

$$\sum_{a \in \Omega_a} p(a) = 1 \,. \tag{A.28}$$

Remark the reference to the set Ω_a will often be omitted for conciseness: $\sum_a p(a) := \sum_{a \in \Omega_a} p(a)$.

Definition A.1.6 *Let $P : \Omega_a \times \Omega_b \to \mathbb{R}$ be the probability measure on $\Omega_a \times \Omega_b$, the* ***joint probability distribution*** *of $\hat{a}$ and $\hat{b}$, or* ***joint distribution*** *in short, is the function $p : \mathbb{R} \to [0, 1]$ defined by*

$$p(a, b) := P(\hat{a} = a \cap \hat{b} = b) \,. \tag{A.29}$$

When summed over all possible realizations of $\hat{a}$ and $\hat{b}$, it gives a probability of 1:

$$p(a, b) \in [0, 1] : \sum_a \sum_b p(a, b) = 1 \,. \tag{A.30}$$

Whereas the summation over a single random variable defines the **marginal distribution** (of the other variable):

$$p(a) := \sum_b p(a, b) \,; \tag{A.31a}$$

$$p(b) := \sum_a p(a, b) \,. \tag{A.31b}$$

Where the first line of the above is read 'the marginal distribution of a' for instance.

Marginal distributions can be used to define the **conditional distribution** of a variable given another:

$$p(a|b) := \frac{p(a, b)}{\sum_a p(a, b)} = \frac{p(a, b)}{p(b)} \,. \tag{A.32a}$$

$$p(b|a) := \frac{p(a, b)}{\sum_a p(a, b)} = \frac{p(a, b)}{p(a)} \,. \tag{A.32b}$$

Where the second line of the above is read 'the conditional distribution of b given a' for instance.

From the definition, the following holds

$$p(a, b) = p(a|b)p(b) = p(b|a)p(a) \,. \tag{A.33}$$

So that the marginal and conditional distributions are linked by **Bayes theorem**:

$$p(a|b) = \frac{p(b|a)p(a)}{p(b)} \,. \tag{A.34}$$

A special case of marginal distribution is the mixture. The distribution of a random variable $\hat{a}$ may be conditioned on a second variable $q \in [0, 1]$ so that its distribution has the form:

$$p(a) = q p_1(a) + (1 - q) p_2(a) \,, \tag{A.35}$$

where p_1 and p_2 are different distributions. Such a distribution is called a **mixture** of distributions p_1 and p_2 and q is called the weight of the mixture. Mixture can be defined for an arbitrarily large amount of distributions

$$p(a) = \sum_i q_i p_i(a) \,, \tag{A.36}$$

where the weights q_i obey $q_i \in [0,1]$ and $\sum_i q_i = 1$.

In general, a mixture (or compound) distribution can be seen as a marginal distribution, in which case the weights q_i are interpreted as the probabilities associated with the realizations of a random variable $\hat{b}$. Whence, the mixture is seen as a marginal over $\hat{b}$ of some joint distribution $p(a,b)$:

$$p(a) = \sum_i q_i p_i(a) = \sum_b p(b) p(a|b) \,, \tag{A.37}$$

under the identification $i \mapsto b$: $q_i \equiv p(b)$ and $p_i(a) \equiv p(a|b)$.

Another special case of joint distributions is independent distributions.

Definition A.1.7 *Two random variable $\hat{a}$ and $\hat{b}$ are* ***independent*** *if and only if their joint distribution satisfies*

$$\forall a, b: \quad p(a,b) = p(a)p(b) \,. \tag{A.38}$$

Otherwise, they are ***dependent*** *(or* ***correlated****).*

In the case of independent random variables, the following holds from the definitions:

$$p(a|b) = p(a) \,; \tag{A.39a}$$
$$p(b|a) = p(b) \,. \tag{A.39b}$$

From these formulae, independence can also be phrased as follows:

$$\forall a \in \Omega_a, \,, \forall b, b' \in \Omega_b, \quad p(a|b) = p(a|b') \,; \tag{A.40a}$$
$$\forall a, a' \in \Omega_a, \,, \forall b \in \Omega_b, \quad p(b|a) = p(b|a') \,. \tag{A.40b}$$

When clear from the context, these will be concisely phrased as

$$\forall a, a', \quad p(b|a) = p(b|a') \,; \tag{A.41a}$$
$$\forall b, b', \quad p(a|b) = p(a|b') \,. \tag{A.41b}$$

A less strong notion than independence is conditional independence: the random variables $\hat{a}$ and $\hat{b}$ are independent of each other but share a common cause represented by a variable $\hat{z}$ with realizations $z \in \Omega_z$.

Definition A.1.8 *Two random variables $\hat{a}$ and $\hat{b}$ are* ***conditionally independent*** *[165] given a third random variable $\hat{z}$ if and only if their joint distribution satisfies*

$$\forall a, b, z: \quad p(a,b|z) = p(a|z)p(b|z) \,. \tag{A.42}$$

[165]: Dawid (1979), *Conditional Independence in Statistical Theory*.

This definition has several equivalent definitions holding for all values

a, b, z:

$$p(a|b,z) = p(a|z)\ ; \tag{A.43a}$$
$$p(b|a,z) = p(b|z)\ ; \tag{A.43b}$$

and

$$\forall b', \quad p(a|b,z) = p(a|b',z)\ ; \tag{A.44a}$$
$$\forall a', \quad p(b|a,z) = p(b|a',z)\ . \tag{A.44b}$$

A.2. Some Elements on Local Correlations

Some distributions can be no-signaling but still have a common cause, for example, the kind of distribution obtained by local parties measuring part of a shared system. Well-studied instances of such distribution are those admitting a common cause described by a classical variable as well, the so-called *local hidden variable models* [166].

[166]: Bell (1964), *On the Einstein Podolsky Rosen paradox*.

Definition A.2.1 (Local Hidden Variable Model) *A no-signaling distribution $p(a,b|x,y)$ as in Definition 1.2.1 is said to have a* ***local hidden variable model*** *(LHVM) if there exists a random variable λ such that the distribution factories as a mixture of conditionally independent distributions:*

$$p(a,b|x,y) = \sum_{\lambda} p(a|x,\lambda)p(b|y,\lambda)\ . \tag{A.45}$$

A no-signaling distribution admitting an LHVM is concisely called ***local****.*

In a nutshell, local hidden variable models certify that there exists a local common cause (whose influence is represented by λ) that explains the correlations between local outcomes a and b. The states of a theory that can be explained in terms of an LHVM for all possible local measurements are called *(Bell-)local*. The compliance to a hidden-variable model can be certified with a bound on correlations called a *Bell inequality*. In the case of quantum theory, a well-known result is that some states of quantum theory violate some Bell inequalities. For example, the CHSH inequality, noted S, has a value of at most 2 for distributions with an LHVM, $S_{LHVM} \leq 2$, but this bound is surpassed by some distributions obtained with a bipartite state of two qubits and can go up to $S_{Quantum} = 2\sqrt{2}$ in the case of maximally entangled states, like the *Bell states*.

Any quantum state violating a Bell inequality is called a non-local state. The maximal departure from a Bell inequality with quantum theory is quantified in terms of the *Tsirelson bound* [167]. States saturating this bound are called maximally non-local. Remark that the no-signaling condition also puts an upper bound on correlations, so the Tsirelson bound lies between the Bell-local and no-signaling bounds. In the case of CHSH, the values are:

[167]: Tsirelson (1980), *Quantum generalizations of Bell's inequality*.

$$S_{LHVM} = 2 < S_{Quantum} = 2\sqrt{2} < S_{NS} = 4\ . \tag{A.46}$$

Note that a hypothetical pair of closed boxes whose inner workings obey a different theory than quantum physics so that their outcomes distribution saturates the no-signaling bound for CHSH has been studied in the literature. It is referred to as a *Popescu-Rohrlich (PR) box* [63].

[63]: Popescu et al. (1994), *Quantum non-locality as an axiom*.

In quantum theory, the characterization of the local states of systems is an especially difficult task, which is its own subfield (quantum non-locality, see Reference [56] for a review). A necessary[5] condition for non-locality is that the state is entangled... However, characterizing entangled states is a difficult task, with entire books devoted to it (e.g., Reference [74]). The key thing to keep in mind is that the two concepts are distinct: entangled states are not automatically non-local; entangled states that cannot violate a Bell inequality, even when local operations and classical communications (LOCC) are allowed, are called *bound entangled* [169].

[56]: Brunner et al. (2014), *Bell nonlocality*.

5: This assumes a single preparation of a joint system in a given state. Tasks requiring the differentiation of various joint states produced by two local preparation procedures can also show non-local behavior, although the prepared states are pure and unentangled. This phenomenon is called Non-Locality Without Entanglement (NLWE). See Reference [168] for an introduction.

[168]: Croke et al. (2017), *Difficulty of distinguishing product states locally*.

[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.

[169]: Horodecki et al. (1998), *Mixed-State Entanglement and Distillation: Is there a "Bound" Entanglement in Nature?*

A special case of LHVM is obtained when the hidden variable λ only takes one value. In that trivial case, the correlations are independent in the mathematical sense with x only affecting a and y only b.

Definition A.2.2 *Let Alice and Bob be two parties in a process as in Definition 1.2.1. Their experiments are said* ***independent*** *if and only if the joint distribution is no signaling and their outcomes are independent:*

$$p(a, b|x, y) = p(a|x)p(b|y) \ . \tag{A.47}$$

Remark that no-signaling has to be imposed otherwise Alice's outcome a can still depend on Bob's setting y and vice-versa. This is the kind of correlation one may expect from two experimentalists in closed labs that are causally disconnected; as if each of the local labs in which each party conducted their measurement have always been shielded from the rest of the universe. These correlations are the ones exhibited by bipartite systems in which the preparation, evolution, and measurement stages are all composed in parallel, so that no interaction at all happens between Alice and Bob subsystems. In quantum theory, a situation in which the state, evolution, and effects are in pure tensor products is guaranteed to produce such correlations.

Independence is the original notion of "locality" for quantum theory that was used in the context of the EPR paradox [170]. For this reason, it is sometimes dubbed *Einstein locality* [73, p. 302][67] as opposed to *Bell locality*. The important thing to remember is that if interventions in closed boxes are called local, yet the correlations observed by the local parties performing these interventions are not automatically (Bell-)local. Local correlations are defined by a heuristic requirement on the correlations. This heuristic sits in between the related notions of no-signaling and independent correlations[6].

[170]: Einstein et al. (1935), *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*

[73]: Holevo (2011), *Probabilistic and Statistical Aspect of Quantum Theory*.

[67]: D'Ariano et al. (2014), *Determinism without causality*.

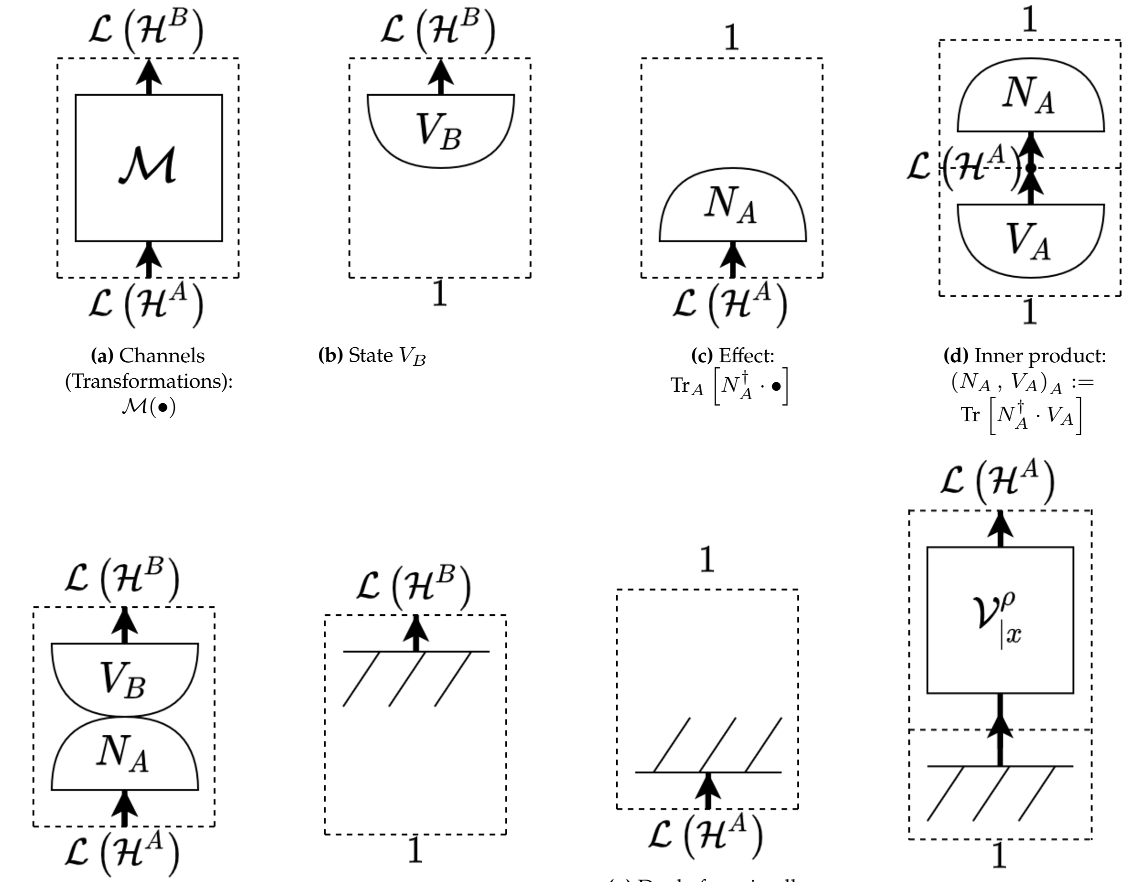
6: This point is stressed because all these notions have been called 'local' at some point in the literature, see Ref. [171] for disambiguation of the different notions.

[171]: Eberhard (1978), *Bell's theorem and the different concepts of locality*.

A.3. Graphical Methods and Turning States and Effects into Operations

All the usual objects of quantum theory fall under this general definition of a quantum operation or intervention. Meaning all can be represented as some special case of linear maps. Following [81], the graphical methods used in Chapter 1 are formalized into diagrammatic rules in order to picture these transformations more easily.

[81]: Coecke et al. (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*.



(a) Channels (Transformations): $\mathcal{M}(\bullet)$

(b) State V_B

(c) Effect: $\mathrm{Tr}_A\left[N_A^\dagger \cdot \bullet\right]$

(d) Inner product: $(N_A\,,\,V_A)_A := \mathrm{Tr}\left[N_A^\dagger \cdot V_A\right]$

(e) Outer Product (Dyad): $\mathrm{Tr}\left[N_A^\dagger \bullet\right] \otimes V_B$

(f) Maximally mixed state: $\frac{\mathbb{1}}{d_B}$

(g) Dual of maximally mixed state; the trace: $\mathrm{Tr}_A\left[\mathbb{1}_A \cdot \bullet\right] = \mathrm{Tr}_A\left[\bullet\right]$

(h) State preparation conditioned by x: $\mathcal{V}^\rho_{|x}(\frac{\mathbb{1}}{d}) = \rho^A_{|x}$

Figure A.1.: Some common diagrammatic representation of local and deterministic quantum operations

Refer to Figure A.1. As set up within the process framework, systems are represented as thick wires with a label, so that each thick wire is associated with a Hilbert space of operators $\mathcal{L}\left(\mathcal{H}^X\right)$ on which the state of the system X is expressed. Quantum operations are linear maps $\mathcal{M}$ from an input space $\mathcal{L}\left(\mathcal{H}^A\right)$ to an output space $\mathcal{L}\left(\mathcal{H}^B\right)$. By consequence, these are represented as a box in between two wires as in Figure A.1a with the $\mathcal{M}$ written within it. The whole diagram in the dashed frame is the linear map, with the bottom of the frame associated with the input space and the top of it associated with the output space.

The thin wires, representing classical data and used for settings and outcomes, will often be omitted in favor of indices in the diagrams. For example, in Figure A.1h an incoming wire with setting x has been omitted since the reference to the setting is already made in the label $\mathcal{V}^\rho_{|x}$.

Within such diagrammatic heuristics, it is easy to argue that preparation and measurement are special cases of linear maps, i.e. quantum interventions. Indeed, in this picture, a state preparation corresponds to a deterministic operation with an input space of dimension 1, a *trivial input space*. Graphically, the preparation is indicated with a bottom half-circle with the state of the prepared state substituting the name of the map, as in Figure A.1b. A special symbol is reserved to refer to the maximally mixed state: a ground symbol (dashed lines as in Figure A.1f. A state can equivalently be seen as a deterministic operation $\mathcal{V} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ applied on a maximally mixed state in A_0, i.e.

$$\mathcal{V}\left(\frac{\mathbb{1}_{A_0}}{d_{A_0}}\right) = \rho \,. \tag{A.48}$$

This is because a state space reduced to a subspace consisting of the maximally mixed state is effectively a trivial space, as no measurement procedure can extract any useful information from it. This picture can be understood as encoding information on a system (A_0) that was only containing white noise up to that point ($\frac{\mathbb{1}_{A_0}}{d_{A_0}}$) through a procedure $\mathcal{V}$ resulting in a modified system A_1 in state ρ.

In the same way, a measurement corresponds to an instrument with a trivial output space. An instrument with trivial output space reduces back to the usual definition of a POVM; the action of a map $\mathcal{E}_a \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right) = \mathbb{C}\right)$ becomes equivalent to the action of a POVM element $E_a \in \mathcal{L}\left(\mathcal{H}^{A}\right)$ under the identification

$$\forall \rho,\ \mathcal{E}_a(\rho) = \mathrm{Tr}\left[E_a\ \rho\right] \,. \tag{A.49}$$

In other words, if the map is reduced to a functional, $\mathcal{E}_a \in \mathcal{L}\left(\mathcal{H}^{A_0}\right)^*$, it goes back to the usual form of a destructive measurement, $\mathrm{Tr}\left[E_a \cdot\right]$, where $E_a \geq 0 : \sum_a E_a = \mathbb{1}$ is a vector in $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$ representing an element of the dual space $\mathcal{L}\left(\mathcal{H}^{A_0}\right)^*$ through the inner product $(E_a\,,\,\cdot) := \mathrm{Tr}\left[E_a^\dagger \cdot\right]$.

As the dual space is (anti-)isomorphic to the direct, $\mathbb{1}$ also plays a distinguished role there: the *discarding* operation [81]. It amounts to ignoring the output state of an operation, to discard it. Concretely, it is obtained by averaging over all outcomes of any destructive measurement. This indeed yields a discarding operation: the state has been destroyed and at the same time no information about it has been kept. Because of the POVM condition $\sum_a E_a = \mathbb{1}$, the discarding is indeed a functional represented by $\mathbb{1}$, which amounts to taking the trace:

$$\mathcal{N}_a(\cdot) = \sum_a \mathrm{Tr}\left[E_a \cdot\right] = \mathrm{Tr}\left[\mathbb{1} \cdot\right] = \mathrm{Tr}\left[\cdot\right] \,. \tag{A.50}$$

As with preparations, the measurement procedure can be seen as an operation $\mathcal{N}_a \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_0}\right), \mathcal{L}\left(\mathcal{H}^{A_1}\right)\right)$ followed by a discarding, or "applied on the maximally mixed state in the dual space", which leads to the following identification of an effect with an instrument

$$E_a = \mathcal{N}_a^*(\mathbb{1}_{A_1}) \,. \tag{A.51}$$

In the above, $*$ indicated the adjoint in $\mathcal{L}\left(\mathcal{H}^{A_0}\right)$ so that the map $\mathcal{N}_a^* \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^{A_1}\right), \mathcal{L}\left(\mathcal{H}^{A_0}\right)\right)$ is uniquely defined by

$$\left(\sigma_{A_1}\,,\, \mathcal{N}_a(\rho_{A_0})\right)_{A_0} = \left(\mathcal{N}_a^*(\sigma_{A_1})\,,\, \rho_{A_0}\right)\ \forall \sigma, \rho \,. \tag{A.52}$$

Note that the destructive measurements, here in the form of a POVM, reflect randomized state preparation, i.e. those depending on a setting x such that it results in different output states $\rho_{|x}$ depending on its value,

$$\rho_{|x} = \mathcal{V}_{|x}\left(\frac{\mathbb{1}_{A_0}}{d_{A_0}}\right) \,. \tag{A.53}$$

Compared to Equation A.51, these two formulas are similar, with the only difference being that the measurement is defined in the dual space of the preparation. This duality is known as a *state and effect pair*. The important thing to notice is that averaging over the settings of a state preparation,

$$\sum_x \mathcal{V}_{|x}\left(\frac{\mathbb{1}_{A_0}}{d_{A_0}}\right) = \sum_x \rho_{|x} := \rho\,, \tag{A.54}$$

does not have to lead to the maximally mixed state necessarily. But, on the other hand, averaging over the outcomes of a measurement does:

$$\sum_a \mathcal{N}_a^*(\mathbb{1}_{A_1}) = \sum_a E_a = \mathbb{1}_{A_0}\,. \tag{A.55}$$

This is due to the condition that the elements of a POVM sum up to the identity.

Because of that, it must hold that the adjoint of the elements of a quantum instrument, $\mathcal{N}_a^*$ sum up to a unital map,

$$\sum_a \mathcal{N}_a^*(\mathbb{1}_{A_1}) = \left(\sum_a \mathcal{N}_a^*\right)(\mathbb{1}_{A_1}) = \mathcal{N}^*(\mathbb{1}_{A_1}) = \mathbb{1}_{A_0}\,. \tag{A.56}$$

And since the adjoint of a unital map is Trace-Preserving (TP), this explains why a quantum channel must be TP.

This whole discussion has been made so to motivate two ideas. The first is that a local intervention without input or output is equivalent to an intervention between trivial input and output. And the second idea is that a quantum operation is a versatile enough tool to represent any quantum procedure which moreover puts every object on the same footing: everything is a CP map of some kind. For example, in quantum information, a communication protocol is usually represented as a preparation of states $\rho_{|x}$ conditioned by setting x, followed by transmission through a channel $\mathcal{M}$, and terminated by a measurement $\{E_a\}$ yielding outcomes a, the distribution of which is given by the Born rule

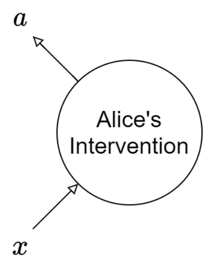
$$p(a|x) = \big(E_a\,,\,\mathcal{M}\left(\rho_{|x}\right)\big)\,. \tag{A.57}$$

The point is that the channel is a linear map, whereas the state and effect are operators. Seen as operations, all these objects are CP maps composed together and acting on trivial input and output,

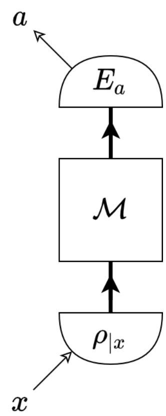
$$p(a|x) = \left(\mathbb{1}_{A_1}\,,\,\left(\mathcal{N}_a^{(E)}\circ\mathcal{M}\circ\mathcal{V}_{|x}^{(\rho)}\right)\left(\frac{\mathbb{1}_{A_0}}{d_{A_0}}\right)\right)\,. \tag{A.58}$$

(Here the superscript notation refers to the operator the CP map is representing). Using this kind of construction to put all the objects in the same kind of mathematical space is one of the key ingredients of the theory of higher-order quantum processes presented here.

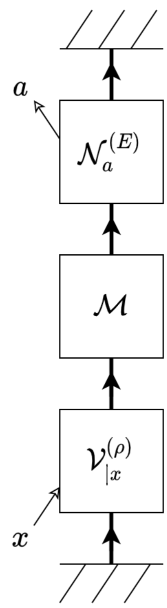
Remark that this kind of construction has been studied in much more depth in the categorical framework of quantum mechanics. Especially, this picture of local quantum theory is axiomatically obtained as a diagrammatic language under the name of the *CPM* [126] or *doubling* constructions [81].



(a) Local experiments $p(a|x)$...



(b) ...are described by some quantum protocols, (A.57), in which the object...



(c) ...are all specific instances of quantum operations (boxes) between trivial input and output systems, (A.58).

Figure A.2.: Quantum Theory formulated as CP maps.

[126]: Selinger (2007), *Dagger Compact Closed Categories and Completely Positive Maps: (Extended Abstract)*.

[81]: Coecke et al. (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*.

Appendices to Chapter 2 | B.

I have not yet any clear view as to the extent to which we are at liberty arbitrarily to create imaginaries and endow them with supernatural properties.

John Graves on quaternions*

B.1. Some Elements on the Multipartite Process Formalism

The development of the mathematical tools in the main text consists of setting the theory ruling the parties' intervention to be quantum theory. That is, to impose quantum theory as being the way to represent how parties can prepare and measure systems locally to interact with their environment in order to obtain the probability distributions. The resulting global theory obtained when assuming quantum theory locally, the process matrix formalism [5], is the theory of higher-order quantum processes (first defined under the name of 'quantum supermaps' [8]), i.e. the theory of nested, or higher-order, quantum interventions.

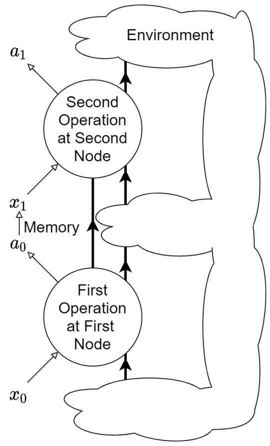
The postulates are indeed lenient enough to accommodate the idea of a higher-order intervention, as will now be discussed. It should be first stressed that the process formalism is agnostic with respect to the exact form of the systems, as well as to any form of global background besides what the environment provides. By looking at diagrams alone, one may be tempted to label each thick wire as some point in space and time, but these are examples of global things of the backgrounds that need not be assumed, and can potentially flaw the interpretation. The input system can indeed be composed of several subsystems, each entering Alice's lab at a different point and at a different time (with respect to Alice's local clock[1]) and the same way the output system can actually be several subsystems released back to the environment at different times. In between these different times, Alice can correlate the systems she release at a later time with the ones she received at an earlier time because she is allowed to. She can indeed do anything locally, including keeping a memory of her previous classical settings and outcomes in the form of a side system she keeps in her closed lab. This kind of system that Alice fully controls (for which Alice is a *global* party) and that she can always add in parallel to any scenario in which she is involved (for which she is a *local* party), are called the ancillary systems of Alice[2] or side systems.

1: A notion that can be sensible only if the assumed local theory allows for it!

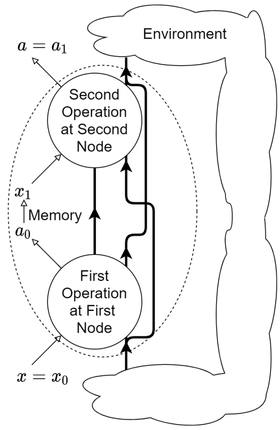
2: Or *ancilla* in short.

Assuming that the system of a party splits into subsystems according to a notion of space and time is a special case of the general possibility that it splits according to something she can measure by some strategy. Consider the situation in Figure B.1: a party can perform multiple interventions at multiples *nodes*; however, her nodes are not exactly local parties since the

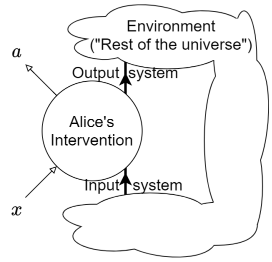
* Quoted from Farenick (2000), *Algebras of Linear Transformations* [162].



(a) A multi-round intervention still assumes that Alice is a local party...



(b) ...so it can be encompassed in a closed box which so to rephrase it as...



(c) ...a single local intervention of Alice.

Figure B.1.: A scenario allowed by the process framework: multi-round interventions. B.1a: Alice acts twice on a system, which is assumed to return into the environment in between, so the state of the second system she received can be in general different but correlated to the state of her first output system. Each operation of Alice is referred to as a *node* in her *network of operations*. In between her nodes, she keeps a *memory*: constituted by her previous settings and outcomes (thin wire), as well as *a side system* (or *ancilla*; thick wire). Her overall situation although multi-round, is still local. B.1b: Her multiple interventions can be gathered as a single overall intervention... B.1c: ...Equivalent to a single-round intervention.

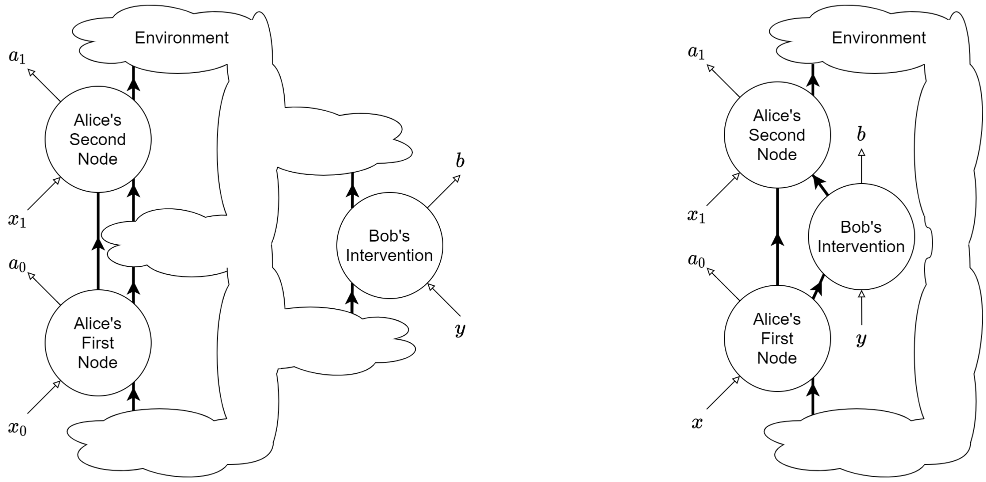
setting of the second operation may depend on the setting of the first. Therefore, it is be termed a *multi-round* intervention.

The multi-round intervention is a special case of an even more general situation: the higher-order intervention. Consider a bipartite process. It can be that, on the one hand, party Alice has an intervention that splits into two rounds as in Figure B.2a. On the other hand, the environment can turn out to be such that the output of Alice's first intervention is exactly the first input of Bob's, and at the same time, his output is the input of Alice's second intervention, as in Figure B.2b. In that case, Bob's intervention is entirely under Alice's control; she sees it as a random process between her first input and second output. From Bob's perspective, Alice's intervention is global in the same manner that the environment is. In such a scenario, Alice's intervention is called higher-order with respect to the one of Bob. Remark that the environment is a general instance of higher-order intervention; it is always assumed global as opposed to the local parties.

Note that more developed approaches to the splitting of systems into subsystems have been investigated in the context of higher-order quantum processes. The process formalism has been extended to consider these splitting as subsystems not associated with a definite time, but rather as *time-delocalized subsystems* [28]. In the unitary case, constraints on the splitting into subsystems can be treated with an extension of the circuit formalism: routed quantum circuits [172].

[28]: Oreshkov (2019), *Time-delocalized quantum subsystems and operations: on the existence of processes with indefinite causal structure in quantum mechanics*.

[172]: Vanrietvelde et al. (2021), *Routed quantum circuits*.



(a) A scenario in which Alice's intervention splits into two rounds.

(b) A scenario in which Alice's intervention encompasses Bob's: from the point of view of the process, she acts in a higher order than him.

Figure B.2.: The higher-order intervention scenario

B.2. Directed Arrows and Some Remarks on Antilinearity

The Choi-Jamiołkowski isomorphism of Definition 2.2.1 is not natural, that is, basis-independent. Formulated as Equation (2.6), the correspondence identifies an element of $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right)$ with one of $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right) \cong \mathcal{L}\left(\mathcal{H}^A\right) \otimes \mathcal{L}\left(\mathcal{H}^B\right)$. However, the space of linear maps $\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^B\right)\right)$ is itself a Hilbert space naturally isomorphic to $\mathcal{L}\left(\mathcal{H}^A\right) \otimes \mathcal{L}\left(\mathcal{H}^B\right)^*$ where $\mathcal{L}\left(\mathcal{H}^B\right)^* := \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^B\right), \mathbb{C}\right)$ is the algebraic dual of $\mathcal{L}\left(\mathcal{H}^B\right)$. The CJ correspondence, therefore, hides an identification of one of the spaces, $\mathcal{L}\left(\mathcal{H}^B\right)$, with its dual. Nevertheless, such identification has to be antilinear and basis-dependent, making the CJ correspondence itself basis-dependent and antilinear on one of its arguments as a consequence. This hidden antilinearity manifests itself as the transpose in the reverse direction of the isomorphism, Equation (2.7); this is indeed a basis-dependent map that identifies the standard basis with (the representation in the direct space of) its dual.

This whole 'problem' of keeping track of transposes would be alleviated by properly specifying that the Choi operator lives in two fundamentally different spaces, dual to each other, but that would spoil the interpretation of M_{AB} as a bipartite state. It could also be avoided by defining the isomorphism with an extra partial transpose on either of the two systems, but that would spoil Choi theorem: the maps identified with the PSD cone would then be the *Completely co-Positive (CcP) maps* (that is, CP

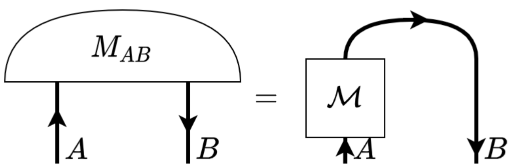


Figure B.3.: Graphical depiction of the CJ isomorphism: a transformation $\mathcal{M}$ is turned into a bipartite effect M. The maximally entangled state can be seen as 'bending a wire' to provide an antilinear connection between the two spaces (see Appendix B.2 for more details).

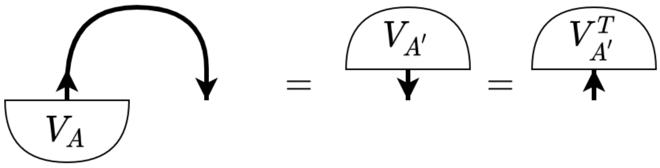


Figure B.4.: Graphical representation of the antilinear connection provided by the maximally entangled bipartite operator. As with the CJ isomorphism, applying it on a state 'flips it down', meaning that it transforms it into an *effect in the dual space* (with the arrow pointing down). The first equality is (B.1). In the second equality, interpreting it as an effect of the direct space (arrow pointing up) requires applying the antilinear connection again, which exactly corresponds to the transpose under this choice of connection.

[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.

maps followed by a transposition; see Chapter 11 of Reference [74] for instance).

But what does the transpose mean? For operators V_A, it is related to the following property of the maximally entangled state:

$$\mathrm{Tr}_A\left[\left(\sum_{i,j}|i\rangle\langle j|_A\otimes|i\rangle\langle j|_{A'}\right)^\dagger(V_A\otimes\mathbb{1}_{A'})\right]=V_{A'}^T\,. \tag{B.1}$$

This provides a representation of the dual Hilbert space of $\mathcal{H}^A$ into $\mathcal{H}^{A'}$ as the inner product is mapped antilinearly[3], $(N_A\,,\,V_A)_A\mapsto\left(N_{A'}^T\,,\,V_{A'}^T\right)_{A'}=(V_{A'}\,,\,N_{A'})_{A'}$. In other words, $\mathcal{H}^{A'}\cong\left(\mathcal{H}^A\right)^*$ and so the maximally mixed state can be seen as the connection[4] between the two spaces. In other words, taking the (partial) inner product between a state $V_A\in\mathcal{L}\left(\mathcal{H}^A\right)$ with the (unnormalized) maximally entangled state $\sum_{i,j}|i\rangle\langle j|_A\otimes|i\rangle\langle j|_{A'}$ corresponds to applying the antilinear map relating $\mathcal{H}^A$ to the representation of its dual in $\mathcal{H}^{A'}$ [173].

3: As can be seen by the fact that the antilinear and linear parts of the inner product, i.e., left and right parts, have been switched.

4: Connection is used here in the multilinear algebraic sense, meaning in the same way that the Minkowski metric is the connection between lower and upper indices for 4-vectors which also live in dual spaces.

[173]: Uhlmann (2016), *Anti- (conjugate) linearity*.

A consequence of this identification is that the adjoint map $\mathcal{M}^*\in\mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^B\right),\mathcal{L}\left(\mathcal{H}^A\right)\right)$, which has a CJ representation in $\mathcal{L}\left(\mathcal{H}^B\otimes\mathcal{H}^A\right)$, also gets a representation in $\mathcal{L}\left(\mathcal{H}^A\otimes\mathcal{H}^B\right)$ since $\mathcal{H}^B\otimes\mathcal{H}^A\cong\mathcal{H}^A\otimes\mathcal{H}^B$ (which amounts to applying a SWAP gate, or to label the spaces carefully and to always sort them in the same order, as is done in this work[5]). For a map $\mathcal{M}$ with Choi operator M, the adjoint map $\mathcal{M}^*$ has Choi representation

5: By convention, the tensor factors of Hilbert spaces are always organized alphabetically from left to right.

$$M_{AB}^*=\left[\sum_{\nu=0}^{d_B-1}(\mathcal{M}^*\otimes\mathcal{I})\{f_\nu\otimes f_\nu\}\right]^T\,, \tag{B.2}$$

where $\{f_\nu\}_{\nu=0}^{d_B^2-1}=\{|k\rangle\langle l|\}_{k,l=0}^{d_B-1,d_B-1}$ is the standard basis of $\mathcal{L}\left(\mathcal{H}^B\right)$. Using that $(N_B\,,\,\mathcal{M}(V_A))=\left(V_A^T\,,\,(\mathcal{M}^*(N_B))^T\right)$, the reverse direction of the isomorphism yields

$$\mathrm{Tr}\left[(\mathbb{1}_A\otimes N_B)^\dagger\left[M_{AB}(V_A\otimes\mathbb{1}_B)\right]^{T_B}\right]=\mathrm{Tr}\left[\left(V_A^{T_A}\otimes\mathbb{1}_B\right)^\dagger\left[M_{AB}^*(\mathbb{1}_A\otimes N_B)\right]^{T_B}\right]\,. \tag{B.3}$$

Which, after some rewriting, leads to

$$\mathrm{Tr}\left[M_{AB}\left(V_A\otimes\overline{N}_B\right)\right]=\mathrm{Tr}\left[M_{AB}^*\left(\overline{V}_A\otimes N_B\right)\right]\,. \tag{B.4}$$

In the above, $\overline{\cdot}$ indicates the complex conjugation. As V_A and N_B are arbitrary, the equivalence requires

$$M_{AB}^{*} = \overline{M}_{AB} \; . \tag{B.5}$$

Therefore, the adjoint of a map is represented as its conjugate in the CJ picture, $\mathcal{M}^* \mapsto \overline{M}$. As this work is concerned with mappings between self-adjoint operators, their CJ representation is Hermitian, so the transpose of a CJ operator is the representation of its adjoint.

The transposition in the definition of the CJ isomorphism can thus be interpreted as defining it in the dual space, so that the convention of Equation 2.6 is equivalent to

$$M_{AB} := \sum_{k=0}^{d_B-1} \sum_{l=0}^{d_B-1} \left(\mathcal{M}^* \otimes \mathcal{I}\right) \{|k\rangle \langle l| \otimes |k\rangle \langle l|\} \; . \tag{B.6}$$

where this vector, similar to Equation B.1, makes the interpretation of the action Equation 2.7 as a mapping to the dual space,

$$\mathcal{M}(V_A)^T = \mathrm{Tr}_A \left[M_{AB}^{\dagger} \left(V_A \otimes \mathbb{1}_B\right) \right] \; . \tag{B.7}$$

The transpose is used as a means to go back to the direct space. Remark that the dual space happens to always correspond to an output space: this is a feature, not a bug, of the CJ correspondence. In the graphical methods, the arrows on the wires are used to indicate whether a system is interpreted as an input or an output. But mathematically, this amounts to the operators to be defined on (a representation of) either the direct or dual. As a consequence, the direction of the arrow indicates which inner product to use, or equivalently, where to put transposes; see Reference [81, §8.6.3] for a more in-depth discussion. When reading from bottom to top, if the arrow goes in the same direction, the representation is direct, and nothing has to be done. If the arrow goes in the opposite direction, then the representation is dual. The reversed arrow is therefore here to keep track of the transposition to be applied so as to repay the antilinearity of the correspondence.

[81]: Coecke et al. (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*.

Graphically, the correspondence can be understood as if a bent wire, or 'cap' was put on top of the channel, so that it now looks like a bipartite effect; see Figure B.3 This 'cap' is the connection (B.1), which is obtained by taking the inner product with the maximally mixed state. Hence, a cap symbol closing two parallel wires AB in a diagram is to be interpreted as the functional $\left(\phi_{AB}^{+}\,,\,\cdot\right)_{AB} = \mathrm{Tr}\left[\phi_{AB}^{+}\cdot\right]$, where ϕ_{AB}^{+} is the unnormalized maximally entangled state on these two spaces.

With this in mind, the channel-state duality amounts, graphically, to 'transforming boxes into top half-disks' so that every object is interpreted as an effect. The CJ correspondence can indeed be defined for any quantum operation, even the higher-order ones.

These are all the graphical methods needed in this thesis. A more systematic treatment of the graphical methods is given in Ref. [81]; whereas a complete graphical language taking into account this antilinearity issue of the CJ correspondence has been developed in an article that is not developed in this thesis beside this comment [3].

[3]: Carette et al. (2023), *Complete Graphical Language for Hermiticity-Preserving Superoperators*.

B.3. A Short Literature Review of Higher-order Quantum Processes

Here are gathered some of the results and research works that shaped the field of higher-order quantum theory. This brief account is mainly aimed at giving some background information and giving credit to sources not cited in the main text, albeit sharing similar ideas. Be aware that this is a synthesis of my own bibliographic research; this should by no means be taken as an exhaustive nor precise historical account of the research in the field of higher-order processes during this period. In particular, this account focuses mainly on results in quantum information using the CJ isomorphism. At the same time, many similar ideas were emerging in the then-newborn categorical approach as well as other fields like spin networks.

B.3.1. Quantum Networks

Quantum network formalism started gaining the shape presented in this thesis in the study of memory effects in successive applications of a quantum instrument [174–177]. The first milestone was achieved by Kreschtmann and Werner who considered these under the name *quantum channels with memory* [175]. They proposed a first formal definition and proved it encompassed all quantum causal automata. They also studied the difference in terms of channel capacity between the set of channels sharing a memory register and those that do not. Next, Gutoski and Watrous also considered quantum networks (with n nodes) under the name *(n-)quantum strategy* in the context of quantum game theory [12]. They used the Choi-Jamiołkowski representation of the network to optimize strategies using semi-definite programming (SDP), whence the name. Following this approach, Chiribella, Perinotti, and D'Ariano considered the CJ representation of networks under the name *quantum combs* as a method for optimizing quantum circuit architecture [13].

[174]: Macchiavello et al. (2002), *Entanglement-enhanced information transmission over a quantum channel with correlated noise*.
[175]: Kretschmann et al. (2005), *Quantum channels with memory*.
[176]: Datta et al. (2007), *The coding theorem for a class of quantum channels with long-term memory*.
[177]: Plenio et al. (2007), *Spin Chains and Channels with Memory*.
[12]: Gutoski et al. (2007), *Toward a General Theory of Quantum Games*.
[13]: Chiribella et al. (2008), *Quantum Circuit Architecture*.

Soon after, Chiribella *et al.* considered a version of quantum networks that had no first input and no last output under the name *quantum testers* to study memory effects in quantum channel discrimination [94]. Around the same time, Ziman, in the context of process tomography, considered the most general measure on a quantum channel under the name *Process Positive Operator Valued Measure* (PPOVM) [70].

[94]: Chiribella et al. (2008), *Memory Effects in Quantum Channel Discrimination*.
[70]: Ziman (2008), *Process positive-operator-valued measure: A mathematical framework for the description of process tomography experiments*.

The PPOVM is actually a special case of a quantum tester with only one slot, and, as mentioned, quantum testers are special cases of quantum networks. Chiribella *et al.* recognized these different approaches to yield the same objects. For bringing these concepts together, they first conceptualized the notion of a *quantum supermap* as the most general transformation that maps an input quantum operation into an output quantum operation [8], and they proved a realization theorem showing that any supermap can be physically implemented as a quantum network with two nodes. Using this observation, they formalized all approaches under the name of quantum networks[6] [9]. Generalizing the result of Kretschmann and Werner, they showed that a quantum network encompasses many concepts: it is the most general form of a fragment of a quantum circuit; it is the most general transformation from a network

[8]: Chiribella et al. (2008), *Transforming quantum operations: Quantum supermaps*.
6: This synthesis work is the reason why the nomenclature of this work mostly follows the one of the Pavia group, although for historical reasons one can argue in favor of using the name 'quantum channel with memory' instead of 'quantum network' and the name 'quantum strategy' instead of 'quantum comb' for their CJ representation.
[9]: Chiribella et al. (2009), *Theoretical framework for quantum networks*.

to a channel; and it is a causally ordered succession of quantum channels sharing an ancillary memory.

B.3.2. The Quantum Switch

Processes similar to the quantum switch had some early occurrences in the literature in the 2000s, see in particular Reference [178]. Hardy is usually credited for formulating the idea of processes without fixed causal order around that time [53]. The breakthrough came from the Pavia group for the same reason cited in the main text: by probing where the quantum comb formalism could not be used to represent a circuit, they came up with a few examples during that time, all of which implying a 'causal loop' of some sort [4, 19, 65]. Remark that in the following years, it was claimed that the quantum switch was realized in a lab, see for example References [18, 179].

[178]: Oi (2003), *Interference of Quantum Channels*.

[53]: Hardy (2001), *Quantum Theory From Five Reasonable Axioms*.

[4]: Chiribella et al. (2013), *Quantum computations without definite causal structure*.
[19]: Colnaghi et al. (2012), *Quantum computation with programmable connections between gates*.
[65]: D'Ariano et al. (2011), *No Signaling, Entanglement Breaking, and Localizability in Bipartite Channels*.

[18]: Procopio et al. (2015), *Experimental superposition of orders of quantum gates*.
[179]: Rubino et al. (2017), *Experimental verification of an indefinite causal order*.

It was already known at that time that the way around this no-go theorem was to drop the assumption of a fixed causal order. The motivation for such a radical step was known before the example of the quantum switch. According to Kochen-Specker contextuality, any observable quantity cannot be defined outside of its (local) measurement context. The fact that causal structure is fixed for all observers *a prori* appears as a contradiction to this notion, as it has observable effects [15].The causal structure must itself be a variable that presents quantum characteristics; it should be able to be in a coherent superposition of several states, and consequently, there can be uncertainty on it; in such case, there may be *non-fixed causal structure* [16].

[15]: Hardy (2005), *Probability Theories with Dynamic Causal Structure: A New Framework for Quantum Gravity*.

[16]: Hardy (2007), *Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure*.

Remark that the presentation in the main text of the causal non-separability of the quantum switch was made colloquially and using the process formalism. However, the realizability of processes is still an open question. Therefore, the experimental realizations of the switch usually require a specialized framework to be treated, like the *time-delocalized subsystems* [28, 30].

[28]: Oreshkov (2019), *Time-delocalized quantum subsystems and operations: on the existence of processes with indefinite causal structure in quantum mechanics*.
[30]: Wechs et al. (2023), *Existence of processes violating causal inequalities on time-delocalised subsystems*.

On Quantum Causal Models. However, the quantum switch does not actually require such a broad framework to be discussed. It falls into a class of processes called quantum circuits with quantum control. This restricted class has been studied extensively, and its purifiable subset even more. In that latter case, a fully general model of quantum causal models has been developed [180, 181], so the treatment of the quantum switch can be made in a much more fine-grained description. In particular, the sketch of the non-causal separability of the switch presented in the main text has been made rigorous and studied in-depth using the formalism of routed quantum circuits [31, 172].

[180]: Barrett et al. (2019), *Quantum Causal Models*.
[181]: Barrett et al. (2021), *Cyclic quantum causal models*.

[31]: Ormrod et al. (2023), *Causal structure in the presence of sectorial constraints, with application to the quantum switch*.
[172]: Vanrietvelde et al. (2021), *Routed quantum circuits*.

It is then legitimate to wonder why this thesis is not considering these results. The issue is that most of these works assume a purified picture, in which the process can be identified with unitary operations. However, because of a no-go theorem of Araújo et al, the purification of certain higher-order processes is impossible[7] [182]. In particular, the OCB example is one of these no-purifiable processes. Hence, to keep the formalism broad enough to consider violations of causal inequalities

7: At least when assuming a global past and future common to all parties.

[182]: Araújo et al. (2017), *A purification postulate for quantum mechanics with indefinite causal order*.

of the OCB-kind, the process formalism considered in this thesis has not been restricted to the purifiable processes, contrary to what is done in many of these works. The reconciliation of these two approaches is nonetheless an exciting direction for future inquiries.

B.3.3. The Process Matrix Formalism

The derivation of the process formalism was inspired by the proofs of the Gleason theorem for POVM made by Busch as well as Caves and collaborators [111, 112]. In addition, the work of Barnum and collaborators [123] shaped the idea of the global process emerging from the compatibility of local operations. It should be noted that this group rederived quantum theory in an approach very similar to the process matrix but did not obtain the process matrix formalism because they assumed a fixed spacetime background and therefore a fixed causal order [57].

[111]: Busch (2003), *Quantum States and Generalized Observables: A Simple Proof of Gleason's Theorem*.
[112]: Caves et al. (2004), *Gleason-Type Derivations of the Quantum Probability Rule for Generalized Measurements*.

[123]: Barnum et al. (2005), *Influence-free states on compound quantum systems*.

[57]: Barnum et al. (2010), *Local Quantum Measurement and No-Signaling Imply Quantum Correlations*.

The process formalism was developed a year after the no-go theorem of Chiribella [4] by Oreshkov, Costa, and Brukner [5]. Its original motivation was quite different; whereas quantum combs were intended to implement general tools for analyzing the properties of every physically possible quantum circuit, *process matrices* were aimed to probe more fundamental aspects of nature, as needed in the search for a theory of quantum gravity per example. It aimed to provide a more general framework to quantum theory that could, according to Hardy's ideas (among others), treat the causal structure as *non-fixed*, which comes from the fundamentally contextual[8] behavior of quantum theory, but also as *dynamical* which is motivated by the theory of general relativity [16].

[4]: Chiribella et al. (2013), *Quantum computations without definite causal structure*.

[5]: Oreshkov et al. (2012), *Quantum correlations with no causal order*.

8: In the sense of Kochen-Speckers.

[16]: Hardy (2007), *Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure*.

Appendices to Chapter 3 C.

We're scientists. We only know things until someone shows us we're wrong.

Elvi Okoye to Fayez Okoye-Sarkis in: James S. A. Corey (2021), *The Expanse 9: Leviathan Falls*

C.1. Mathematical Methods

Projective characterization methods are extensively based on the decomposition of operators into bases that are formed by tensor products of Hermitian operators. The simplest example of such a basis is the *Pauli basis*, Definition 2.3.4.

In this section, decompositions in bases similar to the Pauli basis are reviewed. Then, the projectors on subspaces spanned by subsets of these bases, abstracted under the name *projectors on operator systems* in Definition 3.2.7, are presented alongside some basic properties of superoperator projectors.

C.1.1. Decomposition of Hermitian Operators

Operator systems are represented as real subspaces of Hermitian matrices. This section collects three basic results on how to decompose such operators because they are extensively used in the proofs presented in this chapter. For a more systematic treatment, see for example References [96, 162].

First, there always exists a Hermitian basis of $\mathcal{L}(\mathcal{H})$, so that $\forall W \in \mathcal{L}(\mathcal{H})$,

$$W = \sum_i q_i \, \sigma_i \, , \tag{C.1}$$

where $q_i \in \mathbb{C}$ and $\sigma_i^\dagger = \sigma_i$.

Second, because $\mathcal{L}\left(\mathcal{H}^A\right) \otimes \mathcal{L}\left(\mathcal{H}^B\right) \cong \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$, it is true that

$$\forall V_A \in \mathcal{L}\left(\mathcal{H}^A\right), \forall N_B \in \mathcal{L}\left(\mathcal{H}^B\right): \quad (V_A \otimes N_B)^\dagger = V_A^\dagger \otimes N_B^\dagger \, . \tag{C.2}$$

This, in turn, implies that a factorizable self-adjoint basis exists, so that all $W_{AB} \in \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ can be decomposed as

$$W_{AB} = \sum_{ij} q_{ij} \sigma_i^A \otimes \sigma_j^B \, , \tag{C.3}$$

where $q_{ij} \in \mathbb{C}$ $\{\sigma_i^A\}$ and $\{\sigma_j^B\}$ are Hermitian bases like Equation C.15 for respectively spaces $\mathcal{L}\left(\mathcal{H}^A\right)$ and $\mathcal{L}\left(\mathcal{H}^B\right)$. Remark that since the basis elements are self-adjoint, whenever W is self-adjoint the weights q_{ij} are real.

[96]: Roman (2008), *Advanced Linear Algebra*.
[162]: Farenick (2000), *Algebras of Linear Transformations*.

Third, a self-adjoint operator can always be expressed as the difference between two positive operators,

$$\forall W = W^\dagger,\, \exists W^+, W^- \geq 0 : \quad W = W^+ - W^- \,, \tag{C.4}$$

[159]: Hall (2013), *Quantum Theory for Mathematicians*.
[164]: Horn et al. (2013), *Matrix Analysis*.

This is a consequence of the spectral theorem [159, 164]. This last observation can be extended to operator systems to define them as the difference between two unnormalized elements of a state structure.

Lemma C.1.1 *Any trace-normalized self-adjoint operator in an operator system supporting state structure $\mathscr{A}$ can be represented as an affine difference between two elements of the state structure $\mathscr{A}$. In symbols,*

$$\begin{gathered}\forall V : V = V^\dagger,\, \mathcal{P}_A\left\{V\right\} = V,\, \mathit{Tr}\left[V\right] = c_A, \\ \exists V_1, V_2 \in \mathscr{A}, \exists q_1, q_2 \in \mathbb{R}^+,\, q_1 - q_2 = 1 \\ \textit{such that} \\ V = q_1 V_1 - q_2 V_2 \,.\end{gathered} \tag{C.5}$$

Proof. V factors as

$$V = V^+ - V^- \,. \tag{C.6}$$

These two parts are positive so they cannot have negative traces. If any part has a zero trace, then because of

$$c_A = \mathrm{Tr}\left[V\right] = \mathrm{Tr}\left[V^+\right] + \mathrm{Tr}\left[V^-\right] \,, \tag{C.7}$$

either $\mathrm{Tr}\left[V^-\right] = 0$ in which case V is positive and $q_1 = 1$ and $V_1 = V = V^+$. Or $\mathrm{Tr}\left[V^+\right] = 0$ in which case V is positive times a negative constant and $q_2 = 1$ and $V_2 = V$.

In between these two special cases, as a projector on operator systems is linear by definition, and since $V = V^+ - V^-$, it should hold that

$$V = \mathcal{P}_A\left\{V^+\right\} - \mathcal{P}_A\left\{V^-\right\} \,, \tag{C.8}$$

Each part of the difference has support only on the operator system of $\mathscr{A}$. However, the projection may have turned the positive operators V^+ and V^- into Hermitian operators. Hence they need to be split again. Let $\mathcal{P}_A\left\{V^+\right\} = \mathcal{P}_A\left\{V^+\right\}^+ - \mathcal{P}_A\left\{V^+\right\}^-$ and $\mathcal{P}_A\left\{V^-\right\} = \mathcal{P}_A\left\{V^-\right\}^+ - \mathcal{P}_A\left\{V^-\right\}^-$. By defining $\mathcal{P}_A\left\{V^+\right\}^+$ to be the operator built from the projection on the eigenvectors with positive eigenvalues of $\mathcal{P}_A\left\{V^+\right\}$ and $\mathcal{P}_A\left\{V^+\right\}^-$ the one with the negative eigenvalues and by proceeding similarly with $\mathcal{P}_A\left\{V^-\right\}^+$ and $\mathcal{P}_A\left\{V^-\right\}^-$, these four operators are positive and supported on $\mathscr{A}$ only. Define

$$\tilde{V}_1 := \mathcal{P}_A\left\{V^+\right\}^+ + \mathcal{P}_A\left\{V^-\right\}^- \;; \tag{C.9a}$$
$$\tilde{V}_2 := \mathcal{P}_A\left\{V^+\right\}^- + \mathcal{P}_A\left\{V^-\right\}^+ \,. \tag{C.9b}$$
$$\tag{C.9c}$$

These are two positive operators that have support on the operator system of $\mathscr{A}$. If any of those is zero the situation is back to the special cases discussed above, so assuming they are not V can be rewritten as

$$V = \tilde{V}_1 - \tilde{V}_2 \,, \tag{C.10}$$

it only remains to normalize them properly. Define

$$q_1 := \frac{\mathrm{Tr}\left[\tilde{V}_1\right]}{c_A}\,, \quad q_2 := \frac{\mathrm{Tr}\left[\tilde{V}_2\right]}{c_A}\,, \tag{C.11a}$$

$$V_1 := c_A\,\frac{\tilde{V}_1}{\mathrm{Tr}\left[\tilde{V}_1\right]}\,, \quad V_2 := c_A\,\frac{\tilde{V}_2}{\mathrm{Tr}\left[\tilde{V}_2\right]}\,, \tag{C.11b}$$

so that the form of Equation C.5 has been reached:

$$V = q_1 V_1 - q_2 V_2\,. \tag{C.12}$$

To check that it obeys the conditions, on the one hand, $\frac{\mathrm{Tr}\left[\tilde{V}_1\right]}{c_A}$ and $\frac{\mathrm{Tr}\left[\tilde{V}_2\right]}{c_A}$ are positive real numbers hence V_1 and V_2 are positive operators that have support on $\mathscr{A}$ and have a trace norm of c_A. Therefore they are valid elements of $\mathscr{A}$. On the other hand, since $\mathrm{Tr}\left[V\right] = c_A$, $c_A = \mathrm{Tr}\left[\tilde{V}_1\right] - \mathrm{Tr}\left[\tilde{V}_2\right]$ and therefore

$$q_1 - q_2 = \frac{\mathrm{Tr}\left[\tilde{V}_1\right]}{c_A} - \frac{\mathrm{Tr}\left[\tilde{V}_2\right]}{c_A} = 1\,, \tag{C.13}$$

concluding the proof. □

C.1.2. Decomposition in Traceless Basis

There are many ways to construct a basis for the decomposition of Equation C.1. But, since all operator systems contain an element proportional to the identity, it is relevant to use a basis that features the identity as one of its a elements. Because of the orthonormality condition, all the other basis elements $\{\sigma_i\}$ will be traceless since the inner product will require that $(\mathbb{1}\,,\,\sigma_i) = \mathrm{Tr}\left[\mathbb{1}^\dagger\sigma_i\right] = 0$.

That way, any trace-normalized Hermitian operator has a *Bloch vector* [52] or *Fano* [74] decomposition as

$$V = \frac{c}{d}\,\mathbb{1} + \sum_i^{d^2-1} v_i\,\sigma_i\,. \tag{C.14}$$

[52]: Nielsen et al. (2009), *Quantum Computation and Quantum Information*.

[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.

There are many ways of building a Fano decomposition. In accordance with the previous section, the extra condition that this basis is self-adjoint is taken. In two dimensions, an example of such a basis is the Pauli basis (2.33). But such a basis exists for all dimensions: to construct it one can use the d^2-1 generators of the representation of the $\mathfrak{su}(d)$ algebra in $\mathcal{L}\left(\mathcal{H}\right)$ (which are traceless and self-adjoint; see e.g. Ref. [159, Ch. 16]) and the identity matrix (which has a Hilbert-Schmidt inner product with all traceless matrices equal to zero and which is self-adjoint). Yet, in the case of the Pauli basis, the inner product of an element with itself yields the dimension i.e. $\mathrm{Tr}\left[\mathbb{1}^2\right] = \mathrm{Tr}\left[X^2\right] = \mathrm{Tr}\left[Y^2\right] = \mathrm{Tr}\left[Z^2\right] = 2$. A better-behaved basis will be required to be normalized instead so that $(\sigma_i\,,\,\sigma_j) = \mathrm{Tr}\left[\sigma_i^\dagger\cdot\sigma_j\right] = \delta_{i,j}$ [1]. This self-adjoint orthonormal basis, which is sometimes called *a generalised Gell-Mann* [183] or a *conal* [184]

[159]: Hall (2013), *Quantum Theory for Mathematicians*.

1: Where $\delta_{i,j}$ is the Kronecker symbol. Notice that the Pauli basis can be turned into that form simply by dividing each of its elements by $\frac{1}{\sqrt{2}}$.

[183]: Bertlmann et al. (2008), *Bloch vectors for qudits*.

[184]: Arrighi et al. (2003), *Conal representation of quantum states and non-trace-preserving quantum operations*.

basis, satisfies

$$(\sigma_i)^\dagger = \sigma_i\,; \tag{C.15a}$$
$$\sigma_0 := \mathbb{1}/\sqrt{d_A}\,; \tag{C.15b}$$
$$\mathrm{Tr}\left[\sigma_{i\neq 0}\right] = 0\,; \tag{C.15c}$$
$$\left(\sigma_i\,,\,\sigma_j\right) := \mathrm{Tr}\left[\sigma_i \cdot \sigma_j\right] = \delta_{i,j}\,. \tag{C.15d}$$

And is used in the proof of Lemma 3.5.3 as well as the one of Lemma 3.5.1. For this last proof, a characterization of positivity for 2-dimensional subspaces in this basis is used. This is the following result.

Lemma C.1.2 *A Hermitian operator of the form*

$$V = \frac{c}{\sqrt{d}}\mathbb{1} + v\sigma_i = c\sigma_0 + v\sigma_i \tag{C.16}$$

when expressed in a basis of the form (C.15) *is positive semi-definite if*

$$\frac{c}{\sqrt{d}} \geq \left|\frac{v}{\sqrt{2}}\right|\,. \tag{C.17}$$

In particular, V is positive for $c = |v|$ when $d > 1$.

Proof. Assume without loss of generality that $v \geq 0$. Since σ_0 is the identity it commutes with σ_i so they are simultaneously diagonalizable. Let the d eigenvalues of σ_i be sorted in the vector $\{s_0, ..., s_{d-1}\}$ in ascending order[2], so that the k-th diagonal element of the diagonalization of $c\sigma_0 + v\sigma_i$ is equivalent to $\frac{c}{\sqrt{d}} + v \times s_k$. The smallest eigenvalue of $c\sigma_0 + v\sigma_i$ is therefore $\frac{c}{\sqrt{d}} + v \times s_0$.

2: If v was negative, the proof would still hold by sorting them in descending order.

The trace of a matrix is the Schatten 1-norm (see e.g., References [74, 90]) and therefore equivalent to the sum of the eigenvalues. Since the matrix σ_i is traceless but non-zero, $\sum_k s_k = 0$ and s_0 must therefore be negative otherwise all the eigenvalues of σ_i are zero, thus

[74]: Bengtsson et al. (2017), *Geometry of Quantum States: An Introduction to Quantum Entanglement*.
[90]: Watrous (2018), *The Theory of Quantum Information*.

$$s_0 = -\sum_{k\neq 0} s_k \tag{C.18}$$

The Hilbert-Schmidt inner product of a matrix with itself is its Schatten 2-norm. As the matrices σ_i are normalized to 1 with respect to this inner product, their eigenvalues obey the following relation

$$\sqrt{s_0^2 + \sum_{k\neq 0} s_k^2} = 1\,. \tag{C.19}$$

By the triangle inequality, it is straightforward to see that the worst case scenario happens when $-s_0 = s_k = 1/\sqrt{2}$ so a sufficient condition for $\frac{c}{\sqrt{d}} + v \times s_0$ to be positive is indeed that $\frac{c}{\sqrt{d}} \geq \frac{v}{\sqrt{2}}$. □

C.1.3. Superoperator projectors

The concept of a projector can also be defined for superoperators. A linear superoperator $\mathcal{P}_A \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^A\right)\right)$ is a *projector* on a subspace of

$\mathcal{L}\left(\mathcal{H}^A\right)$ if it is idempotent,

$$\mathcal{P}_A \circ \mathcal{P}_A = \mathcal{P}_A \,. \tag{C.20}$$

Two examples of such projectors are the identity map,

$$\mathcal{I}_A : \quad \forall V \in \mathcal{L}\left(\mathcal{H}^A\right) \,, \qquad \mathcal{I}_A(V) = V \,, \tag{C.21}$$

which projects $\mathcal{L}\left(\mathcal{H}^A\right)$ onto itself and the depolarizing superoperator

$$\mathcal{D}_A : \quad \mathcal{D}_A(V) := \frac{\mathbb{1}^A}{d_A} \, \mathrm{Tr}_A\left[V\right] \,, \tag{C.22}$$

which projects $\mathcal{L}\left(\mathcal{H}^A\right)$ onto the span of the identity operator.

An operator $V \in \mathcal{L}\left(\mathcal{H}^A\right)$ belongs to the subspace defined by $\mathcal{P}_A$ if and only if

$$\mathcal{P}_A\left\{V\right\} = V \,. \tag{C.23}$$

An operator that does not belong to the subspace defined by $\mathcal{P}_A$ belongs to the one defined by its complement, noted $\mathcal{P}_A^{\perp}$ and defined as

$$\mathcal{P}_A^{\perp} := \mathcal{I}_A - \mathcal{P}_A \,. \tag{C.24}$$

$\mathcal{P}_A$ projects to a subspace closed under the adjoint if it obeys

$$\mathcal{P}_A\left\{V\right\} = V \Rightarrow \mathcal{P}_A\left\{V^{\dagger}\right\} = V^{\dagger} \,. \tag{C.25}$$

This condition is actually necessary and sufficient since $\mathcal{P}_A\left\{V^{\dagger}\right\} = V^{\dagger} \Rightarrow \mathcal{P}_A\left\{(V^{\dagger})^{\dagger}\right\} = (V^{\dagger})^{\dagger}$ which is the left-hand side of Equation (C.25) since the Hermitian adjoint is an involution. Then, the left-hand side of Equation (C.25) can be inserted into the right-hand side so that

$$\mathcal{P}_A\left\{\mathcal{P}_A\left\{V\right\}^{\dagger}\right\} = \mathcal{P}_A\left\{V\right\}^{\dagger} \,. \tag{C.26}$$

This can be written concisely as $\mathcal{P}_A \circ \dagger \circ \mathcal{P}_A = \dagger \circ \mathcal{P}_A$ where $\dagger$ means 'taking the adjoint in $\mathcal{L}\left(\mathcal{H}^A\right)$'.

A projector is *orthogonal* if it does not increase the norm of operators[3]:

$$\left|\left|\mathcal{P}_A\left\{V\right\}\right|\right|_2 \leq \left|\left|V\right|\right|_2 \,. \tag{C.27}$$

3: Here $\left|\left|V\right|\right|_2 \equiv \sqrt{\mathrm{Tr}\left[V^{\dagger} \cdot V\right]}$ is the Hilbert-Schmidt norm

This condition is equivalent to self-adjointness (see e.g., [96]),

$$\mathrm{Tr}\left[\mathcal{P}_A\left\{V'\right\}^{\dagger} \cdot V\right] = \mathrm{Tr}\left[V'^{\dagger} \cdot \mathcal{P}_A\left\{V\right\}\right] \,, \quad \forall V, V' \,. \tag{C.28}$$

Indicating the adjoint of a map $\mathcal{M} \in \mathcal{L}\left(\mathcal{L}\left(\mathcal{H}^A\right), \mathcal{L}\left(\mathcal{H}^A\right)\right)$ with $*$ so that $(V' \,,\, \mathcal{M}\left(V\right)) = (\mathcal{M}^*(V') \,,\, V)$, this condition can be written concisely as $\mathcal{P}_A = \mathcal{P}_A^*$. All projectors are thereafter assumed orthogonal.

Since the space of superoperators is a Hilbert space of bounded operators, the properties of orthogonal projectors on Hilbert spaces also apply to them. A few results from this rich theory will underly the mathematical properties of the projective characterization methods. See references [71, 96, 106, 162] for a more thorough exposition. The main property, which is the mathematical starting point of Chapter 5, is that the projectors on a

[71]: von Neumann (1932), *Mathematical Foundations of Quantum Mechanics*.
[96]: Roman (2008), *Advanced Linear Algebra*.
[106]: Sinclair et al. (2008), *Finite von Neumann Algebras and Masas*.
[162]: Farenick (2000), *Algebras of Linear Transformations*.

Hilbert space are in bijection with closed subspaces of the Hilbert space. Meaning $\mathcal{P}$ can be identified with $\mathrm{Im}\{\mathcal{P}\}$ and all their properties will be in one-to-one correspondence; this is why Chapter 5 focuses on the algebra of projectors rather than on the inclusions properties of operator systems.

From there, the inclusion of the image of two projectors can be phrased as an algebraic property. This is the content of the following standard result.

Proposition C.1.3 (Inclusion of Projectors) *Let $\mathcal{P}$ and $\mathcal{P}^{'}$ two orthogonal (superoperator) projectors on some Hilbert space of operators $\mathcal{L}(\mathcal{H})$. Then the following are equivalent:*

$$\mathrm{Im}\{\mathcal{P}\} \subseteq \mathrm{Im}\left\{\mathcal{P}^{'}\right\} ; \tag{C.29a}$$
$$\mathcal{P} \circ \mathcal{P}^{'} = \mathcal{P} ; \tag{C.29b}$$
$$\mathcal{P}^{'} \circ \mathcal{P} = \mathcal{P} ; \tag{C.29c}$$
$$||\mathcal{P}\{V\}||_2 \leq ||\mathcal{P}^{'}\{V\}||_2, \quad \forall V \in \mathcal{L}(\mathcal{H}) ; \tag{C.29d}$$
$$0 \leq Tr\left[V^{\dagger} \cdot \left(\mathcal{P}^{'} - \mathcal{P}\right)\{V\}\right], \quad \forall V \in \mathcal{L}(\mathcal{H}) . \tag{C.29e}$$

In general, an orthogonal superoperator projector can be written as a sum of vectors in $\mathcal{L}(\mathcal{H})$ times functional in $\mathcal{L}(\mathcal{H})^*$ like

$$\mathcal{P}\{\cdot\} = \sigma_i \mathrm{Tr}\left[\sigma_i^{\dagger} \cdot\right] . \tag{C.30}$$

Where the orthonormal collection of operators $\{\sigma_i\} \subset \mathcal{L}(\mathcal{H})$ spans the subspace defined by the projector. This is the analog to how a projector on a subspace of $\mathcal{H}$ can be written as $\Pi = \sum_i |\phi_i\rangle\langle\phi_i|$ for a given orthonormal set of vectors $\{\phi_i\} \subset \mathcal{H}$ defining the support of Π.

As the orthogonal projectors should be used to characterize operator systems, which are subspaces that contain the identity, it should be true that their images contain the span of the identity as a subspace. By Proposition C.1.3, a necessary and sufficient condition for that to be true is

$$\mathcal{P}_A \circ \mathcal{D}_A = \mathcal{D}_A , \tag{C.31}$$

with $\mathcal{D}_A$ defined as in Eq. (C.22).

Finally, and as defined in the main text, a **projector on operator systems** is orthogonal, and projects onto a self-adjoint subspace that contains the identity: it obeys conditions (C.20), (C.25), (C.28), and (C.31). Also, note that a property of superoperator projectors is that they are stable under transpose. This is because, on the one hand, they are $\mathbb{C}$-linear maps thus stable under complex conjugation. And, on the other hand, they are stable under Hermitian adjoint by definition. Combining the two yields stability under the transpose.

C.1.4. Examples of Projectors on Operator Systems

Besides $\mathcal{I}$ and $\mathcal{D}$, an example of a projector on an operator system is the 'dephasing' or 'quantum-to-classical' map Δ that sends an operator to its diagonal subspace with respect to a given basis[4] Consider $\mathcal{H}$ to be of dimension two. Expressing the qubits states $\rho \in \mathcal{L}(\mathcal{H})$ in the Pauli basis (2.33), the dephasing superoperator projector with respect to this basis has the form

$$\Delta = \frac{\mathbb{1}}{2}\mathrm{Tr}\left[\cdot\right] + \frac{Z}{2}\mathrm{Tr}\left[Z\cdot\right] . \tag{C.32}$$

4: The use of the symbol Δ instead of a calligraphic letter is made to stick with the literature.

It can be checked that it obeys the different requirements of Definition 3.2.7: it is an idempotent map because the Pauli matrices obey the relation

$$\sigma_i \cdot \sigma_j = \delta_{i,j}\mathbb{1} + \epsilon_{i,j,k} i \sigma_k \tag{C.33}$$

for $\sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z$, $\delta_{i,j}$ the Kronecker symbol, and $\epsilon_{i,j,k}$ the Levi-Civita symbol; It is Hermitian-preserving because the Pauli basis is hermitian; It is self-adjoint because its CJ operator[5] is

$$M^{(\Delta)} = \mathbb{1} \otimes \mathbb{1} + Z \otimes Z\,, \tag{C.34}$$

5: In order to compute it, the maximally entangled state in Pauli basis is given by Equation C.36 below.

which is symmetric and therefore self-adjoint as the adjoint * of linear maps corresponds to the transpose ^{T} of their Choi operators under CJ isomorphism; and it contains the depolarizing superoperator $\mathcal{D}\{\cdot\} := \frac{\mathbb{1}}{d}\ \mathrm{Tr}\left[\cdot\right]$ as can be seen from direct inspection. Notice in passing that the Choi operator of Δ is positive, therefore it is a completely positive map.

6: This was brought to my attention by Nicola Pinzani, to whom I am grateful.

However, projectors on operator systems are usually not even positive-preserving as shown in the following example[6]. The counterexample is built from the map

$$\mathcal{P}^{\mathbb{R}}\left\{\cdot\right\} := \frac{\mathbb{1}}{2}\mathrm{Tr}\left[\cdot\right] + \frac{X}{2}\mathrm{Tr}\left[X\cdot\right] + \frac{Z}{2}\mathrm{Tr}\left[Z\cdot\right] , \tag{C.35}$$

7: Meaning that it is the largest subspace whose basis can be expressed using real numbers only.

which projects onto the "real" subspace of the Bloch sphere[7].

This map is positive-preserving but nonetheless not CP. Consider the bipartite maximally entangled state $\phi^+ = \sum_{i,j=0}^{1,1} 1/2 \left|i\right\rangle\left\langle j\right|_A \otimes \left|i\right\rangle\left\langle j\right|_B$. This is a state in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ thus a positive operator. In the Pauli basis, it has the form

$$\phi^+ = \frac{1}{2}\left(\mathbb{1}_A \otimes \mathbb{1}_B + X_A \otimes X_B - Y_A \otimes Y_B + Z_A \otimes Z_B\right) . \tag{C.36}$$

Applying the projector $\mathcal{P}^{\mathbb{R}}\left\{\cdot\right\}$ on one side, say A, yields the state

$$\left(\mathcal{P}^{\mathbb{R}}_A \otimes \mathcal{I}_B\right)\left\{\phi^+\right\} = \frac{1}{2}\left(\mathbb{1}_A \otimes \mathbb{1}_B + X_A \otimes X_B + Z_A \otimes Z_B\right) , \tag{C.37}$$

which is no longer positive. Moreover, remark that $\mathcal{P}^{\mathbb{R}}_A \otimes \mathcal{I}_B$ is also a projector on operator system, but space $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ this time. This constitutes an example of a non-positive-preserving projector on operator system. For the same reason, the projectors $\mathcal{I}_A \otimes \mathcal{P}^{\mathbb{R}}_B$ and $\mathcal{P}^{\mathbb{R}}_A \otimes \mathcal{P}^{\mathbb{R}}_B$ are also non-positive-preserving. It appears that the positivity-preservation of projectors $\mathcal{P}$ is not a property preserved by their quasi-orthogonal complement $\overline{\mathcal{P}}$.

C.2. Proofs

C.2.1. Proof of Theorem 3.3.2

Theorem 3.3.2 states that the set of operators V such that

$$\forall N \in \mathscr{A} \ : \ \mathrm{Tr}\left[V^\dagger \cdot N\right] = 1\,, \tag{3.29}$$

and taking each element E_a of every resolution of N to a positive number between 0 and 1, i.e.,

$$\mathrm{Tr}\left[V^\dagger \cdot E_i\right] \in [0,1]\,, \tag{3.30}$$

is a state structure noted $\overline{\mathscr{A}}$ characterized by the following conditions:

$$V \in \overline{\mathscr{A}} \iff$$

$$V \geq 0\,, \tag{3.31a}$$

$$\mathrm{Tr}\left[V\right] = \frac{d_A}{c_A} =: c_{\overline{A}}\,, \tag{3.31b}$$

$$\mathcal{P}_{\overline{A}} := \{\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A\}\,(V) = V\,. \tag{3.31c}$$

Proof. As $c_A/d_A\mathbb{1}$ is a valid element of $\mathscr{A} \subseteq \mathcal{L}\left(\mathcal{H}^A\right)$, any element V of $\overline{\mathscr{A}}$ must satisfy $1 = (V\,,\, c_A/d_A\mathbb{1}) = \mathrm{Tr}\left[V^\dagger \cdot c_A/d_A\mathbb{1}\right] = c_A/d_A\mathrm{Tr}\left[V\right]$. Since V is arbitrary, this fixes the normalization for all elements of $\overline{\mathscr{A}}$: $d_A/c_A = \mathrm{Tr}\left[V\right] =: c_{\overline{A}}$, this is condition (3.31b).

Positivity follows from the requirement of element-wise positivity with each resolution. For any V, let $\{|e_j\rangle\}$ is an orthonormal basis of $\mathcal{H}^A$ so that V is diagonal in this basis. Then its eigendecomposition is given by $V = \sum_j a_j\,|e_j\rangle\!\langle e_j|$, where $a_j \in \mathbb{C}$ are the eigenvalues of V. Since $c_A/d_A\mathbb{1} = c_A/d_A\sum_j |e_j\rangle\!\langle e_j|$, a resolution of $c_A/d_A\mathbb{1}$ can be defined as $\{E_j = c_A/d_A\,|e_j\rangle\!\langle e_j|\}_{j=0}^{d_A-1}$. That each element in the resolution gives a number between 0 and 1 reads $\forall j, \langle V\,,\, E_j\rangle \in [0,1]$. But $\langle V\,,\, E_j\rangle = \mathrm{Tr}\left[\left(\sum_i a_i\,|e_i\rangle\!\langle e_i|\right)^\dagger \cdot \left(c_A/d_A\,|e_j\rangle\!\langle e_j|\right)\right] = c_A/d_A\,\sum_i\,\overline{a_i}\,\delta_{i,j}$, where $\overline{\cdot}$ indicates complex conjugation and $\delta_{i,j}$ is the Kronecker delta. Hence, $\forall j, \langle V\,,\, E_j\rangle = c_A/d_A\,\overline{a_j} \in [0,1]$, and since c_A/d_A is a positive real constant, this means that each a_j is real and greater or equal to zero. Therefore, V is positive semi-definite since it has a positive spectrum, condition (3.31a).

Finally, the projective condition remains. First, $1/c_A\mathbb{1}$ must belong to $\overline{\mathscr{A}}$ as it is positive, properly normalized, and respects $\mathrm{Tr}\left[1/c_A\mathbb{1} \cdot N\right] = 1$ for all $N \in \mathscr{A}$. Assuming that $\overline{\mathscr{A}}$ is a deterministic state structure with projector $\overline{\mathcal{P}}_A := \mathcal{I} - \mathcal{P}_A + \mathcal{D}$, the if part follows: Any positive and properly normalized operator V in $\mathcal{L}\left(\mathcal{H}^A\right)$ on which the projector $\overline{\mathcal{P}}_A$ is applied obeys $\left\langle \overline{\mathcal{P}}_A\left\{V\right\}\,,\, N\right\rangle = 1$ for all $N \in \mathscr{A}$ because $\left\langle \overline{\mathcal{P}}_A\left\{V\right\}\,,\, N\right\rangle \equiv \langle(\mathcal{I} - \mathcal{P}_A)\left\{V\right\}, N\rangle + \langle\mathcal{D}\{V\}\,, N\rangle$. The first member on the right part of the equality vanishes because it belongs to the orthogonal complement of $\mathscr{A}$; the second member is normalized so that $\langle\mathcal{D}\{V\}\,, N\rangle = \langle\mathrm{Tr}\left[V\right]/d_A\mathbb{1}\,, N\rangle = \left\langle \frac{d_A}{c_A d_A}\mathbb{1}\,, N\right\rangle = 1/c_A\mathrm{Tr}\left[N\right] = 1$.

The only if part is proven by a counterexample: assume that there is a positive X with trace norm $c_{\overline{A}} = d_A/c_A$ that does not belong to the space characterized by $\overline{\mathcal{P}}_A$. Then, $\left(\mathcal{I} - \overline{\mathcal{P}}_A\right)\{X\} = X$. From the definition of the projector, $\mathcal{I} - \overline{\mathcal{P}}_A = \mathcal{P}_A - \mathcal{D}$. Now, letting $N = c_A/d_A\,\mathbb{1}$ gives $(X\,,\, N) =$

$(X, c_A/d_A \mathbb{1}) = 1$. Yet, by applying the projector on X it should be that $((\mathcal{P}_A - \mathcal{D})\{X\}, c_A/d_A\mathbb{1}) = 1$, hence $(\mathcal{P}\{X\}, c_A/d_A\mathbb{1}) - 1 = 1$ but $\mathcal{P}_A$ is a self-adjoint projector so $(\mathcal{P}\{A\}, \mathbb{1}) = 2d_A/c_A$ becomes $\langle X, \mathcal{P}_A\{\mathbb{1}\}\rangle = \langle X, \mathbb{1}\rangle = \text{Tr}[X] = 2d_A/c_A$. However, the trace norm of X was assumed to be $c_{\overline{A}} := d_A/c_A$. By consequence, a contradiction has been reached. This proves the necessity of condition (3.31c), concluding the proof. □

C.2.2. Proof of Theorem 3.4.1

Theorem 3.4.1 states that the set of all admissible and structure-preserving maps between state structures $\mathscr{A}$ and $\mathscr{B}$ as defined by Definition 3.4.3 are, in CJ representation, the set noted $\mathscr{A} \to \mathscr{B}$ whose elements M obey

$$M \geq 0\,, \tag{3.70a}$$

$$\text{Tr}[M] = c_{\overline{A}} c_B = \frac{c_B}{c_A} d_A\,, \tag{3.70b}$$

$$\mathcal{P}_{A\to B}\{M\} = M\,. \tag{3.70c}$$

[32]: Castro-Ruiz et al. (2018), *Dynamics of Quantum Causal Structures*.

Proof. Following [32], the second requirement of Definition 3.4.3 states that $\mathcal{M}(V)$ is a valid element of $\mathscr{B}$:

$$\mathcal{M}(V) \geq 0\,, \tag{C.40a}$$

$$\text{Tr}[\mathcal{M}(V)] = c_B\,, \tag{C.40b}$$

$$\mathcal{P}_B \circ \mathcal{M} \circ \mathcal{P}_A = \mathcal{M} \circ \mathcal{P}_A\,. \tag{C.40c}$$

The third requirement further imposes condition (3.70a) by definition. Because of the first requirement, $\mathcal{M}$ is linear so it has a CJ representation. Denote by M the CJ representation of $\mathcal{M}$, complete positivity imposes condition (3.70a). Using the reverse definition of CJ isomorphism, Equation 2.7, condition (C.40b) becomes (subscripts have been put on M and V for clarity)

$$\text{Tr}[M_{AB} \cdot (V_A \otimes \mathbb{1}_B)] = c_B\,, \tag{C.41}$$

from which the normalization condition (3.70b) follows as V_A can be $c_A\mathbb{1}/d_A$.

Finally, as $\mathcal{M}(V) = (\text{Tr}_A[M \cdot (V \otimes \mathbb{1}_B)])^T$ and $\mathcal{M}(V) \in \mathscr{B}$, it should be true that, $\forall N \in \overline{\mathscr{B}}$,

$$\begin{aligned} 1 = \text{Tr}_B[N \cdot \mathcal{M}(V)] \\ = \text{Tr}_B\left[N \cdot (\text{Tr}_A[M \cdot (V \otimes \mathbb{1}_B)])^T\right] \\ = \left(\text{Tr}_B\left[\text{Tr}_A[M \cdot (V \otimes \mathbb{1}_B)] \cdot N^T\right]\right)^T \\ = \text{Tr}\left[M \cdot (V \otimes N^T)\right]\,. \end{aligned} \tag{C.42}$$

Where M is positive and normalized, and where $V \in \mathscr{A}$ and $N \in \overline{\mathscr{B}}$ are arbitrary. This last equation, as it is linear, also holds for all affine combinations of elements in $\mathscr{A}$ and $\overline{\mathscr{B}}$,

$$1 = \text{Tr}\left[M \cdot (V \otimes N^T)\right] \Rightarrow \text{Tr}\left[M \cdot \left(\sum_i q_i (V_i \otimes N_i^T)\right)\right] = 1\,, \tag{C.43}$$

$\forall\{q_i\} \subset \mathbb{R} : \sum_i q_i = 1$, $\forall\{V_i\} \subset \mathscr{A}$, $\forall\{N_i\} \subset \overline{\mathscr{B}}$. By Lemma 3.5.1, the affine combinations of $\mathscr{A}$ and $\overline{\mathscr{B}}$ are themselves a state structure characterized by

$$\begin{aligned} W : & \\ W &\geq 0\,, && \text{(C.44a)} \\ \mathrm{Tr}\,[W] &= c_A c_{\overline{B}}\,, && \text{(C.44b)} \\ \left(\mathcal{P}_A \otimes \overline{\mathcal{P}}_B\right)\{W\} &= W\,, && \text{(C.44c)} \end{aligned}$$

and noted by $\mathscr{A} \otimes \overline{\mathscr{B}}$ according to Definition 3.4.1. Hence, the consequence of Equation C.42 is that the set of all M obeying it is the set of deterministic functionals on $\overline{\mathscr{A} \otimes \overline{\mathscr{B}}}$. Therefore, by Proposition 3.3.2, it is characterized by a projector $\overline{\mathcal{P}_A \otimes \overline{\mathcal{P}}_B}$ yielding the projective condition (3.70c). Finally, notice that $\mathcal{P}_{A\to B}$ can be expressed as tensor product and negation of projectors on operators systems. As a consequence, it is a valid projector on operator system as is proven explicitly in Appendix D.3.2. □

C.2.3. Proof of Lemma 3.5.1

The proof of Theorem 3.4.1 relies on the statement that the tensor composite state structure as in Definition 3.4.1 belongs to the affine span of the composed state structures, i.e.,

$$\left\{\sum_i q_i\, V_i \otimes N_i | V_i \in \mathscr{A},\, N_i \in \mathscr{B},\, q_i \in \mathbb{R} : \sum_i q_i = 1\right\} \supset \mathscr{A} \otimes \mathscr{B}\,. \tag{3.81}$$

Proof. To prove that any $W \in \mathscr{A} \otimes \mathscr{B}$ decomposes into $\sum_i q_i\, V_i \otimes N_i$, the self-adjointness of W is used. Because W is self-adjoint, it can be written in a Hermitian basis as a real-linear sum like Equation C.3,

$$W = \sum_{i=0,j=0}^{n_A,n_B} q_{ij}\,\sigma_i^A \otimes \sigma_j^B\,, \tag{C.45}$$

so that the basis element are of the form (C.15) and are restricted to the support of $\mathscr{A} \otimes \mathscr{B}$ and where the number of non-zero coefficients is $(n_A+1)(n_B+1)$ so that $n_A + 1 \leq d_{\mathscr{A}} \leq d_A$ and $n_B + 1 \leq d_{\mathscr{B}} \leq d_B$.

According to Lemma C.1.2, the elements in the decomposition can be made positive by adding the identity times a positive constant to them. This constant can actually be chosen to be $|q_{ij}|$. In addition, each of these positive elements can be made so that they have a trace of $c_A c_B$. Let

$$\begin{aligned} & \tilde{q}_{00} := 1\,, && \text{(C.46a)} \\ (i,j) \neq (0,0) \in [0,n_A]\times[0,n_B]\,,\quad & \tilde{q}_{ij} := \frac{\sqrt{d_A}}{c_A}\frac{\sqrt{d_B}}{c_B}|q_{ij}|\,, && \text{(C.46b)} \\ (i,j) \neq (n_A,n_B) \in [n_A,2n_A]\times[n_B,2n_B]\,,\quad & \tilde{q}_{ij} := -\tilde{q}_{i-n_A\,j-n_B}\,, && \text{(C.46c)} \end{aligned}$$

and

$$W_{00} = \frac{c_A c_B}{\sqrt{d_A}\sqrt{d_B}} \left(\sigma_0^A \otimes \sigma_0^B\right) , \tag{C.47a}$$

$$(i,j) \neq (0,0) \in \; [0, n_A] \times [0, n_B] \, , \quad W_{ij} = \frac{c_A c_B}{\sqrt{d_A}\sqrt{d_B}} \left(\sigma_0^A \otimes \sigma_0^B + \frac{q_{ij}}{|q_{ij}|}\sigma_i^A \otimes \sigma_j^B\right) , \tag{C.47b}$$

$$(i,j) \neq (n_A, n_B) \in \; [n_A, 2n_A] \times [n_B, 2n_B] \, , \quad W_{ij} = \frac{c_A c_B}{\sqrt{d_A}\sqrt{d_B}} \left(\sigma_0^A \otimes \sigma_0^B\right) . \tag{C.47c}$$

That way,

$$W = \sum_{i=0,j=0}^{2n_A n_B - 1} \tilde{q}_{ij} W_{ij} \, , \tag{C.48}$$

where each W_{ij} is a valid element of $\mathscr{A} \otimes \mathscr{B}$ and $\sum_{ij} \tilde{q}_{ij} = 1$ since $\mathrm{Tr}\,[W] = (n_A+1)(n_B+1)(c_A c_B) - ((n_A+1)(n_B+1)-1)c_A c_B$. The decomposition in $(n_A+1)(n_B+1)$ Hermitian operators thus have been turned into an affine sum of $2(n_A+1)(n_B+1)-1$ elements of $\mathscr{A} \otimes \mathscr{B}$ in accordance with Lemma C.1.1.

It remains to prove that each element W_{ij} can be split into an affine sum of tensor products of elements of $\mathscr{A}$ and $\mathscr{B}$. This is already the case for W_{00} and the W_{ij} for which $(i,j) > (n_A, n_B)$ since all these elements but the tensor product of the identity elements of each state structure:

$$\frac{c_A c_B}{\sqrt{d_A}\sqrt{d_B}} \left(\sigma_0^A \otimes \sigma_0^B\right) = \left(\frac{c_A}{d_A} \mathbb{1}_A\right) \otimes \left(\frac{c_B}{d_B} \mathbb{1}_B\right) . \tag{C.49}$$

To show it for the remaining composite operators of the form $W_{ij} = \frac{c_A c_B}{\sqrt{d_A}\sqrt{d_B}} \left(\sigma_0^A \otimes \sigma_0^B + \frac{q_{ij}}{|q_{ij}|}\sigma_i^A \otimes \sigma_j^B\right)$, the following is used:

$$W_{ij} = \frac{1}{2}\Big[\frac{c_A}{\sqrt{d_A}}\left(\sigma_0^A + \sigma_i^A\right) \otimes \frac{c_B}{\sqrt{d_B}}\left(\sigma_0^B + \frac{q_{ij}}{|q_{ij}|}\sigma_j^B\right)\Big] + \frac{1}{2}\Big[\frac{c_A}{\sqrt{d_A}}\left(\sigma_0^A - \sigma_i^A\right) \otimes \frac{c_B}{\sqrt{d_B}}\left(\sigma_0^B - \frac{q_{ij}}{|q_{ij}|}\sigma_j^B\right)\Big] . \tag{C.50}$$

This is a convex sum of two tensor products. By construction, each tensor product feature a valid element of $_A$ tensored with one of $\mathscr{B}$[8]. A convex sum is an affine sum, therefore each such W_{ij} can be decomposed into an affine sum of tensor products. As an affine sum of affine sums is an affine sum, this provides a constructive proof that any $W \in \mathscr{A} \otimes \mathscr{B}$ can be split into an affine sums of elements of the form $V \otimes N$ where $V \in \mathscr{A}$ and $N \in \mathscr{B}$. □

8: For example, $\frac{c_A}{\sqrt{d_A}}\left(\sigma_0^A - \sigma_i^A\right)$ is positive by Lemma C.1.2, has support on the operator system of $\mathscr{A}$, and has a trace equal to c_A.

C.2.4. Proof of Lemma 3.5.3

Lemma 3.5.3 states that a necessary and sufficient condition for

$$\mathrm{Tr}_A\left[(V \otimes N) \cdot W\right] = \frac{\mathrm{Tr}\,[V]\,\mathrm{Tr}_A\,[W]}{d_A} \cdot N \, , \tag{3.98}$$

to hold for all $V \in \overline{\mathscr{A}}$ and $N \in \mathcal{L}\left(\mathcal{H}^B\right)$ is that

$$\left(\mathcal{P}_A \otimes \mathcal{I}_B\right)\{W\} = W \, . \tag{C.51}$$

9: Which exists for all dimensions since it is the $\mathfrak{su}(d)$ generators and the identity matrix. See subsection C.1.1.

Proof. Let $\left\{\sigma_i^X\right\}$ be an orthonormal basis of $\mathcal{L}\left(\mathcal{H}^X\right)$ so that[9]

$$\left(\sigma_i^X\right)^\dagger = \sigma_i^X \, ; \tag{C.52a}$$

$$\sigma_0^X := \mathbb{1}/\sqrt{d_A} \, ; \tag{C.52b}$$

$$\mathrm{Tr}\left[\sigma_{i\neq 0}^X\right] = 0 \, ; \tag{C.52c}$$

$$\left(\sigma_i^X \, , \, \sigma_j^X\right) := \mathrm{Tr}\left[\sigma_i^X \cdot \sigma_j^X\right] = \delta_{i,j} \, . \tag{C.52d}$$

And choose this basis on $\mathcal{L}\left(\mathcal{H}^A\right)$ such that the $n < d_A^2$ elements after σ_0^A, $\{\sigma_1^A, \sigma_2^A, \dots, \sigma_n^A\}$, form a basis of $\overline{\mathscr{A}} \setminus \{\mathbb{1} =: \sqrt{d_A}\sigma_0\}$. This implies that if $\mathcal{P}_A$ is the projector on $\mathscr{A}$, then $\sum_{i=0}^{d_A^2-1} \mathcal{P}_A\left\{\sigma_i^A\right\} = \sum_{i=0}^{n} \mathcal{P}_A\left\{\sigma_i^A\right\}$. Expand W and V using this basis, let $W = \sum_{i,j}^{d_A^2-1, d_B^2-1} w_{ij} \; \sigma_i^A \otimes \sigma_j^B$; $V = c_{\overline{A}}/\sqrt{d_A} \; \sigma_0^A + \sum_{k=1}^n v_k \; \sigma_k^A$. By direct computation:

$$\mathrm{Tr}_A\left[W\right] = d_A \sum_{j=0}^{d_B^2-1} w_{0j} \; \sigma_j^B \, ; \tag{C.53}$$

$$\mathrm{Tr}\left[V\right] = c_{\overline{A}} \, ; \tag{C.54}$$

$$\mathrm{Tr}_A\left[W \cdot (V \otimes N)\right] = \left(\sum_{i=0}^{n} \sum_{j=0}^{d_B^2-1} d_A \; w_{ij} \; v_i \sigma_j^B\right) \cdot N \, . \tag{C.55}$$

Combining Eqs. (C.53) and (C.54) in the right-hand side of Eq. (3.98) and (C.55) in its left-hand side yields

$$\begin{gathered}\mathrm{Tr}_A\left[W \cdot (V \otimes N)\right] = \frac{\mathrm{Tr}_A\left[W\right] \mathrm{Tr}\left[V\right]}{d_A} \cdot N \Rightarrow \\ \left(\sum_{i=0}^{n} \sum_{j=0}^{d_B^2-1} d_A \; w_{ij} \; v_i \sigma_j^B\right) \cdot N = \frac{\left(d_A \; \sum_{j=0}^{d_B^2-1} w_{0j} \; \sigma_j^B\right)\left(c_{\overline{A}}\right)}{d_A} \cdot N \, ,\end{gathered} \tag{C.56}$$

the equality is true for

$$\left(\sum_{i>0}^{n} \sum_{j=0}^{d_B^2-1} w_{ij} \; v_i \; \sigma_j^B\right) \cdot N = 0 \, . \tag{C.57}$$

This reduces to $\sum_{i>0}^{n} \sum_{j=0}^{d_B^2-1} w_{ij} \; v_i \; \sigma_j^B = 0$ because N is arbitrary. On the other hand, this equation has to hold for all V in $\mathscr{A}$, so one can consider the family $V_i \equiv \sigma_i^A$. Each V_i then induces a condition $\sum_j \; w_{ij} \; \sigma_j^B = 0$ for $i \in 1, \dots, n$. Because the σ_j^B's are orthonormal vectors, and W is arbitrary, this condition is true for a given V_i if and only if $w_{ij} = 0 \; \forall j$ whenever i corresponds to a basis element of $\mathscr{A}$, i.e., whenever $0 < i \leq n$. The condition for the equality to hold is therefore that W has the form $\sum_{j=0}^{d_B^2-1}\left(w_{0j} \; \sigma_0^A \otimes \sigma_j^B + \sum_{i>n}^{d_A^2-1} w_{ij} \; \sigma_i^A \otimes \sigma_j^B\right)$. This is equivalent to requiring Equation (C.51) on M, concluding the proof. □

C.3. Comments

C.3.1. On the Definition of a Resolution

Following the idea that a quantum instrument and their CJ representation are 'probabilistic resolutions' of the set of quantum channels, the concept of a resolution of the identity has been generalized into the concept of a resolution of a state structure in Definition 3.2.4.

Therefore Definition 3.2.4 is the postulate, made implicitly in the theory of higher-order processes, that all convex decomposition of a unit effect into positive operators must be identifiable with a given probabilistic intervention.

The justification of this postulate is beyond the scope of this thesis. For completeness, the following discussion presents some elements about a potential justification by finding a decomposition of the resolution into a known to be realizable experimental procedure.

While it is known that all POVM have an in-principle physical realization through Naimark dilation (see e.g. References [52, 90]) and, at the level above, that all quantum instruments have an in-principle realization through Stinespring dilation [76], this is not guaranteed to be true for the higher-orders. What about the realization of the resolutions of other state structures as a concrete lab procedure?

[52]: Nielsen et al. (2009), *Quantum Computation and Quantum Information*.
[90]: Watrous (2018), *The Theory of Quantum Information*.

[76]: Ozawa (1984), *Quantum measuring processes of continuous observables*.

For instance, can the resolution of a single partite process be decomposed as an ordered succession of a state preparation followed by a measurement and sharing an ancillary memory channel? And what about bipartite processes? This is to say, even if a deterministic operation appears feasible in a lab (a single partite process matrix is simply a state preparation followed by a measurement and sharing an ancillary channel), its probabilistic resolutions may require more resources (like backward in time signaling, indefinite causal order, postselection, etc.).

An interesting thing to keep in mind is that the resolutions are always allowed to be outside of the support of the state structure, as long as they sum up to an element of it. This is already the case for quantum effects, which can be any positive operator as long as they belong to a collection that sums up to the identity. As resolutions are mathematically compatible with the full structure of the theory, and indeed yield a well-defined probabilistic theory they should be allowed by the no-restriction hypothesis. But, in light of the result cited above, one may wonder if the no-restriction hypothesis is too lenient in that case? An answer to this question would require more research on the realizability of higher-order quantum processes. This direction is left open for future research. Without a clear answer, it should be noted that the hypothesis that all the allowed probabilistic operations correspond to all possible resolutions of state structures may induce extra hypotheses on what is feasible in a local lab. The concept of the resolution of a state structure is taken purely as a mathematical definition for the purpose of obtaining a general mathematical model.

C.3.2. On the Definition of a Frame Function on a State Structure

The definition of frame function on state structure requires assuming a generalized notion of Gleason non-contextuality as well as a $\sigma-$additivity of the frame functions on resolutions of state structures. For completeness, here are presented some potential issues that may result from such assumptions.

[51]: Shrapnel et al. (2018), *Causation does not explain contextuality*.
[60]: Shrapnel et al. (2018), *Updating the Born rule*.
[58]: Spekkens (2005), *Contextuality for preparations, transformations, and unsharp measurements*.
[59]: Spekkens (2014), *The Status of Determinism in Proofs of the Impossibility of a Noncontextual Model of Quantum Theory*.

The first assumption, called instrument non-contextuality in [51, 60], is actually an adaptation of Gleason non-contextuality (later formalized as Spekkens' measurement non-contextuality; see e.g., [58, 59]) to the case of higher-order interventions. In the context of Definition 3.3.1, this is the assumption that only the outcome is important for assigning a probability to an element of a resolution. Hence, it is an assumption that is not specific to state structures, so higher-order quantum processes, but any kind of process that can be treated using the formalism of Definition 1.1.1.

At the level of resolutions of frame functions, it is a non-trivial assumption. If a specific outcome $\tilde{a}$ appears in two different observations, it corresponds to two probabilistic operations like $\{N_{a_1}\}_{a_1\in\Omega_{a_1}}$ and $\{M_{a_2}\}_{a_2\in\Omega_{a_2}}$ (these observations are assumed independent of a choice of setting to simplify the notation). When this is the case, one can randomize between these two observations so that the new global observation takes values in $\Omega_{a|x}$ such that it results in a new operation $\{E_a\}_{a\in\Omega_{a|x}}$ with $\Omega_{a|x=0} = \Omega_{a_1}$, $\Omega_{a|x=1} = \Omega_{a_2}$ and $\tilde{a}$ can be observed in both conditional subsets. In other words, depending on the setting, the observation of the party is either represented by $\{N_{a_1}\}$ when $x = 0$ or by $\{M_{a_2}\}$ when $x = 1$. The point is that the outcome $\tilde{a}$ has been mapped to two different operators, either $E_{\tilde{a}|x=0} = N_{\tilde{a}}$ or $E_{\tilde{a}|x=1} = M_{\tilde{a}}$. The analog of non-contextuality here is that the obtained probability, for all frame function f, is independent of which of these operators was chosen,

$$p(\tilde{a}|x = 0) = p(\tilde{a}|x = 1) \,. \tag{C.58}$$

This is true if and only if

$$f(E_{\tilde{a}|x=0}) = f(E_{\tilde{a}|x=1}) \,. \tag{C.59}$$

10: Which is also an hypothesis of the model, see Appendix C.3.1.

Since the resolutions can run over the entirety of $\mathcal{L}\left(\mathcal{H}^A\right)$, and that f can also be resolved[10], this amounts to requiring that $E_{\tilde{a}|x=0} = E_{\tilde{a}|x=1}$ = $E_{\tilde{a}}$. In a sentence, the assumption states that each different operator corresponds to a different outcome, and if the same operator appears in two different resolutions, the outcome distribution for all frame functions is independent of the resolution it was taken from.

[59]: Spekkens (2014), *The Status of Determinism in Proofs of the Impossibility of a Noncontextual Model of Quantum Theory*.
[155]: Grudka et al. (2008), *Is There Contextuality for a Single Qubit?*

In general, this kind of non-contextuality conditions for generalizations of POVMs are hard to justify with respect to contextuality for PVMs. Non-repeatability blurs the notion of deterministic assignment of measurement results. And, if one purifies the unsharp effects to sharp effects so as to make the assignments deterministic, some effects that differed only by their choice of setting can be mapped to different projectors and, therefore, to different statistics; one induces contextual effects by purifying despite having assumed them to be non-contextual, see in particular References [59, 155] for a discussion on that point. Since frame functions on state structures are equivalent to a subset of frame functions on quantum effects constrained by condition (3.24a), these critics also apply to Definition 3.3.1. The study of Gleason-kind contextuality for frame functions is left open as a direction for future research.

The second assumption is also a stronger requirement than the original Gleason theorem because the probabilistic measure has been shifted from a PVM to a POVM. As the overlap between effects in a POVM can be non-zero, assuming they obey the σ-additivity condition (3.24c) is an extra condition compared to the PVM case.

Indeed, nothing guarantees that the union of two effects with non-zero overlap splits homogeneously like $f(E_i+E_j) = f(E_i)+f(E_j)$ [112]. This point has been first formulated concerning Busch's proof of Gleason's theorem for POVM (see Reference [115] for review). But Busch's proof only assumes $f(\mathbb{1}) = 1$. Compared to Equation (3.24a), it appears that an extra assumption was added by using the same form of frame functions by formal analogy alone. Gleason-kind proofs are based on an idea of σ-finite measure, which is order-preserving with respect to the partial order in the set of positive operators, i.e.

[112]: Caves et al. (2004), *Gleason-Type Derivations of the Quantum Probability Rule for Generalized Measurements*.

[115]: Wright et al. (2019), *A Gleason-type theorem for qubits based on mixtures of projective measurements*.

$$E_i \leq E_j \Rightarrow f(E_i) \leq f(E_j)\,, \tag{C.60}$$

this property leads to the weakening of Equation (3.24c), $\sum_i f(E_i) = f(\sum_i E_i)$ for $\sum_i E_i \leq \mathbb{1}$ which justifies coarse-graining: summing operators amounts to summing their respective probabilities. However, Definition 3.3.1 assumes $\sum_i f(E_i) = f(\sum_i E_i)$ for $\sum_i E_i \leq N$ for all $N \in \mathscr{A}$ and not only $\mathbb{1}$; the frame functions on operator system appear much more constrained than frame functions on POVMs. This does not appear as a trivial extra condition since not all $N \leq \mathbb{1}$ nor not all $N \geq \mathbb{1}$. Whether this assumption has implications is also left open for future research.

C.3.3. On the Freedom of Choice Assumption in Quantum Operations

Under the statistical interpretation, a choice of setting is always seen as something that can be done deterministically, whether the choice of outcome is always probabilistic. This is the statement that at the lowest order, the preparation procedure can always result in a preselected state in a deterministic, and therefore repeatable, way, but, on the contrary, the measurement procedure cannot result in a postselected effect in a repeatable way. The freedom of choice assumption asserts that the choices of settings of different parties are independent of each other. In the generalized state and effect pair picture presented above, this assumption was actually sneaked in by demanding the environment to be chosen independently of the operation the party chooses to perform.

By writing that the averaged operation is given by

$$N := \sum_x p(x) \sum_a N_{a|x} \tag{C.61}$$

the unconditional distribution of x was used. This assumes that the average over the setting of the party is not conditional on the choice of the environment y, so that $p(x|y) = p(x|y') = p(x)$ for any two y, y'. Hence the settings are independent, $p(x, y) = p(x)p(y)$. Indeed, a general operator $N_{a,x}$ representing the joint distribution of outcome a and setting y obeys

$$\left(N_{a,x}\ ,\ V_{|y}\right) = p(a, x|y)\ . \tag{C.62}$$

By the definition of a conditional distribution,

$$p(a, x|y) = p(a|x, y)p(x|y)\ , \tag{C.63}$$

so because

$$p(a|x, y) = \left(N_{a|x}\ ,\ V_{|y}\right) = \mathrm{Tr}\left[N_{a|x}^{\dagger} \cdot V_{|y}\right]\ , \tag{C.64}$$

the relation between the operators $N_{a|x}$ and $N_{a,x}$ is obtained using the linearity of the trace in Equation C.63

$$\mathrm{Tr}\left[N_{a,x}^{\dagger} \cdot V_{|y}\right] = \mathrm{Tr}\left[\left(p(x|y)\, N_{a|x}\right)^{\dagger} \cdot V_{|y}\right]\ . \tag{C.65}$$

As $V_{|y}$ can be the identity operator and that $N_{a,x}$ and $N_{a|x}$ take value over the whole of $\mathcal{L}\left(\mathcal{H}^{A}\right)$ (since these are effects), it is necessary that

$$N_{a,x} = p(x|y)N_{a|x}\ . \tag{C.66}$$

Finally, it is direct to see that $p(x, y) = p(x)p(y)$ is enough to weaken the above relation into Equation C.61[11]

11: Proving it at the level of distributions alone,

$$p(a|x, y)p(x|y) = p(a|x, y)p(x, y)/p(y). \tag{C.67}$$

And $p(x, y) = p(x)p(y)$. Therefore,

$$p(a|x, y)p(x|y) = p(a|x, y)p(x)\ . \tag{C.68}$$

By seeing the environment as an independent party, the freedom of choice assumption indeed requires the setting y of this new party to be uncorrelated with the setting x of the party performing the operation. In a sentence: the local choice of an operation is independent of the global environment.

C.3.4. On the Definition of Parallel Composition as the No-Signaling Composition

The smallest composition that can be witnessed by the projective characterization techniques, a linear method, is the set of no-signaling compositions. But some smaller sets are also interesting, like the separable states defined by the minimal tensor product. Considering these subsets makes sense for quantum states because of entanglement and non-locality [56, 169].

However, the exact meaning of entanglement in higher-order state structure remains to be clarified. For instance, what does it mean when a no-signaling bipartite channel has an entangled CJ representation? When it is not, the separability entails that it can be represented by local operations and shared randomness (LOSR). However, when it is entangled, it cannot be that it has a realization using local operation and shared entanglement (LOSE) since there exist no-signaling channels that do not [61, 62, 64, 65, 185].

[61]: Beckman et al. (2001), *Causal and localizable quantum operations*.
[62]: Eggeling et al. (2002), *Semicausal operations are semilocalizable*.
[64]: Piani et al. (2006), *Properties of quantum nonsignaling boxes*.
[65]: D'Ariano et al. (2011), *No Signaling, Entanglement Breaking, and Localizability in Bipartite Channels*.
[185]: Horodecki et al. (2016), *Relativistic Causality vs. No-Signaling as the limiting paradigm for correlations in physical theories*.

The kind of channel realizable by LOSE brings another interesting subset of $\mathscr{A} \otimes \mathscr{B}$ to consider the set of 'localizable' bipartite composite operators. For the set of bipartite quantum channels, this is the set of local channels sharing entanglement [61, 64]. However the generalization to arbitrary state structures opens the question of how to generalize the notion of localizability to compositions of arbitrary state structures. This question has been considered in much more detail in Reference [36].

[36]: Simmons et al. (2022), *Higher-order causal theories are models of BV-logic*.

Naturally, this brings the extra question of how to define local operations and classical communications (LOCC) paradigm for arbitrary pairs of state structures. This was discussed in References [186, 187].

[186]: Akibue et al. (2017), *Entanglement-assisted classical communication can simulate classical communication without causal order*.
[187]: Kunjwal et al. (2023), *Trading Causal Order for Locality*.

Remark in passing that the admissibility of higher-order processes hides on the assumption that the set $\mathscr{A} \otimes \mathscr{B}$ is fully realized by local parties at all levels. Otherwise, the higher-order environment these local parties could share would actually be bigger than the state structure $\overline{\mathscr{A}} \parr \overline{\mathscr{B}}$. For instance, if all that the parties could realize were product states, then the most general map normalized on this would not have to be CP, Positivity on Pure Tensors would suffice [123]. On physical heuristics, one could require that the set of local operations be restricted to the localizable ones at some 'highest level' of the hierarchy instead. Would that prevent any non-trivial higher-order transformations from being defined over that level?

[123]: Barnum et al. (2005), *Influence-free states on compound quantum systems*.

All of these questions have no answer to the best of my knowledge. They are left as interesting, open directions for further developments.

C.3.5. No-Signaling Bipartite Biased Quantum States are Separables.

In the example of biased quantum theory of subsection 3.6.1, it is claimed that the set $\mathscr{A} \otimes \mathscr{B}$ of bipartite no-signaling states cannot show non-local effects. This is because this state space is entirely composed of separable states.

Proposition C.3.1 (Bipartite Biased Quantum States are Separable) *Let $\mathscr{A}$ and $\mathscr{B}$ be state structures of biased quantum states as in Definition 3.6.1. Then, their no-signaling composition as in Definition 3.4.1 is a minimal tensor product. In other words, any no-signaling bipartite biased quantum state is separable.*

Proof. To see it, first notice that the space spanned by the Pauli matrices minus the Y one is the space of symmetric operators with respect to transpose in the computational basis,

$$\mathrm{Sym}(\mathcal{L}\left(\mathcal{H}^A\right)) := \mathrm{Span}\left\{\mathbb{1}_A, X_A, Z_A\right\} . \tag{C.69}$$

And notice that the operator system supporting $\mathscr{A} \otimes \mathscr{B}$ is similar to a tensor product of spaces of symmetric matrices, i.e. its support is similar to the nine elements in

$$\mathrm{Sym}(\mathcal{L}\left(\mathcal{H}^A\right)) \otimes \mathrm{Sym}(\mathcal{L}\left(\mathcal{H}^B\right)) = \mathrm{Span}\left\{\{\mathbb{1}, X, Z\}_A \otimes \{\mathbb{1}, X, Z\}_B\right\} . \tag{C.70}$$

Let H be the following unitary transform[12].

$$H := \frac{Y + Z}{\sqrt{2}} \tag{C.71}$$

12: Which properties are analog to the Hadamard gate (see e.g., Section 4.2 in Reference [52]) but this matrix changes the computational basis (the eigenbasis of Z) into the $\{|\pm i\rangle\}$ basis (the eigenbasis of Y) instead of changing it into the $\{|\pm\rangle\}$ basis (the eigenbasis of X).

[52]: Nielsen et al. (2009), *Quantum Computation and Quantum Information*.

whose adjoint action leaves the identity matrix invariant, maps the Pauli X to $HXH = -X$, and maps the Pauli Y matrix to the Z and vice-versa, i.e. $HYH = Z$ and $HZH = Y$. The similarity is obtained by mapping all elements $W \in \mathscr{A} \otimes \mathscr{B}$ to $(H \otimes H)^{\dagger} W (H \otimes H)$ so that

$$\mathrm{Sym}(\mathcal{L}(\mathcal{H}^A)) \otimes \mathrm{Sym}(\mathcal{L}(\mathcal{H}^B)) = (H \otimes H)^{\dagger} \mathscr{A} \otimes \mathscr{B} (H \otimes H) \ . \quad \text{(C.72)}$$

Now the space of symmetric matrices spanned by these operators is invariant under transposes and partial transposes[13]. Hence, it is a subspace of the state space of two qubits in which all elements have a Positive Partial Transpose (PPT). In this dimension, this is a necessary and sufficient condition for separability by the Peres-Horodecki criterion [188, 189]. Finally, since the unitary map between the two state spaces is separable as well, it cannot map separable states to entangled states. Therefore, $\mathscr{A} \otimes \mathscr{B}$ have to be separable. □

13: Actually, the only non-partial-transpose-invariant element of the space of symmetric operators over vectors of dimension 2x2 in the Pauli basis is the matrix $Y \otimes Y$.

[188]: Peres (1996), *Separability Criterion for Density Matrices*.

[189]: Horodecki et al. (1996), *Separability of mixed states: necessary and sufficient conditions*.

Appendices to Chapter 5 | D.

Forget this world and all its troubles and if possible its multitudinous Charlatans –everything in short but the Enchantress of Numbers.

Ada Lovelace

D.1. Mathematical Methods

Some known results about the abstract mathematical structures evoked in the main text are gathered here.

Some set theory is used. When it is done, abstract sets are noted using capital letters $S, L, \ldots$ and their elements are noted using lowercase letters $a, b, c, \ldots$. From the context, there should be no risk of confusion with operators on a Hilbert space.

D.1.1. Lattices

The operator systems on which the algebra of projectors is built are subspaces of the real vector space of self-adjoint operators which is itself embedded in the complex vector spaces of operators, $\mathcal{L}(\mathcal{H})$. It is well-known that the set of subspaces of a vector space forms a special algebraic structure named a lattice. Here some facts about these structures are recollected. It is then shown under which operations the algebra of commuting projectors on operator systems in a space $\mathcal{L}(\mathcal{H})$ forms a Boolean lattice. More precisely, it is shown that the set of operator systems sharing the same basis is a Boolean lattice characterized by projectors whose algebra is homomorphic to this lattice. For a complete review, see Chapter 8 of Reference [150] for instance.

[150]: Jacobson (2009), *Basic Algebra*.

The set of subspaces of a space is a special kind of set called a *partially ordered set*.

Definition D.1.1 (Partial order) *A* ***partially ordered set*** *is a set together with a binary relation $a \subseteq b$ defined between any two of its elements a and b and satisfying the following rules:*

1. *$a \subseteq a$ (reflexivity);*
2. *If $a \subseteq b$ and $b \subseteq a$, then $a = b$ (anti-symmetry);*
3. *If $a \subseteq b$ and $b \subseteq c$, then $a \subseteq c$ (transitivity).*

The $\subseteq$ operation for subspace translates the property of a subspace being contained into another one. The greatest (respectively, lowest) element of a subset $\{a_i\}$ is the element a_j verifying $a_j \supseteq a_i$ $(a_j \subseteq a_i)$ $\forall i$. It is related to another structure called a *lattice*. A lattice is a partially ordered set in which two elements have a lowest upper and greatest lower bounds. This requirement can be shown as equivalent to the following properties:

Definition D.1.2 (Lattice and Sublattice) *A **lattice** is a set L together with two binary compositions $S \times S \to S$, the **cap**[1] $\cap$ and the **cup**[2] $\cup$, satisfying the following rules (a, b, c are elements of S):*

1: Also called 'meet', 'and', or 'additive conjunction'.
2: Also called 'union', 'join', 'or', or 'additive disjunction'.

1. *$a \cap b = b \cap a$;*
2. *$(a \cap b) \cap c = a \cap (b \cap c)$;*
3. *$a \cap a = a$;*
4. *$(a \cap b) \cup a = a$;*
5. *If a statement S can be deducted from the axioms, the dual statement S' obtained by replacing $\cap$ by $\cup$ througouht S can be deducted (Principle of duality).*

*A subset K of L is called a **sublattice** if it is closed under the compositions $\cap$ and $\cup$.*

The join and meet of a lattice are related to a partial order by the following identification:

$$a \subseteq b \iff a \cap b = a \iff a \cup b = b \,, \tag{D.1}$$

in terms of subspaces, this equation reads 'a subspace is contained within another if and only if its intersection with the other space is equal to itself and if and only if its union with the other space is equal to the other space'. These lattices are not totally arbitrary; they actually follow certain special rules.

Definition D.1.3 (Properties of Lattices) *A lattice is:*

- ***Complete** if every subset have a greatest and lowest element. In which case, the **zero** 0 and **unit** 1 of the lattice are defined as, respectively, the greatest and lowest element of the whole lattice,*

$$0 := \bigcap_{a \in L} a \,; \tag{D.2a}$$
$$1 := \bigcup_{a \in L} a \,; \tag{D.2b}$$

- ***Distributive** if it obeys the distributive law,*

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \,; \tag{D.3}$$

- ***Modular** if*

$$a \cap (b \cup c) = b \cup (a \cap c) \tag{D.4}$$

whenever $a \supseteq b$. Equivalently, a lattice L is modular if and only if whenever $\exists a, b, c$ such that

$$a \supseteq b, \text{ and } a \cap c = b \cap c, \text{ and } a \cup c = b \cup c \;\Rightarrow a = b \,; \tag{D.5}$$

- ***Complemented** if for any $a \in L$ there exists $\overline{a} \in L$ such that*

$$0 = a \cap \overline{a} \,; \tag{D.6a}$$
$$1 = a \cup \overline{a} \,. \tag{D.6b}$$

- ***Boolean**[3] if it is complemented and distributive.*

3: This structure is also called 'Boolean algebra'.

*In addition, a **lattice homomorphism** is a map $\phi : L \to L'$ such that $\phi(a \cap b) = \phi(a) \cap \phi(b)$ and $\phi(a \cup b) = \phi(a) \cup \phi(b)$. If the map is bijective then it is a **lattice isomorphism**[4].*

4: Note that ϕ being a lattice isomorphism is equivalent to the maps ϕ and ϕ^{-1} being both order-preserving.

D.2. The Projector Algebra

The projector algebra is relative to how the Hilbert space is partitioned between parties. To start with, this section deals with single-partite Hilbert space. Single-partite projectors are considered when discussing the local state structure of a single party as well as the global state structure seen by a set of parties.

D.2.1. The Lattice of Commuting Projectors on Operator Systems

In the set of all subspaces of a given space, the inclusion of a subspace into another one is a partial order relation. As discussed in subsection C.1.3 around Proposition C.1.3, instead of being phrased in terms of subspaces, these inclusions can be translated into operations on the projectors. By restricting the set of subspaces to either included subspaces or subspaces with no overlap, it is then possible to promote this inclusion relation $\subseteq$ defined by Equation C.29 into an operation on projectors as the intersection $\cap$ defined in the main text by

$$\forall \mathcal{P}, \mathcal{P}^{'}, \quad \mathrm{Im}\left\{\mathcal{P} \cap \mathcal{P}^{'}\right\} = \mathrm{Im}\{\mathcal{P}\} \cap \mathrm{Im}\left\{\mathcal{P}^{'}\right\} . \tag{5.6}$$

The link between the two is then $\mathcal{P} \cap \mathcal{P}^{'} = \mathcal{P} \iff \mathcal{P} \subseteq \mathcal{P}^{'}$. This amounts to restricting the set of all projectors to only commuting projectors so that the inclusion relation can be promoted to an operation under which this set is closed. It is almost direct to show that, like with the inclusion, it is the composition operation[5] $\circ$ that corresponds to the intersection $\cap$,

$$\mathcal{P} \cap \mathcal{P}^{'} \equiv \mathcal{P} \circ \mathcal{P}^{'} . \tag{5.7}$$

5: See e.g. Theorem 2.26 in Ref. [96].
[96]: Roman (2008), *Advanced Linear Algebra*.

Thus, commuting projectors also have a natural 'conjunction' operation interpretable as a logic 'and' and called the *intersection* or the *cap*. It can also be interpreted as the 'multiplication' since it is the role inherited from the algebra of superoperators. This is indeed a binary operation that is distributive over the addition and is compatible with scalar multiplication. Moreover, it is both associative and commutative. When acting on sets of projectors, this operation inherits the idempotent property from the projector definition:

$$\mathcal{P}_A^2 := \mathcal{P}_A \cap \mathcal{P}_A \equiv \mathcal{P}_A \circ \mathcal{P}_A = \mathcal{P}_A . \tag{D.7}$$

In the above equation, the shorthand notation for 'squaring' under the 'multiplication' $\cap$ has been defined: $\mathcal{P}_A \cap \mathcal{P}_A \equiv \mathcal{P}_A^2$.

When the set of projectors is further restricted to projectors on operator systems[6] as in Definition 3.2.7, the set is actually stable under the intersection.

6: Recall that these are superoperators obeying conditions

$$\mathcal{P}_A \circ \mathcal{P}_A = \mathcal{P}_A ; \tag{3.20a}$$
$$\mathcal{P}_A^* = \mathcal{P}_A ; \tag{3.20b}$$
$$\mathcal{P}_A \circ \dagger \circ \mathcal{P}_A = \dagger \circ \mathcal{P}_A ; \tag{3.20c}$$
$$\mathcal{P}_A \circ \mathcal{D}_A = \mathcal{D}_A . \tag{3.20d}$$

Proposition D.2.1 (The cap yields valid projector on operator system.) *If $\mathcal{P}_A$ and $\mathcal{P}'_A$ are a pair of projectors that obey Equations* (3.20) *then $\mathcal{P}_A \cap \mathcal{P}'_A$ obey condition* (3.20) *as well, provided $\mathcal{P}_A$ and $\mathcal{P}'_A$ commute.*
In addition, $\mathcal{P}_A \cap \mathcal{P}'_A$ commutes with both $\mathcal{P}_A$ and $\mathcal{P}'_A$ as well as any other projector that commutes with the two of them.

Proof. The preservation of commuting relations is inherited from the associativity of $\circ$, for example:

$$\left(\mathcal{P}_A \cap \mathcal{P}'_A\right) \cap \mathcal{P}_A = \left(\mathcal{P}_A \circ \mathcal{P}'_A\right) \circ \mathcal{P}_A = \mathcal{P}_A \circ \mathcal{P}'_A \circ \mathcal{P}_A = \mathcal{P}_A \circ \mathcal{P}_A \circ \mathcal{P}'_A = \mathcal{P}_A^2 \circ \mathcal{P}_A = \mathcal{P}_A \cap \mathcal{P}_A\,. \tag{D.9}$$

Condition (3.20a) comes from the associativity and commutativity of the $\cap$ as well as the idempotence of each projector:

$$\left(\mathcal{P}_A \cap \mathcal{P}'_A\right)^2 = \mathcal{P}_A \circ \mathcal{P}'_A \circ \mathcal{P}_A \circ \mathcal{P}'_A = \mathcal{P}_A \circ \mathcal{P}_A \circ \mathcal{P}'_A \circ \mathcal{P}'_A = \mathcal{P}_A \circ \mathcal{P}'_A = \mathcal{P}_A \cap \mathcal{P}'_A\,. \tag{D.10}$$

provided $\mathcal{P}_A$ and $\mathcal{P}'_A$ commute. From $\mathcal{P}_A \cap \mathcal{P}'_A = \mathcal{P}_A \circ \mathcal{P}'_A$, it is direct to see that $(\mathcal{P}_A \circ \mathcal{P}'_A) \circ \dagger = \dagger \circ (\mathcal{P}_A \circ \mathcal{P}'_A)$ using associativity. From $(\mathcal{P}_A \circ \mathcal{P}'_A)^* = (\mathcal{P}'_A)^* \circ (\mathcal{P}_A)^*$, $(\mathcal{P}'_A)^* \circ (\mathcal{P}_A)^* = \mathcal{P}_A \cap \mathcal{P}'_A$ if the two projectors commute and are self-adjoint. Finally, the last condition is also direct from associativity: $\left(\mathcal{P}_A \cap \mathcal{P}'_A\right) \cap \mathcal{D}_A = \mathcal{P}_A \cap \left(\mathcal{P}'_A \cap \mathcal{D}_A\right)$. ☐

The set can thus be 'completed' by all possible intersections of elements. A property inherited from Definition 3.2.7, that was used in the proof above is that

$$\mathcal{P}_A \cap \mathcal{D}_A = \mathcal{D}_A\,. \tag{D.11}$$

Whence the set of commuting projectors on operator systems have a 'least' element given by the depolarizing superoperator[7] (C.22).

7: $\mathcal{D}_A: \quad \mathcal{D}_A(V) := \frac{\mathbb{1}^A}{d_A}\,\mathrm{Tr}_A\,[V]\,.$

Projectors on operator systems naturally feature the 'intersection' operation under which they are stable and have a least element. But since the superoperators form an algebra, it is tempting to also use the second operation of the algebra, the addition '+' defined by

$$\left(\mathcal{P}_A + \mathcal{P}'_A\right)\{V\} := \mathcal{P}_A\{V\} + \mathcal{P}'_A\{V\} \quad \forall V \in \mathcal{L}\left(\mathcal{H}^A\right)\,, \tag{D.12}$$

in the set of projectors on operator systems. But an issue revealed by squaring is that the operation '+' does not necessarily map projectors to projectors:

$$\left(\mathcal{P}_A + \mathcal{P}'_A\right)^2 = \mathcal{P}_A^2 + \mathcal{P}'^2_A + 2\left(\mathcal{P}_A \cap \mathcal{P}'_A\right)\,, \tag{D.13}$$

because the last term of the right-hand side is not always zero. Nonetheless, another operation that has all the properties of the addition while preserving idempotency is the union of two projectors: the *union* $\cup$ (see e.g. References [190] and [96, Thm. 2.26]). This is inspired by set theory: as $\mathcal{P}_A$ maps to a subspace $\mathrm{Im}\{\mathcal{P}_A\} \subset \mathcal{L}\left(\mathcal{H}^A\right)$ and $\mathcal{P}'_A$ to $\mathrm{Im}\left\{\mathcal{P}'_A\right\} \subset \mathcal{L}\left(\mathcal{H}^A\right)$, an addition of projectors counts the overlap between these subspaces twice, since it is contained in each. A proper addition '$\cup$' should only

[190]: Piziak et al. (1999), *Constructing projections on sums and intersections.*

count the overlap once. Therefore $\mathrm{Im}\left\{\mathcal{P}_A \cup \mathcal{P}'_A\right\}$ should be equivalent to the exclusive disjunction of subspaces $\mathrm{Im}\{\mathcal{P}_A\} \oplus \mathrm{Im}\left\{\mathcal{P}'_A\right\}$. This requirement is realized by the 'disjunction' operation presented in the main text, which can be interpreted as an addition or as logic 'or', nicknamed the *cup*:

$$\mathcal{P}_A \cup \mathcal{P}'_A = \mathcal{P}_A + \mathcal{P}'_A - \mathcal{P}_A \cap \mathcal{P}'_A \,. \tag{5.9}$$

As the name hints, $\mathcal{P}_A \cup \mathcal{P}'_A$ characterizes the union of the underlying operator systems. As is the case with the cap, the cup of two projectors on operator systems is a valid projector on operator system.

Proposition D.2.2 (The cup yields valid projector on operator system.) *If $\mathcal{P}_A$ and $\mathcal{P}'_A$ are a commuting pair of projectors that obey Equations* (3.20) *then $\mathcal{P}_A \cup \mathcal{P}'_A$ obey condition* (3.20) *as well.*
In addition, $\mathcal{P}_A \cup \mathcal{P}'_A$ commutes with $\mathcal{P}_A$ and $\mathcal{P}'_A$ as well as any projector that commutes with the both of them.

Proof. As with the cap, the preservation of commutation is again inherited from the properties of the algebra of operations on superoperators $+$ and $\circ$. Let $\mathcal{P}''_A$ be a projector that commutes with both $\mathcal{P}_A$ and $\mathcal{P}'_A$, then

$$\left(\mathcal{P}_A \cup \mathcal{P}'_A\right) \cap \mathcal{P}''_A = \left(\mathcal{P}_A + \mathcal{P}'_A - \left(\mathcal{P}_A \cap \mathcal{P}'_A\right)\right) \cap \mathcal{P}''_A = \mathcal{P}_A \circ \mathcal{P}''_A + \mathcal{P}'_A \circ \mathcal{P}''_A - \left(\mathcal{P}_A \cap \mathcal{P}'_A\right) \circ \mathcal{P}''_A \,, \tag{D.14}$$

because composition linear maps distributes over their addition, and from there the commutation of each element in the sum can be used (especially the commutation of $\mathcal{P}''_A$ with $\mathcal{P}_A \cap \mathcal{P}'_A$ that comes from Proposition D.2.1.

Conditions (3.20c) and (3.20b) follow because the addition and the cap preserve these properties, whereas conditions (3.20a) and (3.20d) follow from direct computation using the definition, and the fact and $\cup$ define a distributive lattice (as is proven below) so that the cap distributes over the cup like in Equation D.18. First,

$$\begin{gathered}\left(\mathcal{P}_A \cup \mathcal{P}'_A\right)^2 = \left(\mathcal{P}_A + \mathcal{P}'_A - \left(\mathcal{P}_A \cap \mathcal{P}'_A\right)\right) \cap \left(\mathcal{P}_A + \mathcal{P}'_A - \left(\mathcal{P}_A \cap \mathcal{P}'_A\right)\right) = \\ (\mathcal{P}_A)^2 + \left(\mathcal{P}'_A\right)^2 - 5\left(\mathcal{P}_A \cap \mathcal{P}'_A\right) + 4\left(\mathcal{P}_A \cap \mathcal{P}'_A\right) = \mathcal{P}_A + \mathcal{P}'_A - \left(\mathcal{P}_A \cap \mathcal{P}'_A\right) \,,\end{gathered} \tag{D.15}$$

therefore

$$\left(\mathcal{P}_A \cup \mathcal{P}'_A\right)^2 = \mathcal{P}_A \cup \mathcal{P}'_A \,. \tag{D.16}$$

Then, $\mathcal{D}_A \cap \left(\mathcal{P}_A \cup \mathcal{P}'_A\right) = (\mathcal{D}_A \cap \mathcal{P}_A) \cup \left(\mathcal{D}_A \cap \mathcal{P}'_A\right) = \mathcal{D}_A \cup \mathcal{D}_A$, thus

$$\mathcal{D}_A \cap \left(\mathcal{P}_A \cup \mathcal{P}'_A\right) = \mathcal{D}_A \,. \tag{D.17}$$

Hence, the cup of two commuting projectors on operator systems is a projector on operator system. □

Under the cap and cup operations, the set of projectors is a ring. $\cap$ and $\cup$ can indeed be seen as some multiplication and addition of projectors as the former distributes over the latter:

$$\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \cap \mathcal{P}_A^{''} = \left(\mathcal{P}_A \cap \mathcal{P}_A^{''}\right) \cup \left(\mathcal{P}_A^{'} \cap \mathcal{P}_A^{''}\right) . \tag{D.18}$$

Indeed, $\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \cap \mathcal{P}_A^{''} = \left(\mathcal{P}_A + \mathcal{P}_A^{'} - (\mathcal{P}_A \cap \mathcal{P}_A^{'})\right) \cap \mathcal{P}_A^{''} = (\mathcal{P}_A \cap \mathcal{P}_A^{''}) + (\mathcal{P}_A^{'} \cap \mathcal{P}_A^{''}) - (\mathcal{P}_A \cap \mathcal{P}_A^{'}) \cap \mathcal{P}_A^{''}$, and $(\mathcal{P}_A \cap \mathcal{P}_A^{'}) \cap \mathcal{P}_A^{''} = (\mathcal{P}_A \cap \mathcal{P}_A^{'}) \cap (\mathcal{P}_A^{''} \cap \mathcal{P}_A^{''}) = (\mathcal{P}_A \cap \mathcal{P}_A^{''}) \cap (\mathcal{P}_A^{'} \cap \mathcal{P}_A^{''})$.

The projectors $\mathcal{I}$ and $\mathcal{D}$ play a special role in it: one can identify the identity superoperator $\mathcal{I}$ as the multiplicative identity or '**unit**' of the algebra of projectors since for all projectors $\mathcal{P}$,

$$\mathcal{P}_A \cap \mathcal{I}_A = \mathcal{P}_A . \tag{D.19}$$

As for the depolarizing superoperator $\mathcal{D}$, it is the additive identity,

$$\mathcal{P}_A \cup \mathcal{D}_A = \mathcal{P}_A . \tag{D.20}$$

To actually show the ring structure, it remains to collect these results: first, the distributivity law (D.18) holds. Second, from associativity, it should be clear that the set $\{\mathcal{P}\}$ together with operation $\cap$ is a monoid with unit $\mathcal{I}$. Third, it should also be clear from associativity and commutativity that the set $\{\mathcal{P}\}$ together with operation $\cup$ is an Abelian group with unit $\mathcal{D}$. The only missing thing in the definition is the inverse with respect to this group.

But this set is already constrained enough to uniquely define what the inverse should be. Indeed, by condition (D.7) this is a special kind of ring whose elements are all idempotent, called a Boolean ring. These kinds of rings are also lattices. Because their elements are all commuting, it can be shown that they obey the Definition D.1.2. For example, distributivity also holds when replacing the cap with the cup,

$$\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \cup \mathcal{P}_A^{''} = \left(\mathcal{P}_A \cup \mathcal{P}_A^{''}\right) \cap \left(\mathcal{P}_A^{'} \cup \mathcal{P}_A^{''}\right) , \tag{D.21}$$

as required by the principle of duality. This is explicitly proven as follows:

$$\begin{aligned}
\left(\mathcal{P}_A \cup \mathcal{P}_A^{''}\right) \cap \left(\mathcal{P}_A^{'} \cup \mathcal{P}_A^{''}\right) &= \left(\mathcal{P}_A + \mathcal{P}_A^{''} - \left(\mathcal{P}_A \cap \mathcal{P}_A^{''}\right)\right) \cap \left(\mathcal{P}_A^{'} + \mathcal{P}_A^{''} - \left(\mathcal{P}_A^{'} \cap \mathcal{P}_A^{''}\right)\right) \\
&= \left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) + \left(\mathcal{P}_A \cap \mathcal{P}_A^{''}\right) - \left(\mathcal{P}_A \cap \mathcal{P}_A^{'} \cap \mathcal{P}_A^{''}\right) + \left(\mathcal{P}_A^{'} \cap \mathcal{P}_A^{''}\right) + \mathcal{P}_A^{''} - \left(\mathcal{P}_A^{'} \cap \mathcal{P}_A^{''}\right) \\
- \left(\mathcal{P}_A \cap \mathcal{P}_A^{'} \cap \mathcal{P}_A^{''}\right) - \left(\mathcal{P}_A \cap \mathcal{P}_A^{''}\right) &+ \left(\mathcal{P}_A \cap \mathcal{P}_A^{'} \cap \mathcal{P}_A^{''}\right) = \left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) + \mathcal{P}_A^{''} - \left(\mathcal{P}_A \cap \mathcal{P}_A^{'} \cap \mathcal{P}_A^{''}\right) \\
&= \left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \cup \mathcal{P}_A^{''} .
\end{aligned} \tag{D.22}$$

This lattice is distributive from Equation D.18; from distributivity, it is direct to check that it is modular as well. Moreover, it is also a complete lattice[8]. This is because the identity and depolarizing superoperators also play a particular role with respect to the other operation: the *absorbant* of the operation[9]:

8: Although some technicalities in the definition of completeness have been avoided by the projectors being defined over a finite-dimensional Hilbert space.

9: Both lines hold by the definition of an operator system.

$$\mathcal{D}_A \cap \mathcal{P}_A = \mathcal{D}_A \ ; \tag{D.23a}$$

$$\mathcal{I}_A \cup \mathcal{P}_A = \mathcal{I}_A \ . \tag{D.23b}$$

$\mathcal{D}$ is the absorbant of $\cap$ and $\mathcal{I}$ is the one of $\cup$. In lattice terms, this is simply the fact that $\mathcal{I}$ is the greatest element of the lattice, i.e. the 'unit', and that $\mathcal{D}$ is its smallest, i.e. the 'zero'.

Because of that property, it should be clear that for any element of the algebra, the following is true:

$$\mathcal{D}_A \subseteq \mathcal{P}_A \subseteq \mathcal{I}_A \ . \tag{D.24}$$

As a consequence, this is an algebra of idempotent elements equipped with a partial ordering $\subseteq$ that have a common greatest element $\mathcal{I}$ and a common least element $\mathcal{D}$. Moreover, the intersection and union of any two elements $\mathcal{P}, \mathcal{P}^{'}$ are uniquely defined elements of the algebra. To sort these new elements with respect to the partial order, the lattice structure is taken advantage of to write the following

$$\mathcal{P}_A \cap \mathcal{P}^{'}_A \subseteq \mathcal{P}_A \subseteq \mathcal{P}_A \cup \mathcal{P}^{'}_A \ , \tag{D.25}$$

and the analog with $\mathcal{P}^{'}_A$.

As defined in the main text, such lattices are what is implied when referring to the 'algebra of projectors' because, as will be shown in the next section, these lattices are complemented. In the remainder of this chapter, the term 'projector' used alone always refers to an element of such a Boolean algebra of projectors on operator systems. Accordingly, the term 'algebra' alone always refers to such a Boolean algebra which is defined in a space that should be clear from the context.

Summarizing, in this section, the observation that subspaces of a Hilbert space are in one-to-one correspondence with their projectors was applied to the case of operator systems. Since their projectors are projectors on operator systems, it was then asked under which rules these sets are stable. In order to do so, the observation that superoperators are an algebra over $\mathbb{C}$ under operations $\{+, \circ\}$ motivated the observation that projectors on operator systems form a lattice under operations $\{\cup, \cap\}$. Moreover, this lattice was shown to be a Boolean ring, which happens to be complete, distributive, and modular as a lattice. This Boolean ring is in correspondence with the state structures that can be defined on a Hilbert space, so all these properties actually make the set of state structures quite a tame and easy-to-characterize partially ordered set.

In the following of the mathematical method section: the lattice of the projective characterization will be defined. The goal is to show that the different rules on projector derived in Chapter 3 are valid operations in the algebra of operators, meaning that adding them to the algebra may augment the number of elements in the algebra but globally the algebra will still be a Boolean ring. What is more, this new set will be shown to be a sublattice of the global projector algebra. To prove it, it must be shown that on the one hand, each new rule added will preserve the algebra. This means that each added rule will map valid projectors to valid projectors and commuting sets of projectors to commuting sets of projectors. This is the algebraic counterpart of saying that the different rules $\{\bar{\cdot}, \otimes, \prec, \succ, \parr, \to\}$ to define new state structures out of given state structures result in valid state structures. But proving it at the level of the algebra of projectors will allow to derive the relations between all these different rules from algebraic manipulations only. The end goal is to use the rules to rewrite any projector into a form that makes the signaling relations of the objects it characterizes apparent.

D.2.2. A Comment on the Size of the Lattice

Before proceeding to the systematic breakdown of the algebraic properties of each operation on projector(s) that appeared in the last chapter[10], a short comment on the size of the projector algebra is in order.

10: I.e., $\{\bar{\cdot}, \otimes, \prec, \succ, \invamp, \rightarrow\}$.

Defined as such, the projector algebra is in one-to-one correspondence with the set of all simultaneously diagonalizable operator systems on a given space[11]. The set of all operator systems characterized using the projective rules is a subset of this set. Since it extends the type system by adding a bunch of new rules, namely the intersection $\cap$, the union $\cup$, and the one-way signaling composition $\prec$, the set of valid expressions in the type system must, in turn, be a subset of this subset.

11: And for a given basis defined by the set of commuting projectors.

To upper bound these sets, one can wonder how large the projector algebra is. As observed in the main text, the projector algebra is a lattice of commuting projectors on a finite-dimensional Hilbert space, so it is a finitely generated lattice. That is, the set of non-equivalent different elements must be of finite cardinality.

In the methods of the last chapter, subsection C.1.1, it was mentioned that the projector on operator systems can be identified with subsets of a traceless self-adjoint basis[12] of $\mathcal{L}(\mathcal{H})$ that contains the identity element, σ_0. A given such basis $\{\sigma_\mu\}_{\mu=0}^{d^2-1}$ can be identified with a set of d^2-1 commuting projectors on operator systems by the identification

12: Defined by Equations C.52.

$$\sigma_{\mu\neq 0} \mapsto \mathcal{P}^{(\mu)}\{\cdot\} = \sigma_0 \mathrm{Tr}[\sigma_0 \cdot] + \sigma_\mu \mathrm{Tr}[\sigma_\mu \cdot] \ . \tag{D.26}$$

These projectors on operator systems define 2-dimensional operator systems spanned by $\{\sigma_0, \sigma_\mu\}$. The union of two such projectors, say $\mathcal{P}^{(\mu)}$ and $\mathcal{P}^{(\nu)}$, define the 3-dimensional operator system spanned by $\{\sigma_0, \sigma_\mu, \sigma_\nu\}$.

Considering that all projectors must contain the depolarizing superoperators, that the depolarizing superoperator $\mathcal{D}(\cdot) = \sigma_0 \mathrm{Tr}[\sigma_0 \cdot]$ is a rank-1 superoperator projector itself, and that these rank-2 commuting projectors have zero overlaps, it should be clear that all unions of such projectors span all possible operator systems for this given basis. This implies that the set of all simultaneously diagonalizable operator systems is in one-to-one correspondence with the power set of $\{\sigma_\mu\}_{\mu=1}^{d^2-1}$. Hence it follows that the size of the projector algebra of superoperators on a Hilbert space of dimension d is 2^{d^2-1}. The number of elements in the sublattices considered in Chapter 5 is thus at most exponential with the number of parties. Figuring out the exact size of a signaling lattice defined as in Proposition 5.1.11 is left open for future research.

D.2.3. The Boolean Lattice of Projectors

In the algebra of superoperator projectors, the new projector that naturally appears in Proposition 3.3.2, $\overline{\mathcal{P}}_A$, can be seen as an operation on the original projector $\mathcal{P}_A$ called the **negation** of $\mathcal{P}_A$,

$$\overline{\mathcal{P}} := \mathcal{I} - \mathcal{P} + \mathcal{D} \,, \tag{5.14}$$

whence the 'bar over $\mathcal{P}_A$' notation. This new operation is defined for any projector. As will be shown below, it defines a new projector $\overline{\mathcal{P}}$ which, since the projectors are linear maps, commutes with the original projector. Indeed, $\mathcal{P} \circ \overline{\mathcal{P}} = \mathcal{P} \circ (\mathcal{I} - \mathcal{P} + \mathcal{D}) = (\mathcal{P} \circ \mathcal{I}) - \mathcal{P}^2 + (\mathcal{P} \circ \mathcal{D})$, using the fact that a projector always commutes with itself and the identity, and that, because of Definition 3.2.7, it also commutes with the depolarizing superoperator, then $\mathcal{P} \circ \overline{\mathcal{P}} = (\mathcal{I} \circ \mathcal{P}) - \mathcal{P}^2 + (\mathcal{D} \circ \mathcal{P}) = (\mathcal{I} - \mathcal{P} + \mathcal{D}) \circ \mathcal{P}$. Therefore, $\mathcal{P} \circ \overline{\mathcal{P}} = \overline{\mathcal{P}} \circ \mathcal{P}$. and so the $\cap$ of a projector with its negation is a well-defined expression obeying

$$\mathcal{P} \circ \overline{\mathcal{P}} = \overline{\mathcal{P}} \circ \mathcal{P} \,. \tag{D.27}$$

As it turns out, the negation is actually the complement operation of the lattice (or logic 'not', thereafter referred to as 'negation' [13]) characterized by the condition that the addition of any projector with its negation yields the additive identity, i.e.

13: This terminology is favored over other commonly used terms for this concept like 'inverse', 'dual', or 'complement' because these words are already used for different concepts in quantum information and functional analysis. This choice is made by consequence to avoid ambiguities.

$$\mathcal{P}_A \cup \overline{\mathcal{P}}_A = \mathcal{I}_A \,, \tag{D.28}$$

where $\overline{\mathcal{P}}_A$ is the negation of $\mathcal{P}_A$. This is exactly what the quasi-orthogonal complement of Proposition 3.3.2 is doing. First, compute the cap as $\mathcal{P}_A \cap \overline{\mathcal{P}}_A = \mathcal{P}_A \circ (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) = \mathcal{P}_A - \mathcal{P}_A^2 + \mathcal{D}_A = \mathcal{P}_A - \mathcal{P}_A + \mathcal{D}_A$, so

$$\mathcal{P}_A \cap \overline{\mathcal{P}}_A = \mathcal{D}_A \,. \tag{D.29}$$

Then, it is true for the cup because $\mathcal{P}_A \cup \overline{\mathcal{P}}_A = \mathcal{P}_A + \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A - (\mathcal{P}_A \cap \overline{\mathcal{P}}_A) = \mathcal{I}_A$.

In addition to property (D.28), one can verify from (D.29) that a projector and its negation intersect at the zero of the algebra, which in this case is $\mathcal{D}_A$ (refer to the diagram in Figure D.1). These two rules are actually fundamental in Boolean logic: the first is the law of excluded middle, and the second is the law of noncontradiction. The projector algebra has the characteristics of a model of Boolean logic.

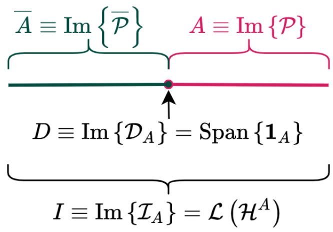


Figure D.1.: Diagrammatic representation of how a space of operator is split between the images of a projector and its negation. In this kind of diagrams, the central dot will always represent the image of the center of the algebra of projectors, i.e. the image of the depolarizing superoperator.

There is now an explanation why the definition of the inverse with respect to the addition of the Boolean ring that was eluded in the previous section. The algebra is more correctly interpreted as a Boolean lattice, and therefore as a model of logic, than as a Boolean ring. Actually, the operation $\cup$ defines a monoid with unit rather than an Abelian group since it is impossible to find an inverse projector on operator system $\mathcal{P}^{-1}$ that would satisfy

$$\mathcal{P}_A \cup \mathcal{P}_A^{-1} = \mathcal{D}_A \,. \tag{D.30}$$

Actually, the correct addition in the Boolean ring is the symmetric difference,

$$\mathcal{P}_A \oplus \mathcal{P}'_A = \left(\mathcal{P}_A \cup \mathcal{P}'_A\right) \cap \left(\overline{\mathcal{P}}_A \cup \overline{\mathcal{P}'_A}\right) \,, \tag{D.31}$$

but using the $\cup$ instead allows the interpretation of the Boolean ring as a Boolean algebra (or lattice) and so to benefit from an extra duality principle as well as the De Morgan law.

The lattice and De Morgan duality: The lattice structure was already shown to be complete, distributive, and modular in the previous section. Under the negation, it is moreover complemented, as the negation is an involution:

$$\overline{\overline{\mathcal{P}}}_A = \mathcal{P}_A \,, \tag{D.32}$$

and it verifies the conditions (D.29) and (D.28).

As a consequence, the De Morgan laws are valid for the Boolean algebra of projectors:

$$\overline{\mathcal{P}_A \cup \mathcal{P}'_A} = \overline{\mathcal{P}_A} \cap \overline{\mathcal{P}'_A} \,, \tag{D.33a}$$

$$\overline{\mathcal{P}_A \cap \mathcal{P}'_A} = \overline{\mathcal{P}_A} \cup \overline{\mathcal{P}'_A} \,. \tag{D.33b}$$

The proof is more involved: $\overline{\mathcal{P}_A \cup \mathcal{P}'_A} = \mathcal{I}_A - \mathcal{P}_A \cup \mathcal{P}'_A + \mathcal{D}_A = \mathcal{I}_A - (\mathcal{P}_A + \mathcal{P}'_A - (\mathcal{P}_A \cap \mathcal{P}'_A)) + \mathcal{D}_A = \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A + \mathcal{D}_A - \mathcal{P}'_A - \mathcal{D}_A + (\mathcal{P}_A \cap \mathcal{P}'_A) - \mathcal{D}_A + \mathcal{D}_A = (\mathcal{I}_A - \mathcal{P}_A + D) \cap (\mathcal{I}_A - \mathcal{P}'_A + \mathcal{D}_A) = \overline{\mathcal{P}}_A \cap \overline{\mathcal{P}'_A}$. The second identity directly ensues.

The inclusion relation $\subseteq$ defined by conditions (5.10) is by consequence reversed when the projectors it involves are negated. Indeed, if $\mathcal{P}'_A \subseteq \mathcal{P}_A \Rightarrow \mathcal{P}'_A \cap \mathcal{P}_A = \mathcal{P}'_A$, negating both sides of the cap yields $\overline{\mathcal{P}'_A} \cap \overline{\mathcal{P}}_A = \overline{\mathcal{P}'_A \cup \mathcal{P}_A}$. But the term under the negation in $\overline{\mathcal{P}'_A \cup \mathcal{P}_A}$ is equivalent to $\mathcal{P}_A$ since the inclusion also implies that $\mathcal{P}'_A \subseteq \mathcal{P}_A \Rightarrow \mathcal{P}'_A \cup \mathcal{P}_A = \mathcal{P}_A$. Whence, $\overline{\mathcal{P}'_A} \cap \overline{\mathcal{P}}_A = \overline{\mathcal{P}'_A \cup \mathcal{P}_A} = \overline{\mathcal{P}}_A$, which by definition is the inclusion $\overline{\mathcal{P}}_A \subseteq \overline{\mathcal{P}'_A}$. This reasoning works in both ways, proving

$$\mathcal{P}'_A \subseteq \mathcal{P}_A \iff \overline{\mathcal{P}}_A \subseteq \overline{\mathcal{P}'_A} \,, \tag{D.34}$$

which is equation (5.16) in the main text. Note that these properties directly follow from the algebra being a Boolean lattice, these were proven for completeness.

The bottom line is that the algebra of projectors is a Boolean algebra under operations $\{\cup, \cap, \overline{\cdot}\}$ with greatest element $\mathcal{I}$ and least $\mathcal{D}$. It remains to show that the negation can be added to the algebra without spoiling its structure of commuting projectors: the new elements $\overline{\mathcal{P}}$ should also be projectors on operator systems and commuting with every other projectors.

Negation yields a valid projector on operator system: Since $\mathcal{I}_A$ and $\mathcal{D}_A$ are Hermitian-preserving (HP) and self-adjoint (SA), the negation of a projector is HP and SA provided the original projector is HP and SA. The idempotency property (3.20a) is preserved from $\overline{\mathcal{P}}^2_A = (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) \cap (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) = \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A - \mathcal{P}_A + \mathcal{P}^2_A - \mathcal{D}_A + \mathcal{D}_A - \mathcal{D}_A + \mathcal{D}_A = \mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A$, hence

$$\overline{\mathcal{P}}^2_A = \overline{\mathcal{P}}_A \,. \tag{D.35}$$

To prove that the negation of a projector on an operator system is a projector on an operator system itself it remains to show that the identity is still contained in the negation, Equation 3.20d,

$$\mathcal{D}_A \cap \overline{\mathcal{P}}_A = \mathcal{D}_A \, . \tag{D.36}$$

This is true from $\mathcal{D}_A \cap \overline{\mathcal{P}}_A = \mathcal{D}_A \cap (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) = \mathcal{D}_A - \mathcal{D}_A + \mathcal{D}_A$. Without the heuristic of Theorem 3.3.2, this could be interpreted as a reason for choosing Equation 5.14 as the definition of negation instead of the orthogonal complement $\mathcal{I}_A - \mathcal{P}_A$ since, in this latter case, identity does not belong to the negated state structure.

Negation preserves commutativity: It should also be clear from $\mathcal{P}_A \cap \overline{\mathcal{P}}_A = \mathcal{P}_A \cap (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) = \mathcal{P}_A - \mathcal{P}_A^2 + \mathcal{D}_A = (\mathcal{I}_A - \mathcal{P}_A + \mathcal{D}_A) \cap \mathcal{P}_A$ that negated projectors commute with the original ones, and, by the same kind of proof, that the negations of two commuting projectors still commute:

$$\mathcal{P}_A \cap \mathcal{P}_A^{'} = \mathcal{P}_A^{'} \cap \mathcal{P}_A \Longrightarrow \overline{\mathcal{P}}_A \cap \overline{\mathcal{P}_A^{'}} = \overline{\mathcal{P}_A^{'}} \cap \overline{\mathcal{P}}_A \, . \tag{D.37}$$

D.3. The Type System/Fragment of Linear Logic

Here are reviewed the properties of the operations $\{\otimes, \rightarrow, \parr\}$ that appeared in the projective characterization of state structures. Together with the negation, these ways of connecting are historically the first ones to have been considered in the theory of higher-order processes. The 'transformation' composition $\rightarrow$ was first introduced in Reference [10], whereas the parr in Reference [33]. Although these two works did not explicitly use these compositions as rules to be applied to the projectors, many of the properties derived here were implicit in them.

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.

[33]: Kissinger et al. (2019), *A categorical semantics for causal structure*.

D.3.1. The Tensor

The first composition operation that is added to the projector algebra is the tensor product because it is inherited as the tensor product of linear maps. In Definition 3.4.1, the tensor product is used at the level of the projectors to consider the no-signaling composition of state structures $\mathscr{A} \otimes \mathscr{B} \subset \mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ when it is known that $\mathscr{A}$ is characterized by a projector $\mathcal{P}_A$ acting on $\mathcal{L}\left(\mathcal{H}^A\right)$ and that $\mathscr{B}$ is by $\mathcal{P}_B$. Here is shown how the tensor product of two algebras of projectors on different spaces is also the algebra of projector on the composite space, translating the fact that $\mathscr{A} \otimes \mathscr{B}$ is a valid state structure if $\mathscr{A}$ and $\mathscr{B}$ are.

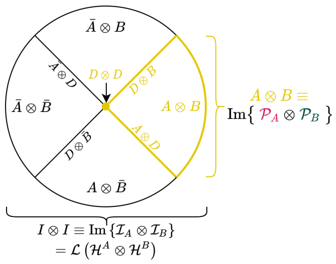


Figure D.2.: Diagrammatic representation of the support of a tensor product of projectors. This is the bipartite version of Figure D.1. The support associated with subsystem A is always on the left side of the tensor product. Note that the intersections and unions are well defined in this diagram: the central dot is associated with $\mathcal{D}_A \cap \mathcal{D}_B$ and every, the bottom left to top right line with $\mathcal{D}_A \otimes \mathcal{I}_B$, and the top left to bottom right one to $\mathcal{I}_A \otimes \mathcal{D}_B$. A quarter of the wheel is associated with a tensor of projectors $\mathcal{P}_A \otimes \mathcal{P}_B$, the other quarters represent the three other subspaces obtainable through projector-wise negation, and the union of the four quarters, the full wheel, represent the full space, associated with the identity projector $\mathcal{I}_A \otimes \mathcal{I}_B$.

Because of the isomorphism $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right) \cong \mathcal{L}\left(\mathcal{H}^A\right) \otimes \mathcal{L}\left(\mathcal{H}^B\right)$, the definition of **no-signaling composition**, nicknamed **tensor**, is straightforward:

$$(\mathcal{P}_A \otimes \mathcal{P}_B) \left\{ \sum_i q_i \left(V_i \otimes U_i\right) \right\} := \sum_i q_i \left(\mathcal{P}_A \left\{V_i\right\} \otimes \mathcal{P}_B \left\{U_i\right\}\right) \, , \tag{D.38}$$

and it should hold for all i such that $q_i \in \mathbb{C}$, $V_i \in \mathcal{L}\left(\mathcal{H}^A\right)$, $U_i \in \mathcal{L}\left(\mathcal{H}^B\right)$ as any operator in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ can be decomposed as such.

By the inherited properties of the tensor product of linear maps, the tensor at the level of superoperators is associative; because of $\mathcal{L}\left(\mathcal{H}^A\right) \otimes \mathcal{L}\left(\mathcal{H}^B\right) \cong \mathcal{L}\left(\mathcal{H}^B\right) \otimes \mathcal{L}\left(\mathcal{H}^A\right)$, it is moreover commutative.

Tensor and cap/cup: The tensor is distributive with respect to the cap,

$$\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \otimes \mathcal{P}_B = \left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cap \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B\right) , \tag{D.39}$$

as well as to the cup,

$$\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \otimes \mathcal{P}_B = \left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B\right) . \tag{D.40}$$

This directly follows from idempotency:

$$\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \otimes \mathcal{P}_B = \left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B\right), \tag{D.41}$$

and because the intersection of projectors is party-wise therefore it commutes with the tensor product. This is due to the fact that the composition of superoperators obeys an interchange law with the tensor product of superoperators [81]:

[81]: Coecke et al. (2017), *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*.

$$\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cap \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right) . \tag{D.42}$$

However, the tensor does not interchange with the cup; rather, the cup distributes:

$$\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \otimes \left(\mathcal{P}_B \cup \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A \otimes \mathcal{P}_B^{'}\right) \cup \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right) . \tag{D.43}$$

This is the case because

$$\begin{aligned}
&\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \otimes \left(\mathcal{P}_B \cup \mathcal{P}_B^{'}\right) = \\
&\left(\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \otimes \mathcal{P}_B\right) + \left(\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \otimes \mathcal{P}_B^{'}\right) - \left(\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right)\right) \\
&\overset{(D.40)}{=} \left(\left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B\right)\right) + \left(\left(\mathcal{P}_A \otimes \mathcal{P}_B^{'}\right) \cup \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right)\right) \\
&\quad - \left(\left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B\right)\right) \cap \left(\left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A \otimes \mathcal{P}_B^{'}\right)\right) \\
&\quad = \left(\left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A \otimes \mathcal{P}_B^{'}\right)\right) \cup \left(\left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right)\right) .
\end{aligned} \tag{D.44}$$

And the union is associative. Note in passing that the following identity has been used,

$$\begin{aligned}
&\left(\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right)\right) \\
&\quad = \left[\left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cap \left(\mathcal{P}_A \otimes \mathcal{P}_B^{'}\right)\right] \cup \left[\left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B\right) \cap \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right)\right] \\
&\quad = \left[\left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B\right)\right] \cap \left[\left(\mathcal{P}_A \otimes \mathcal{P}_B^{'}\right) \cup \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right)\right] ,
\end{aligned} \tag{D.45}$$

the proof of which follow from Equation (D.39) and (D.40). Because of Equation (D.43), the interchange law for the cup yields a lower bound,

$$\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \otimes \left(\mathcal{P}_B \cup \mathcal{P}_B^{'}\right) \supseteq \left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right) , \tag{D.46}$$

which is saturated only when

$$\left[(\mathcal{P}_A \otimes \mathcal{P}_B) \cup \left(\mathcal{P}'_A \otimes \mathcal{P}'_B\right)\right] \supseteq \left[\left(\mathcal{P}'_A \otimes \mathcal{P}_B\right) \cup \left(\mathcal{P}_A \otimes \mathcal{P}'_B\right)\right] . \quad \text{(D.47)}$$

Tensor and negation provides another De Morgan duality: Interestingly, however, the tensor and the negation do not commute with each other: $\overline{\mathcal{P}_A \otimes \mathcal{P}_B} \neq \overline{\mathcal{P}_A} \otimes \overline{\mathcal{P}_B}$. Actually, the latter expression defines a subspace in the former since

$$\begin{aligned}\overline{\mathcal{P}_A \otimes \mathcal{P}_B} \cap \overline{\mathcal{P}_A} \otimes \overline{\mathcal{P}_B} &= (\mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{P}_A \otimes \mathcal{P}_B + \mathcal{D}_A \otimes \mathcal{D}_B) \cap \overline{\mathcal{P}_A} \otimes \overline{\mathcal{P}_B} \\ &= \overline{\mathcal{P}_A} \otimes \overline{\mathcal{P}_B} - \mathcal{D}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \, , \quad \text{(D.48)}\end{aligned}$$

so that

$$\overline{\mathcal{P}_A \otimes \mathcal{P}_B} \cap \overline{\mathcal{P}_A} \otimes \overline{\mathcal{P}_B} = \overline{\mathcal{P}_A} \otimes \overline{\mathcal{P}_B} \, . \quad \text{(D.49)}$$

And from the duality principle,

$$\overline{\mathcal{P}_A \otimes \mathcal{P}_B} \cup \overline{\mathcal{P}_A} \otimes \overline{\mathcal{P}_B} = \overline{\mathcal{P}_A \otimes \mathcal{P}_B} \, . \quad \text{(D.50)}$$

Hence,

$$\overline{\mathcal{P}_A} \otimes \overline{\mathcal{P}_B} \subseteq \overline{\mathcal{P}_A \otimes \mathcal{P}_B} \, ; \quad \text{(D.51)}$$

This identity which, using (D.34), can be recast as

$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} =: \mathcal{P}_A \parr \mathcal{P}_B \, , \quad \text{(D.52)}$$

is Equation 5.27 of the main text.

Because of the interchange law (D.42), relation (D.51) is stable when composed with more parties, i.e.,

$$(\mathcal{P}_A \otimes \mathcal{P}_B) \otimes \mathcal{P}_C \subseteq (\mathcal{P}_A \parr \mathcal{P}_B) \otimes \mathcal{P}_C \, . \quad \text{(D.53)}$$

More generally, the inclusion relation $\subseteq$ induced by conditions (5.11) is stable under the tensor; if $\mathcal{P}_A \subseteq \mathcal{P}'_A$, then the following hold

$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \mathcal{P}'_A \otimes \mathcal{P}_B \, , \quad \text{(D.54a)}$$

$$\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} \subseteq \overline{\overline{\mathcal{P}}'_A \otimes \overline{\mathcal{P}}_B} \, , \quad \text{(D.54b)}$$

$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \overline{\overline{\mathcal{P}}'_A \otimes \overline{\mathcal{P}}_B} \, . \quad \text{(D.54c)}$$

The last equation is a consequence of the first using (D.51). The first equation holds because

$$(\mathcal{P}_A \otimes \mathcal{P}_B) \cap \left(\mathcal{P}'_A \otimes \mathcal{P}_B\right) \overset{\text{(D.39)}}{=} \left(\mathcal{P}_A \cap \mathcal{P}'_A\right) \otimes \mathcal{P}_B \, , \quad \text{(D.55)}$$

and the term in parenthesis is equal to $\mathcal{P}_A$ because $\mathcal{P}_A \subseteq \mathcal{P}'_A$, therefore the first equation holds. The second equation is proven using De Morgan duality:

$$
\begin{aligned}
\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} \cap \overline{\overline{\mathcal{P}}'_A \otimes \overline{\mathcal{P}}_B} &\overset{(D.33a)}{=} \overline{\left(\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B\right) \cup \left(\overline{\mathcal{P}}'_A \otimes \overline{\mathcal{P}}_B\right)} \\
&\overset{(D.40)}{=} \overline{\left(\overline{\mathcal{P}}'_A \cup \overline{\mathcal{P}}_A\right) \otimes \overline{\mathcal{P}}_B} \\
&\overset{(D.33b)}{=} \overline{\overline{\mathcal{P}'_A \cap \mathcal{P}_A} \otimes \overline{\mathcal{P}}_B} \\
&= \overline{\overline{\mathcal{P}}'_A \otimes \overline{\mathcal{P}}_B} \,.
\end{aligned}
\tag{D.56}
$$

Relations (D.51) and (D.54) are important to define the no signaling subset of an arbitrary projector, as explained in subsection 5.1.3 of the main text.

The Tensor yields a valid projector on operator system: Expressions built from the tensor product of Hermitian-preserving projectors are automatically HP since the dagger distributes over the tensor, $(\rho \otimes \sigma)^\dagger = \rho^\dagger \otimes \sigma^\dagger$. If the composed projectors are self-adjoint then so is their tensor product since the Hilbert-Schmidt inner product is $\mathbb{C}$-linear and splits as

$$
\left(\mathcal{P}_A\left\{\rho_A\right\} \otimes \mathcal{P}_B\left\{\sigma_B\right\}\,,\, \eta_A \otimes \chi_B\right)_{AB} = \left(\mathcal{P}_A\left\{\rho_A\right\}\,,\, \eta_A\right)_A \left(\mathcal{P}_B\left\{\sigma_B\right\}\,,\, \chi_B\right)_B
\tag{D.57}
$$

so that $\left(\mathcal{P}_A \otimes \mathcal{P}_B\right)^* = \mathcal{P}_A^* \otimes \mathcal{P}_B^*$. They preserve idempotency,

$$
\left(\mathcal{P}_A \otimes \mathcal{P}_B\right)^2 = \left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cap \left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \overset{(D.42)}{=} \left(\mathcal{P}_A \cap \mathcal{P}_A\right) \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B\right)\,,
\tag{D.58}
$$

because of interchange law, thus

$$
\left(\mathcal{P}_A \otimes \mathcal{P}_B\right)^2 = \left(\mathcal{P}_A \otimes \mathcal{P}_B\right)\,.
\tag{D.59}
$$

A further consequence of the interchange law is that the tensor composition of, respectively, the units and the zeroes on A and B, i.e. $\mathcal{I}_A \otimes \mathcal{I}_B$ and $\mathcal{D}_A \otimes \mathcal{D}_B$, are the respective unit and zero of the projector algebra on $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$, $\mathcal{I}_{AB}$ and $\mathcal{D}_{AB}$. Indeed, they can be used to define the negation of $\mathcal{P}_A \otimes \mathcal{P}_B$, $\overline{\mathcal{P}_A \otimes \mathcal{P}_B}$, and this gives the correct Boolean completions:

$$
\begin{aligned}
\left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cap \overline{\left(\mathcal{P}_A \otimes \mathcal{P}_B\right)} &= \left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cap \left(\mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{P}_A \otimes \mathcal{P}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) \\
&= \left(\mathcal{P}_A \otimes \mathcal{P}_B - \mathcal{P}_A^2 \otimes \mathcal{P}_B^2 + \mathcal{D}_A \otimes \mathcal{D}_B\right) = \mathcal{D}_A \otimes \mathcal{D}_B
\end{aligned}
\tag{D.60}
$$

(at the second equality sign, the interchange law was used to compute the cap of $\mathcal{P}_A \otimes \mathcal{P}_B$ with each of the three components of the negation);

$$
\begin{aligned}
&\left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \overline{\left(\mathcal{P}_A \otimes \mathcal{P}_B\right)} = \left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \left(\mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{P}_A \otimes \mathcal{P}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) \\
&\quad = \mathcal{P}_A \otimes \mathcal{P}_B + \mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{P}_A \otimes \mathcal{P}_B + \mathcal{D}_A \otimes \mathcal{D}_B - \mathcal{D}_A \otimes \mathcal{D}_B = \mathcal{I}_A \otimes \mathcal{I}_B\,.
\end{aligned}
\tag{D.61}
$$

Hence,

$$(\mathcal{P}_A \otimes \mathcal{P}_B) \cap \overline{(\mathcal{P}_A \otimes \mathcal{P}_B)} = \mathcal{D}_A \otimes \mathcal{D}_B =: \mathcal{D}_{AB} \; ; \tag{D.62a}$$

$$(\mathcal{P}_A \otimes \mathcal{P}_B) \cup \overline{(\mathcal{P}_A \otimes \mathcal{P}_B)} = \mathcal{I}_A \otimes \mathcal{I}_B =: \mathcal{I}_{AB} \; . \tag{D.62b}$$

The tensor composition then "preserves the identity" because the identity on a joint system is the tensor product of the identities on each of the spaces being combined, $\mathbb{1}_{AB} = \mathbb{1}_A \otimes \mathbb{1}_B \iff \mathcal{D}_{AB} = \mathcal{D}_A \otimes \mathcal{D}_B$. By the interchange law, this implies that the tensor of projectors on operator systems contains the depolarizing projector (or zero): $\mathcal{D}_{AB} \cap (\mathcal{P}_A \otimes \mathcal{P}_B) = (\mathcal{D}_A \otimes \mathcal{D}_B) \cap (\mathcal{P}_A \otimes \mathcal{P}_B) = (\mathcal{D}_A \cap \mathcal{P}_A) \otimes (\mathcal{D}_B \cap \mathcal{P}_B) = \mathcal{D}_A \otimes \mathcal{D}_B$,

$$\mathcal{D}_{AB} \cap (\mathcal{P}_A \otimes \mathcal{P}_B) = \mathcal{D}_{AB} \; . \tag{D.63}$$

Proving condition (3.20d) for the tensor, which was the last property needed to show that the tensor product of two projectors verifying Equation 3.20 verifies them as well.

The Tensor product preserves commutation: Since, if $\mathcal{P}_A \cap \mathcal{P}_A^{'} = \mathcal{P}_A^{'} \cap \mathcal{P}_A$ and $\mathcal{P}_B \cap \mathcal{P}_B^{'} = \mathcal{P}_B^{'} \cap \mathcal{P}_B$, then

$$(\mathcal{P}_A \otimes \mathcal{P}_B) \cap \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A^{'} \cap \mathcal{P}_A\right) \otimes \left(\mathcal{P}_B^{'} \cap \mathcal{P}_B\right) \; , \tag{D.64}$$

and so

$$(\mathcal{P}_A \otimes \mathcal{P}_B) \cap \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A^{'} \otimes \mathcal{P}_B^{'}\right) \cap (\mathcal{P}_A \otimes \mathcal{P}_B) \; . \tag{D.65}$$

D.3.2. The Transformation

The projector appearing in Theorem 3.4.1 is built from the projectors characterizing its input and output. The corresponding operation is defined as the **transformation** in (5.4), represented by $\rightarrow$:

$$\mathcal{P}_A \rightarrow \mathcal{P}_B := \mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{P}_A \otimes \mathcal{I}_B + \mathcal{P}_A \otimes \mathcal{P}_B - \mathcal{P}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \; . \tag{D.66}$$

The Transformation yields a valid projector on operator system that preserves commutation: This additional operation in the Boolean algebra of projectors is actually secondary since it can be entirely defined using the negation and the no signaling composition (i.e., the tensor):

$$\mathcal{P}_A \rightarrow \mathcal{P}_B \equiv \overline{\mathcal{P}_A \otimes \overline{\mathcal{P}}_B} \; . \tag{D.67}$$

Therefore it will automatically be a valid projector in the sense that it will obey Equations (3.20) if its constituents do, and will accordingly preserve commutation if they do.

14: First proven in the context of higher-order transformations in Reference [10].

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.

Other properties: The transformation is not associative in general. For example, $((\mathcal{P}_A \to \mathcal{P}_B) \to \mathcal{P}_C) \neq (\mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C))$, because the *uncurrying rule*[14] can be applied to the right-hand side:

$$\mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C) = (\mathcal{P}_A \otimes \mathcal{P}_B) \to \mathcal{P}_C \; , \tag{D.68}$$

which is obviously different than $((\mathcal{P}_A \to \mathcal{P}_B) \to \mathcal{P}_C)$ because of Equation D.52. In the algebra, the uncurrying rule relies on the associativity of the tensor:

$$\begin{aligned} \mathcal{P}_A \to (\mathcal{P}_B \to \mathcal{P}_C) = \overline{\mathcal{P}_A \otimes \overline{\left(\overline{\mathcal{P}_B \otimes \overline{\mathcal{P}}_C}\right)}} &= \overline{\mathcal{P}_A \otimes \overline{\overline{\mathcal{P}_B \otimes \overline{\mathcal{P}}_C}}} \overset{(D.32)}{=} \overline{\mathcal{P}_A \otimes \mathcal{P}_B \otimes \overline{\mathcal{P}}_C} \\ &= \overline{(\mathcal{P}_A \otimes \mathcal{P}_B) \otimes \overline{\mathcal{P}}_C} = (\mathcal{P}_A \otimes \mathcal{P}_B) \to \mathcal{P}_C \; . \end{aligned} \tag{D.69}$$

Remark that because of Equation D.54b, the transformation also preserves the inclusion relations:

$$\mathcal{P}_A \subseteq \mathcal{P}'_A \;\Rightarrow\; (\mathcal{P}_A \to \mathcal{P}_B) \subseteq \left(\mathcal{P}'_A \to \mathcal{P}_B\right) \; ; \tag{D.70a}$$

$$\mathcal{P}_B \subseteq \mathcal{P}'_B \;\Rightarrow\; (\mathcal{P}_A \to \mathcal{P}_B) \subseteq \left(\mathcal{P}_A \to \mathcal{P}'_B\right) \; . \tag{D.70b}$$

At the level of Boolean logic, the transformation operation can be understood as a logical implication. Indeed, the transformation is equal to its inverse implication, meaning that it satisfies

$$\mathcal{P}_A \to \mathcal{P}_B = \overline{\mathcal{P}_A} \leftarrow \overline{\mathcal{P}_B} \; , \tag{D.71}$$

which comes from the definition. Additionally, it is equivalent to $\overline{\mathcal{P}_B} \to \overline{\mathcal{P}_A}$ since the order of the systems in the tensor product does not matter as $\mathcal{H}^A \otimes \mathcal{H}^B \cong \mathcal{H}^B \otimes \mathcal{H}^A$.

D.3.3. The Type System

[10]: Perinotti (2017), *Causal Structures and the Classification of Higher Order Quantum Computations*.
[11]: Bisio et al. (2019), *Theoretical framework for higher-order quantum theory*.

The $\to$ operation on projectors recovers the $\to$ type constructor of Bisio and Perinotti type theory [10, 11] (presented in the main text, Section 4.3): if the trivial system is defined as '1', that is to say, the 1-dimensional state structure {1} made of the number 1, one can interpret the measurement (thus the negation of a given state structure) as a transformation into the trivial system. This leads to the identity

$$\overline{\mathcal{P}_A} = \mathcal{P}_A \to 1 \; , \tag{D.72}$$

which justifies the notation $\mathscr{A} \to 1 := \overline{\mathscr{A}}$. The proof is straightforward from the definition since 1 is 1-dimensional: $\mathcal{P}_A \to 1 = \overline{\mathcal{P}_A \otimes \overline{1}} = \overline{\mathcal{P}_A}$. In the same way, one can prove that

$$\mathcal{P}_A = 1 \to \mathcal{P}_A \; . \tag{D.73}$$

Therefore, a state structure can be seen as a transformation from the trivial system to itself and its negation as a transformation from itself to the trivial system. In view of the link between composition and transformation, one may also interpret the former in terms of the latter. A bipartite system in tensor-composed state structures, $\mathscr{A} \otimes \mathscr{B}$ for instance, can actually be seen as characterized by the following transformations

$$\mathcal{P}_A \otimes \mathcal{P}_B = \overline{\mathcal{P}_A \to \overline{\mathcal{P}}_B} = \overline{\overline{\mathcal{P}}_A \leftarrow \mathcal{P}_B}\,. \tag{D.74}$$

This means that a no-signaling composite bipartite system is equivalent to a functional on a transformation from one state structure to the functionals on the other of the other. In the above equation, the direction of the transformation has no influence, which is expected since it is a no-signaling composition.

D.3.4. The Parr

The projector characterizing a two-way signaling composition as in Definition 3.5.2 is derived from the projector characterizing a transformation as

$$\mathcal{P}_A \parr \mathcal{P}_B \equiv \overline{\mathcal{P}}_A \to \mathcal{P}_B := \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B}\,. \tag{D.75}$$

The Parr yields a valid projector on operator system that preserves commutation: Like the transformation, it is a secondary connector so using it is the same as using the negation and tensor. Consequently, it yields valid projectors that preserve commutation.

Other properties: Because the negation is an involution, the parr is associative,

$$\begin{aligned}(\mathcal{P}_A \parr \mathcal{P}_B) \parr \mathcal{P}_C = \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} \parr \mathcal{P}_C = \overline{\overline{\overline{\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B}}} \otimes \overline{\mathcal{P}}_C} = \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B \otimes \overline{\mathcal{P}}_C} = \overline{\overline{\mathcal{P}}_A \otimes \overline{\overline{\overline{\overline{\mathcal{P}}_B \otimes \overline{\mathcal{P}}_C}}}} \\ = \mathcal{P}_A \parr (\mathcal{P}_B \parr \mathcal{P}_C)\,.\end{aligned} \tag{D.76}$$

Notice the intermediate expression which is a symmetric pattern of an overall negation and party-wise negations of a tensor product composition. Because of the idempotency of the negation, every expression involving only the parr connective presents it:

$$\mathcal{P}_A \parr \mathcal{P}_B \parr \ldots \parr \mathcal{P}_K = \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B \otimes \ldots \otimes \overline{\mathcal{P}}_K} \tag{D.77}$$

This form is the one appearing in the definition of the fully signaling superset, Definition 5.1.2.

As this form involves only the tensor product as composition, the parr inherits its commutative characteristic since $\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} \cong \overline{\overline{\mathcal{P}}_B \otimes \overline{\mathcal{P}}_A}$,

$$\mathcal{P}_A \parr \mathcal{P}_B \cong \mathcal{P}_B \parr \mathcal{P}_A\,. \tag{D.78}$$

From the tensor product, it also inherits the order-preservation,

$$
\begin{aligned}
&\mathcal{P}_A \subseteq \mathcal{P}_A^{'} \\
\overset{(5.16)}{\Longleftrightarrow}\ &\overline{\mathcal{P}}_A \supseteq \overline{\mathcal{P}}_A^{'} \\
\overset{(D.54)}{\Rightarrow}\ &\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B \supseteq \overline{\mathcal{P}}_A^{'} \otimes \overline{\mathcal{P}}_B \\
\overset{(5.16)}{\Longleftrightarrow}\ &\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} \subseteq \overline{\overline{\mathcal{P}}_A^{'} \otimes \overline{\mathcal{P}}_B}\,.
\end{aligned}
\tag{D.79}
$$

With the same kind of proof, it can be shown that the parr obeys an interchange law with the cup but a distribution law with the cap:

$$
\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \parr \left(\mathcal{P}_B \cup \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A \parr \mathcal{P}_B\right) \cup \left(\mathcal{P}_A^{'} \parr \mathcal{P}_B^{'}\right)\,, \tag{D.80}
$$

and

$$
\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \parr \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A \parr \mathcal{P}_B\right) \cap \left(\mathcal{P}_A \parr \mathcal{P}_B^{'}\right) \cap \left(\mathcal{P}_A^{'} \parr \mathcal{P}_B\right) \cap \left(\mathcal{P}_A^{'} \parr \mathcal{P}_B^{'}\right)\,. \tag{D.81}
$$

So, the interchange of the cap with the parr yields an upper bound:

$$
\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \parr \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) \subseteq \left(\mathcal{P}_A \parr \mathcal{P}_B\right) \cap \left(\mathcal{P}_A^{'} \parr \mathcal{P}_B^{'}\right)\,, \tag{D.82}
$$

This mirroring with the properties (D.42), (D.43), and (D.46) of the tensor under the substitutions ($\otimes \leftrightarrow \parr$), ($\cap \leftrightarrow \cup$), and ($\supseteq \leftrightarrow \subseteq$) is a nice example of what De Morgan duality implies.

However, the tensor and the parr do not have a meaningful notion of distribution over each other as the right-hand side of the following makes no sense in terms of underlying subspaces:

$$
\mathcal{P}_A \otimes \left(\mathcal{P}_B \parr \mathcal{P}_C\right) \neq \left(\mathcal{P}_A \parr \mathcal{P}_B\right) \otimes \left(\mathcal{P}_A \parr \mathcal{P}_C\right)\,. \tag{D.83}
$$

D.3.5. Multiplicative Additive Linear Logic

Here is shown under which definition that the rules $\{\overline{\cdot}, \cap, \cup, \otimes, \overline{\cdot} \to \cdot\}$ form a model of logic which is almost multiplicative additive linear logic (MALL)[15]. The correspondence with MALL is summarized in Table D.1, where the rules introduced in this thesis are put in correspondence with their usual notation.

15: I am grateful to Titouan Carette for being the first to point out the connection as well as the categorical approach to me.

Table D.1.: Correspondence of the algebraic rules of the projective characterization (proj.) with linear logic (LL)

Name	Symbol		Unit	
	proj.	LL	proj.	LL
Negation	$\overline{\cdot}$	$\cdot^{\perp}$	/	/
Additive conjunction	$\cap$	&	$\mathcal{I}$	T
Additive disjunction	$\cup$	$\oplus$	$\mathcal{D}$	0
Multiplicative conjunction	$\otimes$	$\otimes$	1	1
Multiplicative disjunction	$\parr$	$\parr$	1	$\perp$
Linear Implication	$\to$	$\multimap$	1	/

Linear logic is a formal system of logic, of which MALL is a fragment, that is a restriction of the logic to fewer rules. It can be defined as a *sequent calculus*, that is the formalization of proof systems in which each *proposition* follows under some *structural and inference rules* from other propositions (it is a *sequent* of these propositions, whence the name) see

References [143, 146]. If one sees the projector algebra and the operations on it as the propositions, the starting set of propositions is indeed similar to those of MALL:

[143]: Girard et al. (1989), *Proofs and Types*.
[146]: Girard (1987), *Linear logic*.

1. There is a set of base projectors;
2. For every projector $\mathcal{P}$, there exists a projector $\overline{\mathcal{P}}$;
3. For every projectors $\mathcal{P}_A$ and $\mathcal{P}_A^{'}$, there exist an additive conjunction $\mathcal{P}_A \cap \mathcal{P}_A^{'}$ as well as an additive disjunction $\mathcal{P}_A \cup \mathcal{P}_A^{'}$;
4. For every projectors $\mathcal{P}_A$ and $\mathcal{P}_B$, there exist a multiplicative conjunction $\mathcal{P}_A \otimes \mathcal{P}_B$ as well as a multiplicative disjunction $\overline{\mathcal{P}}_A \rightarrow \mathcal{P}_B$;
5. There are two constants $(\mathcal{I}, \mathcal{D})$ that go with each additive binary connectors;
6. There are two constants $(1, 1)$ that go with each multiplicative binary connectors.

And the properties that can be proven for the projectors are similar to those that can be proven in MALL using the sequent calculus:

1. All binary connectors are commutative;
2. Multiplicative connectors distribute over additive ones;
3. All propositions have a negation obeying the De Morgan rules (5.37);
4. Additive constants are the negation of each other;
5. Multiplicative constants are the negation of each other.

Proposition 1 is always true, as is shown through this appendix. Propositions 2 to 4 are true from the fact that the definition of these various connectors implies closure, which, in the case of projectors, is the conservation of properties (D.7) and (D.11). These were proven for each connector in the previous sections. Proposition 5 follows from equations (D.19) and (D.20). Proposition 6 happens because of the isomorphism $\mathcal{L}(\mathcal{H}) \otimes \mathbb{C} \cong 1$ so that $\mathcal{P} \otimes 1 = \mathcal{P}$ and the same way, $\overline{\overline{\mathcal{P}} \otimes \overline{1}} = \overline{\overline{\mathcal{P}}} = \mathcal{P}$, $\overline{1 \otimes \overline{\mathcal{P}}} = \overline{\overline{\mathcal{P}}} = \mathcal{P}$.

Property 1 is true from the definition. In the case of the mutiplicative connectors, $\mathcal{P}_A \otimes \mathcal{P}_B$ and $\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B}$, the isomorphism $\mathcal{H}^A \otimes \mathcal{H}^B \cong \mathcal{H}^B \otimes \mathcal{H}^A$ should of course be used.

Property 2 follows from Eqs. (D.39) and (D.40) in the case of $\otimes$ and application of the De Morgan rules (D.33) on these two equations can be used to prove the property in the case of $⅋$. Put differently, the multiplicative conjunction (respectively, disjunction) distributes over the additive conjunction (disjunction)

$$\mathcal{P}_A \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) = (\mathcal{P}_A \otimes \mathcal{P}_B) \cap \left(\mathcal{P}_A \otimes \mathcal{P}_B^{'}\right) \; ; \tag{D.84a}$$

$$\mathcal{P}_A ⅋ \left(\mathcal{P}_B \cup \mathcal{P}_B^{'}\right) = (\mathcal{P}_A ⅋ \mathcal{P}_B) \cup \left(\mathcal{P}_A ⅋ \mathcal{P}_B^{'}\right) \; . \tag{D.84b}$$

But as $\otimes$ and $\rightarrow$ are both operations that merge subspaces, the converse obviously does not hold. Indeed, an expression like $\mathcal{P}_X \cap (\mathcal{P}_A \otimes \mathcal{P}_B) = (\mathcal{P}_X \otimes \mathcal{P}_A) \cap (\mathcal{P}_X \otimes \mathcal{P}_B)$ makes no sense as the right-hand side feature a cap between two superoperator projectors defined on different spaces that are not isomorphic in general.

Property 3 was proven at Equation (D.32) for the negation, (D.33) for the additive connectors, and follows from the definition in the case of multiplicative connectors.

Property 4 is the statement $\overline{\mathcal{I}} = \mathcal{I} - \mathcal{I} + \mathcal{D} = \mathcal{D}$, whose converse hold by idempotency or can be proven by the same kind of computation.

Property 5 is the statement $\overline{1} = 1 - 1 + 1$, as in the 1-dimensional case $\mathcal{D} = \mathcal{I} = 1$.

[146]: Girard (1987), *Linear logic*.

There are, however, some discrepancies with MALL, that are now discussed. First, two small issues that indicate a departure from a faithful model of MALL as in Reference [146]:

1. The multiplicative units are equivalent;
2. The additive falsity (i.e. the unit for $\cup$) is not absorbant.

The first issue is why the terminology 'degenerate' was used in the main text to refer to the model of MALL formed by the algebra of projectors. The second issue –that $\mathcal{P}_A \otimes \mathcal{D}_B \neq \mathcal{D}_A \otimes \mathcal{D}_B$– is more severe as it goes against an equality that can be proven in MALL. A way to circumvent this is to redefine the additive falsity as the number 0, but then the trace normalization of all operator systems the projectors characterize should be set to zero which would jeopardize the probabilistic interpretation.

A more substantial problem is the fact that the cap and cup are only well-defined for expression featuring the same set of base projectors: two projectors can only be composed with a cap if they can be embedded in the same space. This is part of the problem evoked when discussing Property 2 above. For the model of logic to work properly, the number of subsystems as well as the dimension of each subsystem must be fixed *a priori*, and each expression is associated with a given (set of) subsystem(s), so that the composition is well-defined. Knowing this information allows one to 'pad' the expressions with identities so as to embed them in the global Hilbert space. For example, an expression like $\mathcal{P}_X \cap (\mathcal{P}_A \otimes \mathcal{P}_B)$ can be made definite by knowing for example that there are two systems A and B, and that $\mathcal{P}_X$ is associated with system B so that the padding reads $\mathcal{P}_X \cong \mathcal{I}_A \otimes \mathcal{P}_B^X$, where $\mathcal{P}_B^X$ is the projector $\mathcal{P}_X$ acting on system B. Another possibility is that $\mathcal{P}_X$ is associated with both systems, so that the padding is trivial $\mathcal{P}_X \cong \mathcal{P}_{AB}^X$. Yet another possibility is that the projector is associated with system A and B, but the global Hilbert space is tripartite: $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C$. In that case, the padding should be done on the two projectors: $\mathcal{P}_X \cong \mathcal{P}_{AB}^X \otimes \mathcal{I}_C, \mathcal{P}_A \otimes \mathcal{P}_B \cong \mathcal{P}_A \otimes \mathcal{P}_B \otimes \mathcal{I}_C$.

[36]: Simmons et al. (2022), *Higher-order causal theories are models of BV-logic*.

Despite these particularities, the connection between the algebra of projectors and MALL is too evident not to be pointed out. In particular, the issues with interpreting the algebra as a model of LL come from the choice of additive connectives as $\cap$ and $\cup$. This choice is motivated by a desire to compare the underlying operator systems associated with different types of higher-order quantum transformations. Another choice proposed in Reference [36] is to use the Cartesian product and direct sum as the additive conjunction and disjunction respectively. In that case, the issues can all be alleviated except for the degeneracy of the multiplicative units... but the additives connectors can no longer be used for comparison.

D.3.6. The Prec

Define the **A-to-B one-way signaling composition**, nicknamed **prec**, as

$$\mathcal{P}_A \prec \mathcal{P}_B := \mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \, . \tag{D.85}$$

The reversed sign $\succ$ (also nicknamed the prec) can be defined accordingly: the **B-to-A one-way signaling composition** is given by

$$\mathcal{P}_A \succ \mathcal{P}_B := \mathcal{P}_A \otimes \mathcal{I}_B - \mathcal{D}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B \, . \tag{D.86}$$

As is the case with the transformation $\rightarrow$, $\mathcal{P}_A \succ \mathcal{P}_B \cong \mathcal{P}_B \prec \mathcal{P}_A$ since $\mathcal{H}^A \otimes \mathcal{H}^B \cong \mathcal{H}^B \otimes \mathcal{H}^A$. Likewise, the connector is in general not commutative, $\mathcal{P}_A \prec \mathcal{P}_B \neq \mathcal{P}_B \prec \mathcal{P}_A$.

The Prec yields a valid projector on operator system and preserves commutation (Lemma 5.1.8): When introduced in the main text, the one-way signaling set $\mathscr{A} \prec \mathscr{B}$ was obtained the most general kind of maps in CJ representation, that is structure-preserving map $\overline{\mathscr{A}} \rightarrow \mathscr{B}$ characterized by projector $\overline{\mathcal{P}}_A \rightarrow \mathcal{P}_B$ but restricted to the subset of maps that obey the no-signaling from B to A condition, obtained as projector $\mathcal{I}_A \otimes \mathcal{P}_B$ from Lemma 3.5.3. In terms of the underlying operator system, the support of $\mathscr{A} \prec \mathscr{B}$ is obtained as the intersection of the following two projectors, Equation (3.104):

$$\mathcal{P}_A \prec \mathcal{P}_B \equiv \left(\overline{\mathcal{P}}_A \rightarrow \mathcal{P}_B\right) \cap \left(\mathcal{I}_A \otimes \mathcal{P}_B\right) \, . \tag{D.87}$$

Because $\mathcal{I}_A, \mathcal{P}_A$, and $\mathcal{P}_B$ are valid projectors on operators systems, meaning they obey Equations (3.20a), and because operations $\overline{\cdot}$, $\otimes$ and $\cap$ all preserve this property as proven in the last sections, $\mathcal{P}_A \prec \mathcal{P}_B$ must obey (3.20a) by construction so its utilization yields valid projectors on operator systems in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$.

For the same reason as above, if $\mathcal{P}_A \cap \mathcal{P}_A^{'} = \mathcal{P}_A^{'} \cap \mathcal{P}_A$ and $\mathcal{P}_B \cap \mathcal{P}_B^{'} = \mathcal{P}_B^{'} \cap \mathcal{P}_B$, then the A-to-B composition preserves these commutations:

$$\left(\mathcal{P}_A \prec \mathcal{P}_B\right) \cap \left(\mathcal{P}_A^{'} \prec \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A^{'} \prec \mathcal{P}_B^{'}\right) \cap \left(\mathcal{P}_A \prec \mathcal{P}_B\right) \, . \tag{D.88}$$

Similarly, because

$$\mathcal{P}_A \succ \mathcal{P}_B := \mathcal{P}_A \otimes \mathcal{I}_B - \mathcal{D}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B \, , \tag{D.89}$$

is an expression derived from connectives that preserve commutation, the B-to-A one-way composition also preserves commutation:

$$\mathcal{P}_A \succ \mathcal{P}_B = \left(\overline{\mathcal{P}}_A \rightarrow \mathcal{P}_B\right) \cap \left(\mathcal{P}_A \otimes \mathcal{I}_B\right) \, . \tag{D.90}$$

Nevertheless, the A-to-B one-way signaling composition is different from the B-to-A one, so it should also be proven that these two composition commutes with each other:

$$\left(\mathcal{P}_A \prec \mathcal{P}_B\right) \cap \left(\mathcal{P}_A^{'} \succ \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A^{'} \succ \mathcal{P}_B^{'}\right) \cap \left(\mathcal{P}_A \prec \mathcal{P}_B\right) \, , \tag{D.91}$$

Using the above reasoning, the commutation of $\prec$ with $\succ$ is equivalent to proving the commutation of the terms in the square brackets in

$$[\mathcal{P}_A \prec \mathcal{P}_B] \cap \left[\mathcal{P}_A^{'} \succ \mathcal{P}_B^{'}\right] = \left[\left(\overline{\mathcal{P}}_A \to \mathcal{P}_B\right) \cap \left(\mathcal{I}_A \otimes \mathcal{P}_B\right)\right] \cap \left[\left(\overline{\mathcal{P}_A^{'}} \to \mathcal{P}_B^{'}\right) \cap \left(\mathcal{P}_A^{'} \otimes \mathcal{I}_B\right)\right] . \tag{D.92}$$

Actually, from the associativity of the $\cap$ and the commutation of the $\to$ with the $\otimes$, the four terms commute pairwise which is enough to prove that the pairs of terms in square brackets commute. The same way,

$$(\mathcal{P}_A \succ \mathcal{P}_B) \cap \left(\mathcal{P}_A^{'} \succ \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A^{'} \succ \mathcal{P}_B^{'}\right) \cap (\mathcal{P}_A \succ \mathcal{P}_B) . \tag{D.93}$$

Interchange law with the additive connectors: Cap and prec satisfy an interchange law,

$$\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \prec \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A \prec \mathcal{P}_B\right) \cap \left(\mathcal{P}_A^{'} \prec \mathcal{P}_B^{'}\right) . \tag{D.94}$$

This can be shown as follows:

$$\begin{aligned}
\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \prec \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) &= \mathcal{I}_A \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) - \overline{\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right)} \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \\
&\overset{\text{(D.33b)}}{=} \mathcal{I}_A \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) - \left(\overline{\mathcal{P}}_A \cup \overline{\mathcal{P}_A^{'}}\right) \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \\
= \mathcal{I}_A \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B &- \overline{\mathcal{P}_A^{'}} \otimes \mathcal{D}_B + \left(\overline{\mathcal{P}}_A \cap \overline{\mathcal{P}_A^{'}}\right) \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \\
= \left(\mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) &\cap \left(\mathcal{I}_A \otimes \mathcal{P}_B^{'} - \overline{\mathcal{P}_A^{'}} \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) \\
&= \left(\mathcal{P}_A \prec \mathcal{P}_B\right) \cap \left(\mathcal{P}_A^{'} \prec \mathcal{P}_B^{'}\right) .
\end{aligned} \tag{D.95}$$

Remark that the grouping at the penultimate line is arbitrary, implying that:

$$\begin{aligned}
\left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \prec \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) &= \left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \prec \left(\mathcal{P}_B^{'} \cap \mathcal{P}_B\right) \\
&= \left(\mathcal{P}_A \cap \mathcal{P}_A^{'}\right) \prec \left(\mathcal{P}_B \cap \mathcal{P}_B^{'}\right) \\
&= \left(\mathcal{P}_A \prec \mathcal{P}_B^{'}\right) \cap \left(\mathcal{P}_A^{'} \prec \mathcal{P}_B\right) .
\end{aligned} \tag{D.96}$$

Therefore, the exact grouping on the right-hand side of Equation (D.94) does not matter as long as the A's and B's are on the correct side of the prec connector. This property, due to the commutativity of the additive connectors, is what causes the non-uniqueness of the normal form presented in Section 5.2. This non-uniqueness is also discussed in Chapter 6.

The cup and the prec satisfy an interchange law as well,

$$\left(\mathcal{P}_A \cup \mathcal{P}_A^{'}\right) \prec \left(\mathcal{P}_B \cup \mathcal{P}_B^{'}\right) = \left(\mathcal{P}_A \prec \mathcal{P}_B\right) \cup \left(\mathcal{P}_A^{'} \prec \mathcal{P}_B^{'}\right) . \tag{D.97}$$

This can be shown in the same way,

$$\begin{aligned}
&\left(\mathcal{P}_A \prec \mathcal{P}_B\right) \cup \left(\mathcal{P}_A' \prec \mathcal{P}_B'\right) = \\
&\left(\mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) \cup \left(\mathcal{I}_A \otimes \mathcal{P}_B' - \overline{\mathcal{P}_A'} \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) \\
&= \left(\mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) + \left(\mathcal{I}_A \otimes \mathcal{P}_B' - \overline{\mathcal{P}_A'} \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) \\
&- \mathcal{I}_A \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B'\right) + \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \overline{\mathcal{P}_A'} \otimes \mathcal{D}_B - \left(\overline{\mathcal{P}}_A \cap \overline{\mathcal{P}_A'}\right) \otimes \mathcal{D}_B - \mathcal{D}_A \otimes \mathcal{D}_B \\
&= \mathcal{I}_A \otimes \mathcal{P}_B + \mathcal{I}_A \otimes \mathcal{P}_B' - \mathcal{I}_A \otimes \left(\mathcal{P}_B \cap \mathcal{P}_B'\right) - \left(\overline{\mathcal{P}}_A \cap \overline{\mathcal{P}_A'}\right) \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \\
&= \mathcal{I}_A \otimes \left(\mathcal{P}_B \cup \mathcal{P}_B'\right) - \overline{\left(\mathcal{P}_A \cup \mathcal{P}_A'\right)} \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \\
&= \left(\mathcal{P}_A \cup \mathcal{P}_A'\right) \prec \left(\mathcal{P}_B \cup \mathcal{P}_B'\right) .
\end{aligned} \tag{D.98}$$

These two interchange laws imply distributivity as a special case because the elements in the algebra are all additively idempotent ($\mathcal{P} = \mathcal{P} \cap \mathcal{P} = \mathcal{P} \cup \mathcal{P}$) so the following can be written: $\left(\mathcal{P}_A \cup \mathcal{P}_A'\right) \prec \mathcal{P}_B = \left(\mathcal{P}_A \cup \mathcal{P}_A'\right) \prec \left(\mathcal{P}_B \cup \mathcal{P}_B\right) = \left(\mathcal{P}_A \prec \mathcal{P}_B\right) \cup \left(\mathcal{P}_A' \prec \mathcal{P}_B\right)$. As a consequence, the following relations hold:

$$(\mathcal{P}_A \cap \mathcal{P}_A') \prec \mathcal{P}_B = (\mathcal{P}_A \prec \mathcal{P}_B) \cap (\mathcal{P}_A' \prec \mathcal{P}_B)\,, \tag{D.99a}$$
$$(\mathcal{P}_A \cup \mathcal{P}_A') \prec \mathcal{P}_B = (\mathcal{P}_A \prec \mathcal{P}_B) \cup (\mathcal{P}_A' \prec \mathcal{P}_B)\,, \tag{D.99b}$$
$$\mathcal{P}_A \prec (\mathcal{P}_B \cup \mathcal{P}_B') = (\mathcal{P}_A \prec \mathcal{P}_B) \cup (\mathcal{P}_A \prec \mathcal{P}_B')\,, \tag{D.99c}$$
$$\mathcal{P}_A \prec (\mathcal{P}_B \cap \mathcal{P}_B') = (\mathcal{P}_A \prec \mathcal{P}_B) \cap (\mathcal{P}_A \prec \mathcal{P}_B')\,. \tag{D.99d}$$

Properties appearing in Proposition 5.1.9: The distributivity of the negation is first proven by direct computation,

$$\begin{aligned}
&\overline{\mathcal{P}_A \prec \mathcal{P}_B} = \\
&\mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{P}_A \prec \mathcal{P}_B + \mathcal{D}_A \otimes \mathcal{D}_B = \mathcal{I}_A \otimes \mathcal{I}_B - (\mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B) + \mathcal{D}_A \otimes \mathcal{D}_B = \\
&\left(\mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{I}_A \otimes \mathcal{P}_B + \mathcal{I}_A \otimes \mathcal{D}_B\right) - \mathcal{P}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B = \left(\mathcal{I}_A \otimes \overline{\mathcal{P}}_B\right) - \overline{\overline{\mathcal{P}}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \\
&= \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B .
\end{aligned} \tag{D.100}$$

The prec has thus a simpler connection with the negation than the no-signaling (the tensor) and the two-way signaling (the parr) compositions as it directly commutes with the negation instead of obeying a De Morgan law,

$$\overline{\mathcal{P}_A \prec \mathcal{P}_B} = \overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B\,. \tag{D.101}$$

One can link one-way signaling $\prec$, and no-signaling $\otimes$ compositions by noticing that

$$(\mathcal{P}_A \prec \mathcal{P}_B) \cap (\mathcal{P}_A \succ \mathcal{P}_B) = \mathcal{P}_A \otimes \mathcal{P}_B\,, \tag{D.102}$$

This equation is proven by developing it, $(\mathcal{P}_A \prec \mathcal{P}_B) \cap (\mathcal{P}_A \succ \mathcal{P}_B) = \left(\mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) \cap \left(\mathcal{P}_A \otimes \mathcal{I}_B - \mathcal{D}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right)$, and noting that the intersection of any two elements but $\left(\mathcal{I}_A \otimes \mathcal{P}_B\right) \cap \left(\mathcal{P}_A \otimes \mathcal{I}_B\right) = \mathcal{P}_A \otimes \mathcal{P}_B$ is giving $\mathcal{D}_A \otimes \mathcal{D}_B$. Thus, the expression reduces to $\mathcal{P}_A \otimes \mathcal{P}_B$ followed by eight occurrences of $\mathcal{D}_A \otimes \mathcal{D}_B$ alternating between a plus and minus sign, therefore canceling each other.

Compared to transformation, one-way signaling composition is also better behaved in the sense that it is associative

$$\mathcal{P}_A \prec (\mathcal{P}_B \prec \mathcal{P}_C) = (\mathcal{P}_A \prec \mathcal{P}_B) \prec \mathcal{P}_C \,. \tag{D.103}$$

So that $\mathcal{P}_A \prec (\mathcal{P}_B \prec \mathcal{P}_C) = \mathcal{P}_A \prec \mathcal{P}_B \prec \mathcal{P}_C$ can be written unambiguously. The distributivity of the negation over the prec is used in the proof:

$$\begin{aligned}
\mathcal{P}_A \prec (\mathcal{P}_B \prec \mathcal{P}_C) &= \mathcal{I}_A \otimes (\mathcal{P}_B \prec \mathcal{P}_C) - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B \otimes \mathcal{D}_C + \mathcal{D}_A \otimes \mathcal{D}_B \otimes \mathcal{D}_C \\
&= \mathcal{I}_A \otimes \mathcal{I}_B \otimes \mathcal{P}_C - \mathcal{I}_A \otimes \overline{\mathcal{P}}_B \otimes \mathcal{D}_C + \mathcal{I}_A \otimes \mathcal{D}_B \otimes \mathcal{D}_C - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B \otimes \mathcal{D}_C + \mathcal{D}_A \otimes \mathcal{D}_B \otimes \mathcal{D}_C \\
&= \left(\mathcal{I}_A \otimes \mathcal{I}_B\right) \otimes \mathcal{P}_C - \left(\mathcal{I}_A \otimes \overline{\mathcal{P}}_B - \mathcal{P}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B\right) \otimes \mathcal{D}_C + \mathcal{D}_A \otimes \mathcal{D}_B \otimes \mathcal{D}_C \\
&= \left(\mathcal{I}_A \otimes \mathcal{I}_B\right) \otimes \mathcal{P}_C - \left(\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B\right) \otimes \mathcal{D}_C + \mathcal{D}_A \otimes \mathcal{D}_B \otimes \mathcal{D}_C \\
&= \left(\mathcal{I}_A \otimes \mathcal{I}_B\right) \otimes \mathcal{P}_C - \overline{\left(\mathcal{P}_A \prec \mathcal{P}_B\right)} \otimes \mathcal{D}_C + (\mathcal{D}_A \otimes \mathcal{D}_B) \otimes \mathcal{D}_C = \left(\mathcal{P}_A \prec \mathcal{P}_B\right) \prec \mathcal{P}_C \,,
\end{aligned} \tag{D.104}$$

where the definition has been used to go to the last line, as well as the commutation of the prec with the negation to go to the penultimate one.

Finally, the preservation of the partial order is direct from the interchange law (D.99) proven above. Let $\mathcal{P}_A \cap \mathcal{P}'_A = \mathcal{P}_A \iff \mathcal{P}_A \subseteq \mathcal{P}'_A$ and $\mathcal{P}_B \cap \mathcal{P}'_B = \mathcal{P}_B \iff \mathcal{P}_B \subseteq \mathcal{P}'_B$, then

$$(\mathcal{P}_A \prec \mathcal{P}_B) \cap \left(\mathcal{P}'_A \prec \mathcal{P}'_B\right) \overset{(D.99)}{=} \left(\mathcal{P}_A \cap \mathcal{P}'_A\right) \prec \left(\mathcal{P}_B \cap \mathcal{P}'_B\right) = \mathcal{P}'_A \prec \mathcal{P}'_B \,, \tag{D.105}$$

which implies that

$$(\mathcal{P}_A \prec \mathcal{P}_B) \subseteq \left(\mathcal{P}'_A \prec \mathcal{P}'_B\right) \,. \tag{D.106}$$

Relation with the parr and inclusion relations with between multiplicative connectors: The parr is linked with the prec in a similar fashion as the prec is linked with the tensor:

$$(\mathcal{P}_A \prec \mathcal{P}_B) \cup (\mathcal{P}_A \succ \mathcal{P}_B) = \mathcal{P}_A \parr \mathcal{P}_B \,. \tag{D.107}$$

This equation has a quick proof using the De Morgan rule and the distributivity of the negation:

$$\begin{aligned}
(\mathcal{P}_A \prec \mathcal{P}_B) \cup (\mathcal{P}_A \succ \mathcal{P}_B) &\overset{(D.32)}{=} \overline{\overline{(\mathcal{P}_A \prec \mathcal{P}_B) \cup (\mathcal{P}_A \succ \mathcal{P}_B)}} \overset{(D.33a)}{=} \overline{\overline{(\mathcal{P}_A \prec \mathcal{P}_B)} \cap \overline{(\mathcal{P}_A \succ \mathcal{P}_B)}} \\
&\overset{(D.101)}{=} \overline{\left(\overline{\mathcal{P}}_A \prec \overline{\mathcal{P}}_B\right) \cap \left(\overline{\mathcal{P}}_A \succ \overline{\mathcal{P}}_B\right)} \overset{(D.102)}{=} \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} = \mathcal{P}_A \parr \mathcal{P}_B \,.
\end{aligned} \tag{D.108}$$

From Equation (D.102), it can also be inferred that

$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \mathcal{P}_A \prec \mathcal{P}_B \,, \tag{D.109a}$$
$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \mathcal{P}_A \succ \mathcal{P}_B \,. \tag{D.109b}$$

It follows from $(\mathcal{P}_A \otimes \mathcal{P}_B) \cap (\mathcal{P}_A \prec \mathcal{P}_B) = (\mathcal{P}_A \prec \mathcal{P}_B) \cap (\mathcal{P}_A \succ \mathcal{P}_B) \cap (\mathcal{P}_A \prec \mathcal{P}_B) = (\mathcal{P}_A \prec \mathcal{P}_B) \cap (\mathcal{P}_A \succ \mathcal{P}_B) = \mathcal{P}_A \otimes \mathcal{P}_B$, and the second

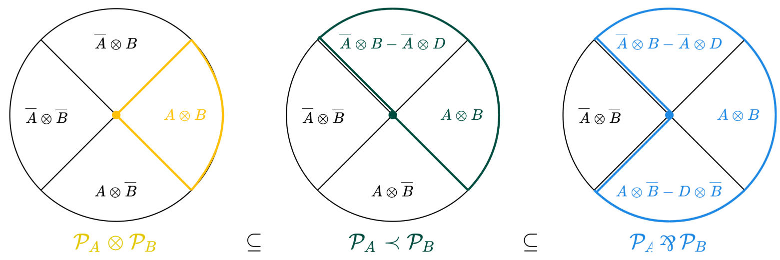


Figure D.3.: Diagrammatic depiction of the image in $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$ of the composition of $\mathcal{P}_A$ with $\mathcal{P}_B$ obtained by the tensor $\otimes$, prec $\prec$, and parr $\parr$ connectors. In the diagram, $A = \mathrm{Im}\{\mathcal{P}_A\}$, $\overline{A} = \mathrm{Im}\{\overline{\mathcal{P}}_A\}$, and the same holds for B, whereas $D = \mathcal{D}_A$ or $= \mathcal{D}_B$ depending on, respectively, whether it is on the left or on the right of the tensor product.

equation is proven analogously. The same way, from Equation (D.107),

$$\mathcal{P}_A \prec \mathcal{P}_B \subseteq \mathcal{P}_A \parr \mathcal{P}_B \,, \tag{D.110a}$$
$$\mathcal{P}_A \succ \mathcal{P}_B \subseteq \mathcal{P}_A \parr \mathcal{P}_B \,. \tag{D.110b}$$

Putting these relations together yields Equations (5.41) in the main text,

$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \mathcal{P}_A \prec \mathcal{P}_B \subseteq \mathcal{P}_A \parr \mathcal{P}_B \,, \tag{D.111a}$$
$$\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \mathcal{P}_A \succ \mathcal{P}_B \subseteq \mathcal{P}_A \parr \mathcal{P}_B \,. \tag{D.111b}$$

The first line is diagrammatically depicted in Figure D.3: the subspace associated with $\overline{\mathcal{P}}_A \to \mathcal{P}_B = \mathcal{P}_A \parr \mathcal{P}_B$ is represented in blue on the right: its definition as $\mathcal{P}_A \parr \mathcal{P}_B = \mathcal{I}_A \otimes \mathcal{I}_B - \overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B$ appears more clearly: the support is the full disc (the image of $\mathcal{I}_A \otimes \mathcal{I}_B$) minus the left quadrant (the image of $\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B$) except the central dot (the image of $\mathcal{D}_A \otimes \mathcal{D}_B$).

The intermediate case, associated with $\mathcal{P}_A \prec \mathcal{P}_B$ is depicted in green on the center; from Equation (3.104), $\mathcal{P}_A \prec \mathcal{P}_B = \left(\overline{\mathcal{P}}_A \to \mathcal{P}_B\right) \cap \left(\mathcal{I}_A \otimes \mathcal{P}_B\right)$, it is obtained as the intersection of the blue part, $\overline{\mathcal{P}}_A \to \mathcal{P}_B$, with $\mathcal{I}_A \otimes \mathcal{P}_B = \left(\mathcal{P}_A \otimes \mathcal{P}_B\right) \cup \left(\overline{\mathcal{P}}_A \otimes \mathcal{P}_B\right)$ which correspond to the right and top quadrants. It can also be constructed from the definition, $\mathcal{P}_A \prec \mathcal{P}_B := \mathcal{I}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B$: the green part made of the right and top quadrants ($\mathcal{I}_A \otimes \mathcal{P}_B$) minus the line separating the left and top quadrants ($\overline{\mathcal{P}}_A \otimes \mathcal{D}_B$) except at the central dot ($\mathcal{D}_A \otimes \mathcal{D}_B$).

The smallest case, associated with $\mathcal{P}_A \otimes \mathcal{P}_B$ is depicted in yellow on the left; as discussed in Proposition 3.5.5 it is effectively defined by the intersection of two no-signaling subspaces that can be defined through Lemma 3.5.3, $\left(\mathcal{I}_A \otimes \mathcal{P}_B\right) \cap \left(\mathcal{P}_A \otimes \mathcal{I}_B\right)$, that is the intersection of the right and top quadrants ($\mathcal{I}_A \otimes \mathcal{P}_B$) with the right and bottom ones ($\mathcal{P}_A \otimes \mathcal{I}_B$), yielding the right quadrant. Remark that the intersection with $\overline{\mathcal{P}}_A \to \mathcal{P}_B$ brings no new constraints, this is the main content of the theorem.

D.4. Accidental Isomorphism in the Case of Quantum Theory and the Proof of Lemma 5.3.1

Observe that the only time Equation (5.27) is thigh, i.e. when identity $\mathcal{P}_A \otimes \mathcal{P}_B = \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B}$ holds, is when the projector is either characterizing the totality of the space or only the span of identity. In equation,

$$\mathcal{P}_A \otimes \mathcal{P}_B = \overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} \iff \mathcal{P}_A = \mathcal{P}_B = \mathcal{I} \text{ or } \mathcal{D} \,. \tag{D.112}$$

This is proven by rewriting $\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B}$ into

$$\begin{aligned}\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} = \mathcal{P}_A \otimes \mathcal{P}_B + &\\ \big(\overline{\mathcal{P}}_A \otimes \mathcal{P}_B - \overline{\mathcal{P}}_A \otimes \mathcal{D}_B - \mathcal{D}_A \otimes \mathcal{P}_B + \mathcal{D}_A \otimes \mathcal{D}_B\big) + &\\ \big(\mathcal{P}_A \otimes \overline{\mathcal{P}}_B - \mathcal{P}_A \otimes \mathcal{D}_B - \mathcal{D}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B\big) \,, &\end{aligned} \tag{D.113}$$

using the definition of negation and algebraic properties. It can then be understood from Figure D.3: the first term of the above is the right quarter of the wheel, the second is the top quarter with its boundary removed, and the third is the bottom also without boundaries. As the three parts share no intersection, the regular addition '+' is equivalent to a conjunction '∪'. Next, more algebraic manipulations lead to

$$\begin{aligned}\overline{\overline{\mathcal{P}}_A \otimes \overline{\mathcal{P}}_B} = \mathcal{P}_A \otimes \mathcal{P}_B + &\\ \big(\overline{\mathcal{P}}_A \otimes \mathcal{P}_B + \mathcal{P}_A \otimes \overline{\mathcal{P}}_B - \mathcal{I}_A \otimes \mathcal{D}_B - \mathcal{D}_A \otimes \mathcal{I}_B\big) \,, &\end{aligned} \tag{D.114}$$

and from this expression, it is direct to check that the term in parenthesis vanishes if and only if either of the conditions in Eq. (D.112) hold.

This has consequences on the set inclusions (D.111), appearing in the main text as Equations (5.41). The situation is more symmetric when these are rephrased as transformations by negating one of the two systems, in that case A:

$$\overline{\mathcal{P}}_A \otimes \mathcal{P}_B \subseteq \overline{\mathcal{P}}_A \prec \mathcal{P}_B \subseteq \mathcal{P}_A \to \mathcal{P}_B \,, \tag{D.115a}$$
$$\overline{\mathcal{P}}_A \otimes \mathcal{P}_B \subseteq \overline{\mathcal{P}}_A \succ \mathcal{P}_B \subseteq \mathcal{P}_A \to \mathcal{P}_B \,. \tag{D.115b}$$

The inclusions become equivalences in the special cases where the projectors are either identity or depolarizing. These imply set isomorphisms that are not without consequences: subsets with different signaling constraints get accidentally equivalent.

Putting an identity on the right side of the $\to$ gives

$$\mathcal{P}_A \to \mathcal{I}_B = \overline{\mathcal{P}_A \otimes \mathcal{D}_B} \,, \tag{D.116}$$

which happens to be equivalent to

$$\overline{\mathcal{P}}_A \prec \mathcal{I}_B = \mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{P}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \,. \tag{D.117}$$

As it can be shown directly from the definitions. In that case, the two-way and one-way signaling compositions coincide. The same way, putting one on the left side gives

$$\begin{aligned}\mathcal{I}_A \to \mathcal{P}_B &= \overline{\mathcal{I}_A \otimes \overline{\mathcal{P}}_B}\\ &= \mathcal{I}_A \otimes \mathcal{I}_B - \mathcal{I}_A \otimes \overline{\mathcal{P}}_B + \mathcal{D}_A \otimes \mathcal{D}_B\\ &= \mathcal{I}_A \otimes \mathcal{P}_B - \mathcal{I}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \,,\end{aligned} \tag{D.118}$$

which is equivalent to

$$\mathcal{D}_A \prec \mathcal{P}_B = \mathcal{I}_A \otimes \mathcal{P}_B - \mathcal{I}_A \otimes \mathcal{D}_B + \mathcal{D}_A \otimes \mathcal{D}_B \, . \tag{D.119}$$

One has then the following identities:

$$\overline{\mathcal{P}}_A \prec \mathcal{I}_B = \mathcal{P}_A \to \mathcal{I}_B \, ; \tag{D.120a}$$
$$\mathcal{D}_A \prec \mathcal{P}_B = \mathcal{I}_A \to \mathcal{P}_B \, . \tag{D.120b}$$

These are the only two cases for which the transformation is equivalent to the prec as the following rewriting shows:

$$\mathcal{P}_A \to \mathcal{P}_B = \overline{\mathcal{P}}_A \prec \mathcal{P}_B + (\mathcal{I}_A - \mathcal{P}_A) \otimes (\mathcal{I}_B - \mathcal{P}_B) \, . \tag{D.121}$$

These relations can be concisely recast as

$$\overline{\mathcal{P}}_A \prec \mathcal{P}_B = \mathcal{P}_A \to \mathcal{P}_B \iff \mathcal{P}_A = \mathcal{I}_A \text{ or } \mathcal{P}_B = \mathcal{I}_B \, . \tag{D.122}$$

In addition to that, one-way signaling composition is equivalent to the no signaling composition in the following cases:

$$\mathcal{P}_A \prec \mathcal{D}_B = \mathcal{P}_A \otimes \mathcal{D}_B \, ; \tag{D.123a}$$
$$\mathcal{I}_A \prec \mathcal{P}_B = \mathcal{I}_A \otimes \mathcal{P}_B \, . \tag{D.123b}$$

Again, this directly follows from the definition by re-expressing it as

$$\overline{\mathcal{P}}_A \prec \mathcal{P}_B = \overline{\mathcal{P}}_A \otimes \mathcal{P}_B + (\mathcal{P}_A - \mathcal{D}_A) \otimes (\mathcal{P}_B - \mathcal{D}_B) \, , \tag{D.124}$$

and this can be concisely recast as

$$\overline{\mathcal{P}}_A \prec \mathcal{P}_B = \mathcal{P}_A \otimes \mathcal{P}_B \iff \mathcal{P}_A = \mathcal{D}_A \text{ or } \mathcal{P}_B = \mathcal{D}_B \, . \tag{D.125}$$

It should be noted that condition (D.125) is stronger than (D.122). Actually, when both conditions are satisfied at once, one has either of the two identities:

$$\mathcal{I}_A \to \mathcal{D}_B = \mathcal{D}_A \prec \mathcal{D}_B = \mathcal{D}_A \otimes \mathcal{D}_B \, ; \tag{D.126a}$$
$$\mathcal{D}_A \to \mathcal{I}_A = \mathcal{I}_A \prec \mathcal{I}_B = \mathcal{I}_A \otimes \mathcal{I}_B \, . \tag{D.126b}$$

The reason this is the case comes from isomorphism (D.112), which reduces the transformation into a no-signaling composition. Indeed,

$$\mathcal{D}_A \prec \mathcal{D}_B \overset{\text{(D.122)}}{=} \mathcal{I}_A \to \mathcal{D}_B = \overline{\mathcal{I}_A \otimes \overline{\mathcal{D}_B}} = \overline{\mathcal{I}_A \otimes \mathcal{I}_B} \overset{\text{(D.112)}}{=} \mathcal{D}_A \otimes \mathcal{D}_B \, . \tag{D.127}$$

And the same way,

$$\mathcal{I}_A \prec \mathcal{I}_B \overset{\text{(D.122)}}{=} \mathcal{D}_A \to \mathcal{I}_B = \overline{\mathcal{D}_A \otimes \overline{\mathcal{I}_B}} = \overline{\mathcal{D}_A \otimes \mathcal{D}_B} \overset{\text{(D.112)}}{=} \mathcal{I}_A \otimes \mathcal{I}_B \, . \tag{D.128}$$

Therefore, the isomorphisms (D.112), (D.122), and (D.125) give the conditions for set equivalences in the composition rules.

One can understand these relations using a diagram like Figure D.4. For a transformation $\mathcal{P}_A \to \mathcal{P}_B$ made of arbitrary projectors (in blue), its different substructures are effectively distinct (no-signaling in yellow,

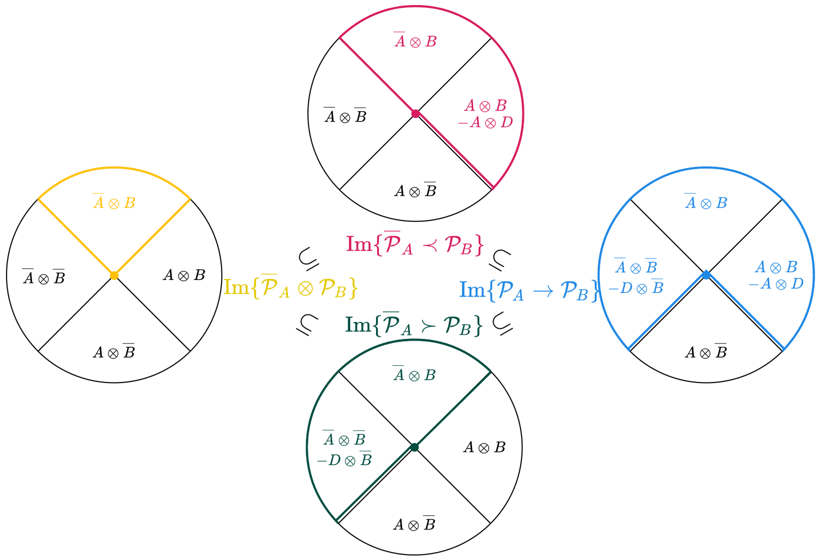


Figure D.4.: Diagrammatic representation of the support of the different projectors representing transformations from A to B with respect to the tensor factors of $\mathcal{L}\left(\mathcal{H}^A \otimes \mathcal{H}^B\right)$. These are the four different ways of composing an effect in $\overline{\mathscr{A}}$ with a state in $\mathscr{B}$. Note that the intersections are well defined; for example, the line '$A \otimes D$' is indeed the intersection $A \otimes B \cap A \otimes \overline{B}$.

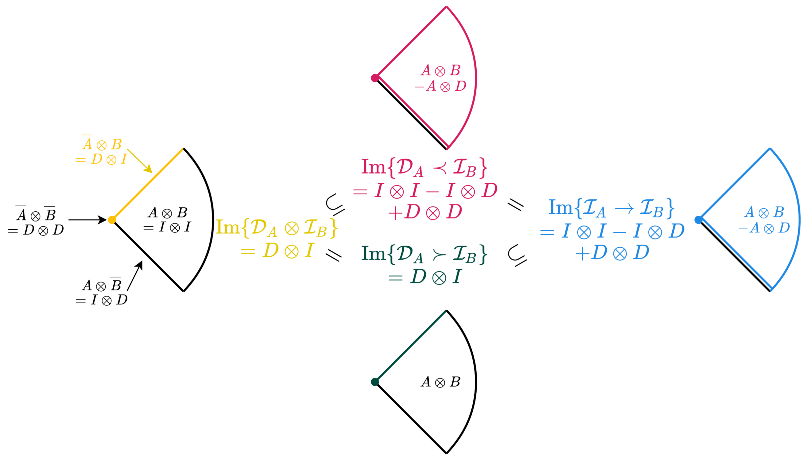


Figure D.5.: Diagrams depicting the subspaces supporting the four different ways of defining a transformation (top). When $\overline{\mathcal{P}}_A = \mathcal{D}_A$ and $\mathcal{P}_B = \mathcal{I}_B$ (bottom), the yellow and green zones are shrunk to the segment $D \otimes B$: Equations akin to (D.123a) and (D.123b) are simultaneously verified so no signaling (yellow) is equivalent to one-way to A (green). At the same time, the pink and blue zones are reduced to $A \otimes B - A \otimes D$: Eqs. (D.120a)and (D.120b) are simultaneously verified so two-way signaling (blue) is equivalent to one-way signaling to B (pink).

A-to-B one-way in pink, and B-to-A one-way in green). Recall that one can infer the inclusion relations from their overlap, e.g. $\overline{\mathcal{P}}_A \otimes \mathcal{P}_B \subsetneq \overline{\mathcal{P}}_A \prec \mathcal{P}_B \subsetneq \mathcal{P}_A \to \mathcal{P}_B$ can be inferred from the fact that the yellow part is contained within the pink one which is contained within the blue one. Another example, $\left(\overline{\mathcal{P}}_A \prec \mathcal{P}_B\right) \cap \left(\overline{\mathcal{P}}_A \succ \mathcal{P}_B\right) = \mathcal{P}_A \to \mathcal{P}_B$ can be inferred from the fact that zone covered by the pink and green areas is equivalent to the blue one.

An accidental isomorphism is present when either the input or the output is a density operator (featuring an identity projector). In the first case, $\mathcal{P}_A = \mathcal{I}_A$ and $\overline{\mathcal{P}}_A = \mathcal{D}_A$, so the area $\overline{A}$ shrinks into a line: the top and left quadrants of the circle are shrunk into the diagonal that goes from bottom left to top right. The yellow zone is thus shrunk into $D \otimes B$, the upper border of $A \otimes B$, and so is the green zone, this equivalence of yellow and green is similar to Equation (D.123a), but with A and B swapped: $\overline{\mathcal{P}}_A = \mathcal{D}_A \Rightarrow \overline{\mathcal{P}}_A \succ \mathcal{P}_B = \mathcal{D}_A \otimes \mathcal{P}_B$. At the same time, the pink zone is shrunk to $A \otimes B - A \otimes D$, i.e. $A \otimes B$ without its bottom border, and so is the blue zone, this equivalence of pink and blue is Equation (D.120b). In the second case, $\mathcal{P}_B = \mathcal{I}_B$ and $\overline{\mathcal{P}}_B = \mathcal{D}_B$, so the area $\overline{B}$ also shrinks to a line: the bottom and left quadrants are shrunk into the diagonal that goes from top left to bottom right. The yellow zone is untouched, but the green one is shrunk into the same zone as $\overline{A} \otimes \overline{B}$ becomes the border $\overline{A} \otimes D$, i.e. the top-left segment. This equivalence when $\mathcal{P}_B = \mathcal{I}_B$ is similar to Equation (D.123b), but with A and B swapped: $\mathcal{P}_B = \mathcal{I}_B \Rightarrow \overline{\mathcal{P}}_A \succ \mathcal{P}_B = \overline{\mathcal{P}}_A \otimes \mathcal{I}_B$. As for the green zone, it is untouched as well, but the blue zone also sees its left quadrant shrunk into the top-left segment, so the green and blue zone becomes equivalent. This is Equation (D.120a).

When $\overline{\mathcal{P}}_A = \mathcal{D}_A$ and $\mathcal{P}_B = \mathcal{I}_B$ (Figure D.5), the yellow and green zones are shrunk to the segment $D \otimes B$. The versions of Equations (D.123a) and (D.123b) where A and B have swapped roles are simultaneously verified, so the diagram depicting the no signaling transformation (yellow) is equivalent to the one depicting the transformation which is one-way signaling to A (green). At the same time, the pink and blue zones are reduced to $A \otimes B - A \otimes D$. Equations (D.120a) and (D.120b) are simultaneously verified so the diagram depicting the two-way signaling transformation (blue) is equivalent to the one depicting the transformation, which is one-way signaling to B (pink).

D.5. Proofs

D.5.1. Proof of Lemma 5.1.6

Let $\mathcal{P}^{(n)}$ be the projector obtained after n 'steps' that can be categorized as 1) do a global negation of the projector, $\mathcal{P}^{(n)} = \overline{\mathcal{P}}^{(n-1)}$; 2) add a base projector on the right using tensor product, $\mathcal{P}^{(n)} = \mathcal{P}^{(n-1)} \otimes \mathcal{P}_X$; 3) add a similarly obtained projector after k steps, $\mathcal{P}^{(k)}$, on the right using tensor product, $\mathcal{P}^{(n)} = \mathcal{P}^{(n-1)} \otimes \mathcal{P}^{(k)}$. This covers all cases as the $\rightarrow$ can be split into a negation and a tensor, $\cdot \rightarrow \cdot \equiv \overline{\cdot \otimes \overline{\cdot}}$, and one can always redefine the tensor factors labeling so that the added system is on the right since $\mathcal{H}^A \otimes \mathcal{H}^B \cong \mathcal{H}^B \otimes \mathcal{H}^A$.

The only non-trivial first step is to choose a base projector, say $\mathcal{P}_A$, in which case the claim trivially holds, $\mathcal{P}_A \subseteq \mathcal{P}_A \subseteq \overline{\overline{\mathcal{P}}}_A$. Suppose it holds after $n-1$ steps during which j base projectors were added, then

$$\widetilde{\mathcal{P}}_A \otimes \ldots \otimes \widetilde{\mathcal{P}}_J \subseteq \mathcal{P}^{(n-1)} \subseteq \overline{\overline{\overline{\widetilde{\mathcal{P}}_A} \otimes \ldots \otimes \overline{\widetilde{\mathcal{P}}_J}}} \,. \tag{D.129}$$

Let $\widetilde{\mathcal{P}}_A \otimes \ldots \otimes \widetilde{\mathcal{P}}_J \equiv \mathcal{P}_{NS}^{(n-1)}$ and $\overline{\overline{\overline{\widetilde{\mathcal{P}}_A} \otimes \ldots \otimes \overline{\widetilde{\mathcal{P}}_J}}} \equiv \mathcal{P}_{FS}^{(n-1)}$. Then, $\mathcal{P}_{NS}^{(n-1)} \subseteq \mathcal{P}^{(n-1)} \subseteq \mathcal{P}_{FS}^{(n-1)}$. The following holds because of Equaitons (5.16)

$$\overline{\mathcal{P}}_{NS}^{(n-1)} \supseteq \overline{\mathcal{P}}^{(n-1)} \supseteq \overline{\mathcal{P}}_{FS}^{(n-1)} \,, \tag{D.130}$$

This corresponds to doing a step of category 1), $\mathcal{P}^{(n)} = \overline{\mathcal{P}}^{(n-1)}$, in which case $\overline{\mathcal{P}}_{NS}^{(n-1)} = \overline{\widetilde{\mathcal{P}}_A \otimes \ldots \otimes \widetilde{\mathcal{P}}_J}$. A negation can be put over every single projector by redefining the tilde, $\widetilde{\mathcal{P}}_A \mapsto \overline{\widetilde{\mathcal{P}}_A}$, which implies that $\overline{\mathcal{P}}_{NS}^{(n-1)}$ is redefined as $\overline{\mathcal{P}}_{NS}^{(n-1)} = \mathcal{P}_{FS}^{(n)} = \overline{\overline{\widetilde{\mathcal{P}}_A} \otimes \ldots \otimes \overline{\widetilde{\mathcal{P}}_J}}$. The same can be done on $\overline{\mathcal{P}}_{FS}^{(n-1)}$, which proves the induction for 1).

Let $\mathcal{P}^{'}$ be an arbitrary projector on operator system, the following holds by (D.54) and (5.27) proven in subsection D.3.1:

$$\mathcal{P}_{NS}^{(n-1)} \otimes \mathcal{P}^{'} \subseteq \mathcal{P}^{(n-1)} \otimes \mathcal{P}^{'} \subseteq \mathcal{P}_{FS}^{(n-1)} \otimes \mathcal{P}^{'} \,, \tag{D.131a}$$

$$\mathcal{P}_{FS}^{(n-1)} \otimes \mathcal{P}^{'} \subseteq \overline{\overline{\mathcal{P}}_{FS}^{(n-1)} \otimes \overline{\mathcal{P}}^{'}} \,, \tag{D.131b}$$

$$\mathcal{P}_{NS}^{(n-1)} \otimes \mathcal{P}^{'} \subseteq \mathcal{P}^{(n-1)} \otimes \mathcal{P}^{'} \subseteq \overline{\overline{\mathcal{P}}_{FS}^{(n-1)} \otimes \overline{\mathcal{P}}^{'}} \,. \tag{D.131c}$$

Since $\mathcal{P}^{'}$ is arbitrary, the first equation corresponds to either doing step 2) or 3). For case 2), $\mathcal{P}^{'} \equiv \mathcal{P}_L$ is the (j+1)-th subsystem added, corresponding to some party L, so that $\mathcal{P}^{(n-1)} \otimes \mathcal{P}_L = \mathcal{P}^{(n)}$. The third equation then reads $\mathcal{P}_{NS}^{(n-1)} \otimes \mathcal{P}_L \subseteq \mathcal{P}^{(n)} \subseteq \overline{\overline{\mathcal{P}}_{FS}^{(n-1)} \otimes \overline{\mathcal{P}}^{'}}$. Set $\mathcal{P}_L \equiv \widetilde{\mathcal{P}}_L$ and the induction is proven. The reasoning is analog in case 3). Note that the added system, $\mathcal{P}^{'} \equiv \mathcal{P}^{(k)}$, is also included in some $\mathcal{P}_{NS}^{(k)} \subseteq \mathcal{P}^{(k)} \subseteq \mathcal{P}_{FS}^{(k)}$ by assumption. Using Equation D.54 again, one has $\mathcal{P}_{NS}^{(n-1)} \otimes \mathcal{P}_{NS}^{(k)} \subseteq \mathcal{P}^{(n-1)} \otimes \mathcal{P}^{(k)} \subseteq \overline{\overline{\mathcal{P}}_{FS}^{(n-1)} \otimes \overline{\mathcal{P}}_{FS}^{(k)}}$. As $\mathcal{P}^{(n)} = \mathcal{P}^{(n-1)} \otimes \mathcal{P}^{(k)}$, the induction is proven by using the fact that the tensor as well as $\overline{\cdot} \to \cdot$ are associative operations to extend the expressions on both side of $\mathcal{P}^{(n-1)} \otimes \mathcal{P}^{(k)}$ and then to define the $\widetilde{\mathcal{P}}$'s according to the sought expression.

D.5.2. Proof of Proposition 5.1.11

This is essentially proven the same way as Lemma 5.1.6, see subsection D.5.1 above, but with two new categories of 'steps': 4) $\mathcal{P}^{(n)} = \mathcal{P}^{(n-1)} \prec \mathcal{P}_L$ and 5) $\mathcal{P}^{(n)} = \mathcal{P}^{(n-1)} \prec \mathcal{P}^{(k)}$. Reducing these steps into a chain of inclusion like $\mathcal{P}_{NS}^{(n-1)} \otimes \mathcal{P}^{'} \subseteq \mathcal{P}^{(n-1)} \prec \mathcal{P}^{'} \subseteq \overline{\overline{\mathcal{P}}_{NS}^{(n-1)} \otimes \overline{\mathcal{P}}^{'}}$ is again obtained by first noticing that $\mathcal{P}_A \subseteq \mathcal{P}_A^{'} \subseteq \mathcal{P}_A^{''} \Rightarrow \mathcal{P}_A \prec \mathcal{P}_B \subseteq \mathcal{P}_A^{'} \prec \mathcal{P}_B \subseteq \mathcal{P}_A^{''} \prec \mathcal{P}_B$ and then by using relations (5.41) so that $\mathcal{P}_A \otimes \mathcal{P}_B \subseteq \mathcal{P}_A \prec \mathcal{P}_B$ and $\mathcal{P}_A^{''} \prec \mathcal{P}_B \subseteq \overline{\overline{\mathcal{P}}_A^{''} \otimes \overline{\mathcal{P}}_B}$.

D.5.3. Proof of Theorem 5.2.1

Let

$$\Gamma = \bigcup_{i=1}^{x} \left(\bigcap_{j=1}^{y_i} \widetilde{\mathcal{P}}_{\sigma_{ij}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\sigma_{ij}(K)} \right) \tag{D.132}$$

and

$$\Xi = \bigcup_{m=1}^{z} \left(\bigcap_{n=1}^{t_m} \widetilde{\mathcal{P}}_{\sigma_{mn}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\sigma_{mn}(K)} \right) \tag{D.133}$$

be two normal forms involving k base projectors. Define the shorthand notation: $\Gamma_i := \left(\bigcap_{j=1}^{y_i} \widetilde{\mathcal{P}}_{\sigma_{ij}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\sigma_{ij}(K)} \right)$ so that

$$\Gamma = \bigcup_{i=1}^{x} \Gamma_i = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_x \,, \tag{D.134}$$

and $\Gamma_{ij} \equiv \widetilde{\mathcal{P}}_{\sigma_{ij}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\sigma_{ij}(K)}$ so that $\Gamma_i = \left(\bigcap_{j=1}^{y_i} \Gamma_{ij}\right)$ and

$$\Gamma = \bigcup_{i=1}^{x} \left(\bigcap_{j=1}^{y_i} \Gamma_{ij}\right) . \tag{D.135}$$

Hence,

$$\Gamma = \bigcup_{i=1}^{x} (\Gamma_{i1} \cap \Gamma_{i2} \cap \ldots \cap \Gamma_{iy_i}) = \left(\bigcap_{j=1}^{y_1} \Gamma_{1j}\right) \cup \left(\bigcap_{j=1}^{y_2} \Gamma_{2j}\right) \cup \ldots \cup \left(\bigcap_{j=1}^{y_x} \Gamma_{xj}\right) ; \tag{D.136}$$

and Ξ_m and Ξ_{mn} are defined accordingly.

First, remark that the negation, intersection, and union of normal forms can be put into normal forms: $\Gamma \cup \Xi$ is a normal form obtained simply by merging the unions:

$$\Gamma \cup \Xi = \left(\bigcup_{i=1}^{x} \Gamma_i\right) \cup \left(\bigcup_{m=1}^{z} \Xi_m\right) = \Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_x \cup \Xi_1 \cup \Xi_2 \cup ... \cup \Xi_z , \tag{D.137}$$

which is a normal form once the redundant terms in the series –the Γ_i's that happen to be equivalent to some Ξ_m's– have been removed by commutativity, associativity, and idempotency. To prove that $\Gamma \cap \Xi$ can be put in normal form requires to use of the distribution law (D.18)

$$\begin{aligned} \Gamma \cap \Xi = \bigcup_{i=1}^{x} \left(\bigcap_{j=1}^{y_i} \Gamma_{ij}\right) \cap \bigcup_{m=1}^{z} \left(\bigcap_{n=1}^{t_m} \Xi_{mn}\right) &= (\Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_x) \cap (\Xi_1 \cup \Xi_2 \cup ... \cup \Xi_z) \\ &= (\Gamma_1 \cap \Xi_1) \cup (\Gamma_2 \cap \Xi_1) \cup \ldots \cup (\Gamma_z \cap \Xi_1) \cup (\Gamma_1 \cap \Xi_2) \cup \ldots \cup (\Gamma_z \cap \Xi_t) . \end{aligned} \tag{D.138}$$

Each $(\Gamma_i \cap \Xi_m)$ term is an intersection of intersections. Therefore, they can be merged as an overall intersection by associativity, and the redundancy can be removed the same way as for the union case above. Then again, the pairwise unions of these terms are in normal form once the redundancies like $(\Gamma_i \cap \Xi_m) = (\Gamma_{i'} \cap \Xi_{m'})$ for some $(i, m) \neq (i', m')$ have been removed.

To prove that $\overline{\Gamma}$ can be put in normal form requires to use of the De Morgan laws,

$$\overline{\Gamma} = \overline{\bigcup_{i=1}^{x} \left(\bigcap_{j=1}^{y_i} \Gamma_{ij}\right)} \overset{\text{(D.33a)}}{=} \bigcap_{i=1}^{x} \overline{\left(\bigcap_{j=1}^{y_i} \Gamma_{ij}\right)} \overset{\text{(D.33b)}}{=} \bigcap_{i=1}^{x} \left(\bigcup_{j=1}^{y_i} \overline{\Gamma_{ij}}\right) , \tag{D.139}$$

then the commutation of the negation with the prec, Equation (D.101), is used on each term,

$$\overline{\Gamma_{ij}} = \overline{\widetilde{\mathcal{P}}_{\sigma_{ij}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\sigma_{ij}(K)}} = \overline{\widetilde{\mathcal{P}}_{\sigma_{ij}(A)}} \prec \ldots \prec \overline{\widetilde{\mathcal{P}}_{\sigma_{ij}(K)}} = \widetilde{\mathcal{P}}'_{\sigma_{ij}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}'_{\sigma_{ij}(K)} = \Gamma'_{ij} . \tag{D.140}$$

Where the base projectors have been redefined so as to incorporate the negation. Thus, the Γ'_{ij} are normal forms, and so their intersections of unions can be put into normal form by using the two properties proven just before this one.

Next, let Υ be a projector on an operator system over l subsystems in $\mathcal{L}\left(\mathcal{H}^L \otimes \ldots \otimes \mathcal{H}^R\right)$ that is in normal form,

$$\Upsilon = \bigcup_{m=1}^{z} \left(\bigcap_{n=1}^{t_m} \widetilde{\mathcal{P}}_{\chi_{mn}(L)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\chi_{mn}(R)} \right) \tag{D.141}$$

where χ_{mn} is an element of the permutation group over l symbols. Then the one-way signaling composition $\Gamma \prec \Upsilon$ is a projector on operator system over $\mathcal{L}\left(\mathcal{H}^A \otimes \ldots \otimes \mathcal{H}^K \otimes \mathcal{H}^L \otimes \ldots \otimes \mathcal{H}^R\right)$ that can be put into a normal form as well. This is proven using the interchange laws (D.94) and (D.97). Define Υ_m and Υ_{mn} like above, let u ranging from 1 to $x \times z$ such that $u = 1$ is identified with $(i, m) = (1, 1)$, $u = 2$ with $(i, m) = (1, 2)$, etc. For each u, let v ranging[16] from 1 to $y_i \times t_m$ such that $v = 1$ is identified with $(j, n) = (1, 1)$, etc., and let $\zeta_{uv} = (\sigma_{ij}, \chi_{mn})$ be an element of the permutation group over $k + l$ elements indexed by u and v. That way, an element like $\Gamma_{ij} \prec \Upsilon_{mn}$ can be rewritten as Θ_{uv} in the following manner:

16: As is the case with Γ and Ξ, each value of the index of unions u defines a new index for the set intersections v_u. Hence, v_u directly depends on the current value of u but this dependence is left implicit, $v := v_u$.

$$\begin{aligned}
\Gamma_{ij} \prec \Upsilon_{mn} &= \left(\widetilde{\mathcal{P}}_{\sigma_{ij}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\sigma_{ij}(K)}\right) \prec \left(\widetilde{\mathcal{P}}_{\chi_{mn}(L)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\chi_{mn}(R)}\right) \\
&= \widetilde{\mathcal{P}}_{\sigma_{ij}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\sigma_{ij}(K)} \prec \widetilde{\mathcal{P}}_{\chi_{mn}(L)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\chi_{mn}(R)} \\
&= \widetilde{\mathcal{P}}_{\zeta_{uv}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\zeta_{uv}(K)} \prec \widetilde{\mathcal{P}}_{\zeta_{uv}(L)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\zeta_{uv}(R)} =: \Theta_{uv} \; ;
\end{aligned} \tag{D.142}$$

This is but a relabelling using associativity of the prec (5.38a) (proven at Equation D.104). But the same way, an element like $\Gamma_{ij} \succ \Upsilon_{mn}$ can also be identified with some $\Theta_{u'v'}$ by finding the permutation $\zeta_{u'v'}$ that exactly corresponds to $\widetilde{\mathcal{P}}_{\zeta_{u'v'}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\zeta_{u'v'}(K)} \prec \widetilde{\mathcal{P}}_{\zeta_{u'v'}(L)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\zeta_{u'v'}(R)} = \widetilde{\mathcal{P}}_{\chi_{mn}(L)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\chi_{mn}(R)} \prec \widetilde{\mathcal{P}}_{\sigma_{ij}(A)} \prec \ldots \prec \widetilde{\mathcal{P}}_{\sigma_{ij}(K)}$. The important thing to notice is by definition, Θ_{uv} is a normal form. The one-way signaling composition of Γ with Υ then reads:

$$\begin{aligned}
\Gamma \prec \Upsilon &= \bigcup_{i=1}^{x} \left(\bigcap_{j=1}^{y_i} \Gamma_{ij} \right) \prec \bigcup_{m=1}^{z} \left(\bigcap_{n=1}^{t_m} \Upsilon_{mn} \right) \overset{\text{(D.97)}}{=} \bigcup_{i=1}^{x} \left(\left(\bigcap_{j=1}^{y_i} \Gamma_{ij} \right) \prec \left(\bigcup_{m=1}^{z} \left(\bigcap_{n=1}^{t_m} \Upsilon_{mn} \right) \right) \right) \\
&\overset{\text{(D.97)}}{=} \bigcup_{i=1}^{x} \left(\bigcup_{m=1}^{z} \left(\left(\bigcap_{j=1}^{y_i} \Gamma_{ij} \right) \prec \left(\bigcap_{n=1}^{t_m} \Upsilon_{mn} \right) \right) \right) \overset{\text{(D.94)}}{=} \bigcup_{i=1}^{x} \left(\bigcup_{m=1}^{z} \left(\bigcap_{j=1}^{y_i} \left(\Gamma_{ij} \prec \left(\bigcap_{n=1}^{t_m} \Upsilon_{mn} \right) \right) \right) \right) \\
&\overset{\text{(D.94)}}{=} \bigcup_{i=1}^{x} \left(\bigcup_{m=1}^{z} \left(\bigcap_{j=1}^{y_i} \left(\bigcap_{n=1}^{t_m} \Gamma_{ij} \prec \Upsilon_{mn} \right) \right) \right) = \bigcup_{(i=1,m=1)}^{(x,z)} \left(\bigcap_{(j=1,n=1)}^{(y_i,t_m)} \Gamma_{ij} \prec \Upsilon_{mn} \right) \\
&= \bigcup_{u=1}^{x \times z} \left(\bigcap_{v=1}^{y_i \times t_m} \Theta_{uv} \right) .
\end{aligned} \tag{D.143}$$

In the above rewriting, the associativities of intersections and unions have been used in the penultimate equality, and then the definition of the Θ_{uv}'s was injected to go the last line. Note that compared to the intersection and union cases, there is no risk of redundancies since the permutations of Γ_{ij} and Υ_{mn} run over different sets ($\{A, \ldots K\}$ and $\{L, \ldots R\}$ respectively). As each Θ_{uv} is a 'prec chain', it is direct to see that the last line is a normal form of $\Gamma \prec \Upsilon$.

Using relations (5.40) (proven around Equations (D.102) and (D.107)), the no-signaling composition of two normal forms, $\Gamma \otimes \Upsilon$, as well as their two-way signaling composition of a normal form into a normal form, $\Gamma \rightarrow \Upsilon$ can be expressed in terms of intersections, unions, negations or precs: $\Gamma \otimes \Upsilon = (\Gamma \prec \Upsilon) \cap (\Gamma \succ \Upsilon)$ and $\Gamma \parr \Upsilon = (\Gamma \prec \Upsilon) \cap (\Gamma \succ \Upsilon)$. And so they can be put in normal form as well because of the above discussion. The same holds for the transformation and any connector that can be derived from $\{\overline{\cdot}, \cap, \cup, \prec, \}$.

Therefore, by showing that the negation of a normal form, the intersection, and union of two normal forms, as well as the one-way signaling composition of two normal forms, can all be rewritten into a normal form. Moreover, the other multiplicative connectors, since they are secondary, can also be put into a normal form. The proof is completed by noticing that a single base projector like $\mathcal{P}_A$ is in a normal form by definition.

D.5.4. Proof of Theorem 5.3.2

Using Lemma 5.3.1, the equivalence between (5.64) and (5.55) is almost immediate: inject the result on each node, $\mathcal{P}^{\text{(n-network)}}_{\mathscr{A}_{\text{channel}}} \equiv (\mathcal{I}_{A_0} \rightarrow \mathcal{I}_{A_1}) \prec \ldots \prec \big(\mathcal{I}_{A_{2n-2}} \rightarrow \mathcal{I}_{A_{2n-1}}\big) \overset{(5.63a)}{=} \big(\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1}\big) \prec \ldots \prec \big(\overline{\mathcal{I}}_{A_{2n-2}} \prec \mathcal{I}_{A_{2n-1}}\big)$, then use the associativity of the prec.

This in turn can be used to recursively prove the equivalence between (5.64) and (5.56). Indeed, observe that the 1-comb is characterized by $\mathcal{I}_{A_0} \rightarrow \mathcal{I}_{A_1}$ which is equivalent to

$$\mathcal{P}^{\text{(2-comb)}}_{\mathscr{A}_{\text{state}}} := \mathcal{I}_{A_0} \rightarrow \mathcal{I}_{A_1} \overset{(5.63a)}{=} \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1}\,. \tag{D.144}$$

Then, each iteration of combs satisfies the latter condition in Lemma 5.3.1 for (5.63a) to hold on the right side of the $\rightarrow$. E.g, for $\mathcal{P}^{\text{(4-comb)}}_{\mathscr{A}_{\text{state}}}$:

$$\begin{aligned}\mathcal{P}^{\text{(4-comb)}}_{\mathscr{A}_{\text{state}}} &:= ((\mathcal{I}_{A_0} \rightarrow \mathcal{I}_{A_1}) \rightarrow \mathcal{I}_{A_2}) \rightarrow \mathcal{I}_{A_3} = \big(\big(\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1}\big) \rightarrow \mathcal{I}_{A_2}\big) \rightarrow \mathcal{I}_{A_3} \\ &= \Big(\overline{\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1}} \prec \mathcal{I}_{A_2}\Big) \rightarrow \mathcal{I}_{A_3} = \big(\mathcal{I}_{A_0} \prec \overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2}\big) \rightarrow \mathcal{I}_{A_3} \\ &= \overline{\mathcal{I}_{A_0} \prec \overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2}} \prec \mathcal{I}_{A_3} = \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \overline{\mathcal{I}}_{A_2} \prec \mathcal{I}_{A_3}\,.\end{aligned} \tag{D.145}$$

Where the associativity of one-way signaling composition (Equation D.103 here) and the distribution of the negation over the prec (Equation D.101 here) has been used to simplify in between each step. The proof for the n-comb directly follows by induction on the above computation: suppose it holds for n, $\mathcal{P}^{\text{(n-network)}}_{\mathscr{A}_{\text{channel}}} = \mathcal{P}^{\text{(2n-comb)}}_{\mathscr{A}_{\text{state}}}$ then for $n+1$ the projector is

$$\begin{aligned}\mathcal{P}^{\text{(2(n+1)-comb)}}_{\mathscr{A}_{\text{state}}} &:= \Big(\mathcal{P}^{\text{(2n-comb)}}_{\mathscr{A}_{\text{state}}} \rightarrow \mathcal{I}_{A_{2n}}\Big) \rightarrow \mathcal{I}_{A_{2n+1}} = \Big(\overline{\mathcal{P}}^{\text{(2n-comb)}}_{\mathscr{A}_{\text{state}}} \prec \mathcal{I}_{A_{2n}}\Big) \rightarrow \mathcal{I}_{A_{2n+1}} \\ &= \overline{\overline{\mathcal{P}}^{\text{(2n-comb)}}_{\mathscr{A}_{\text{state}}} \prec \mathcal{I}_{A_{2n}}} \prec \mathcal{I}_{A_{2n+1}} = \mathcal{P}^{\text{(2n-comb)}}_{\mathscr{A}_{\text{state}}} \prec \overline{\mathcal{I}}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}} \\ &= \mathcal{P}^{\text{(n-network)}}_{\mathscr{A}_{\text{channel}}} \prec \big(\mathcal{I}_{A_{2n}} \rightarrow \mathcal{I}_{A_{2n+1}}\big) \\ &=: \mathcal{P}^{\text{((n+1)-network)}}_{\mathscr{A}_{\text{channel}}}\,,\end{aligned} \tag{D.146}$$

where the hypothesis was injected in between the antepenultimate and penultimate lines as well as identity $\overline{\mathcal{I}}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}} = \big(\mathcal{I}_{A_{2n}} \rightarrow \mathcal{I}_{A_{2n+1}}\big)$.

Next, the equivalence between (5.64) and (5.54) is also proven by induction. It holds by definition for the $n = 1$ case, suppose it holds for n, $\mathcal{P}^{\text{(n-comb)}}_{\mathscr{A}_{\text{channel}}} = \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \ldots \prec \overline{\mathcal{I}}_{A_{2n-2}} \prec \mathcal{I}_{A_{2n-1}}$, and define the relabelling $\mathcal{P}^{\text{(n-comb)}'}_{\mathscr{A}_{\text{channel}}} \equiv \overline{\mathcal{I}}_{A_1} \prec \mathcal{I}_{A_2} \prec \ldots \prec \overline{\mathcal{I}}_{A_{2n-1}} \prec \mathcal{I}_{A_{2n}}$ where all indices have been incremented by 1. Then,

$$\begin{aligned}
\mathcal{P}^{\text{(2(n+1)-comb)}}_{\mathscr{A}_{\text{state}}} :=& \\
&(\ldots(\mathcal{I}_{A_n} \to \mathcal{I}_{A_{n+1}}) \to \ldots) \to \big(\mathcal{I}_{A_0} \to \mathcal{I}_{A_{2n+1}}\big) \\
&= \mathcal{P}^{\text{(n-comb)}'}_{\mathscr{A}_{\text{channel}}} \to \big(\mathcal{I}_{A_0} \to \mathcal{I}_{A_{2n+1}}\big) \\
&\overset{\text{(D.68)}}{=} \big(\mathcal{I}_{A_0} \otimes \mathcal{P}^{\text{(n-comb)}'}_{\mathscr{A}_{\text{channel}}}\big) \to \mathcal{I}_{A_{2n+1}} \\
&\overset{\text{(5.63b)}}{=} \big(\mathcal{I}_{A_0} \prec \mathcal{P}^{\text{(n-comb)}'}_{\mathscr{A}_{\text{channel}}}\big) \to \mathcal{I}_{A_{2n+1}} \\
&\overset{\text{(5.63a)}}{=} \overline{\mathcal{I}_{A_0} \prec \mathcal{P}^{\text{(n-comb)}'}_{\mathscr{A}_{\text{channel}}}} \prec \mathcal{I}_{A_{2n+1}} \\
&= \overline{\mathcal{I}}_{A_0} \prec \overline{\mathcal{P}}^{\text{(n-comb)}'}_{\mathscr{A}_{\text{channel}}} \prec \mathcal{I}_{A_{2n+1}} \\
&= \overline{\mathcal{I}}_{A_0} \prec \overline{\overline{\mathcal{I}}_{A_1} \prec \ldots \prec \mathcal{I}_{A_{2n}}} \prec \mathcal{I}_{A_{2n+1}} \\
&= \overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1} \prec \ldots \prec \overline{\mathcal{I}}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}} \,.
\end{aligned} \tag{D.147}$$

Here, the uncurrying rule (D.68) was used between the second and third lines as a computational shortcut.

Finally, the equivalence between (5.64) and (5.57) follows by induction as well. In the case $n = 1$, it is proven by writing explicitly the content of the projector (5.64),

$$\begin{aligned}
\big(\overline{\mathcal{I}}_{A_0} \prec \mathcal{I}_{A_1}\big)\{M\} = M& \\
(\mathcal{I}_{A_0} \otimes \mathcal{I}_{A_1} - \mathcal{I}_{A_0} \otimes \mathcal{D}_{A_1} + \mathcal{D}_{A_0} \otimes \mathcal{D}_{A_1})\{M\} &= M \\
M - \frac{\mathbb{1}^{A_1}}{d_{A_1}} \operatorname{Tr}_{A_1}[M] + \frac{\mathbb{1}^{A_0}}{d_{A_0}} \operatorname{Tr}_{A_0}\left[\frac{\mathbb{1}^{A_1}}{d_{A_1}} \operatorname{Tr}_{A_1}[M]\right] &= M \\
\frac{\mathbb{1}^{A_0 A_1}}{d_{A_0 A_1}} \operatorname{Tr}_{A_0 A_1}[M] &= \frac{\mathbb{1}^{A_1}}{d_{A_1}} \operatorname{Tr}_{A_1}[M] \\
\operatorname{Tr}_{A_1}[M] &= \frac{1}{d_{A_0}} \operatorname{Tr}_{A_0 A_1}[M]\ \mathbb{1}_{A_0}.
\end{aligned} \tag{D.148}$$

Suppose it holds for n nodes, $\mathcal{P}^{(2n)}\{M^{(n)}\} = M^{(n)} \iff \operatorname{Tr}_{A_1}\left[M^{(1)}\right] = \frac{1}{d_{A_0}} \operatorname{Tr}_{A_0 A_1}\left[M^{(1)}\right] \mathbb{1}_{A_0} \wedge \ldots \wedge \operatorname{Tr}_{A_{2n-1}}[M] = \frac{1}{d_{A_{2n-2}}} \operatorname{Tr}_{A_{2n-2}A_{2n-1}}[M] \otimes \mathbb{1}_{A_{2n-1}}$. Then, for $n+1$ nodes, let $M^{(n+1)} \equiv M$, and

$$\begin{aligned}
M = \mathcal{P}^{(2n+2)}\{M\}& \\
&= \big[\mathcal{P}^{(2n)} \prec \big(\overline{\mathcal{I}}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}}\big)\big]\{M\} \\
&= \big[\big(\mathcal{I}_{A_0} \otimes \ldots \otimes \mathcal{I}_{A_{2n-1}}\big) \otimes \big(\overline{\mathcal{I}}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}}\big) \\
&\quad - \overline{\mathcal{P}}^{(2n)} \otimes \mathcal{D}_{A_{2n}} \otimes \mathcal{D}_{A_{2n+1}} + \mathcal{D}_{A_0} \otimes \mathcal{D}_{A_1} \otimes \ldots \otimes \mathcal{D}_{A_{2n}} \otimes \mathcal{D}_{A_{2n+1}}\big]\{M\} \\
&= \big[\big(\mathcal{I}_{A_0} \otimes \ldots \otimes \mathcal{I}_{A_{2n-1}}\big) \otimes \big(\overline{\mathcal{I}}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}}\big) \\
&\quad - \mathcal{I}_{A_0} \otimes \ldots \otimes \mathcal{I}_{A_{2n-1}} \otimes \mathcal{D}_{A_{2n}} \otimes \mathcal{D}_{A_{2n+1}} + \mathcal{P}^{(2n)} \otimes \mathcal{D}_{A_{2n}} \otimes \mathcal{D}_{A_{2n+1}}\big]\{M\} \,.
\end{aligned} \tag{D.149}$$

Using this last equality, one can regroup terms as

$$
\begin{aligned}
0 = \big[\, \big(\mathcal{I}_{A_0} \otimes \ldots \otimes \mathcal{I}_{A_{2n-1}}\big) \otimes \big(\big(\overline{\mathcal{I}}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}}\big) - \mathcal{I}_{A_{2n}} \otimes \mathcal{I}_{A_{2n+1}}\big) \, \big] \{M\} \\
- \Big[\, \Big(\big(\mathcal{I}_{A_0} \otimes \ldots \otimes \mathcal{I}_{A_{2n-1}}\big) - \mathcal{P}^{(2n)} \Big) \otimes \mathcal{D}_{A_{2n}} \otimes \mathcal{D}_{A_{2n+1}} \Big] \{M\} \, .
\end{aligned}
\tag{D.150}
$$

This defines two projectors with zero intersection, therefore each piece in square brackets must be zero independently of the other. The first piece, $0 = \big[\, \big(\mathcal{I}_{A_0} \otimes \ldots \otimes \mathcal{I}_{A_{2n-1}}\big) \otimes \big(\big(\overline{\mathcal{I}}_{A_{2n}} \prec \mathcal{I}_{A_{2n+1}}\big) - \mathcal{I}_{A_{2n}} \otimes \mathcal{I}_{A_{2n+1}}\big) \, \big] \{M\}$ is exactly equation (D.148) applied on systems $A_{2n+1}A_{2n}$. Whereas the second piece, $0 = \Big[\, \big(\big(\mathcal{I}_{A_0} \otimes \ldots \otimes \mathcal{I}_{A_{2n-1}}\big) - \mathcal{P}^{(2n)} \big) \otimes \mathcal{D}_{A_{2n}} \otimes \mathcal{D}_{A_{2n+1}} \Big] \{M\}$ can be recast into $0 = \big[\big(\big(\mathcal{I}_{A_0} \otimes \ldots \otimes \mathcal{I}_{A_{2n-1}}\big) - \mathcal{P}^{(2n)} \big] \{ \mathrm{Tr}_{A_{2n}A_{2n+1}} [M] \}$. Using $M^{(n)} \equiv \mathrm{Tr}_{A_{2n}A_{2n+1}} [M]$, this last equation must contain by hypothesis the n other causality conditions. Therefore, the $n+1$ causality conditions have been recovered from the projector, completing the proof.

Bibliography

> Toutes ces plumes sur le parchemin qui grincent, tout cela a fait un livre. Tout cela a fait un livre – et moi je ne sais pas lire.*
>
> Arthur Honegger (1938), *Jeanne d'Arc au bûcher* (text by Paul Claudel)

[1] Timothée Hoffreumon and Ognyan Oreshkov. 'The Multi-round Process Matrix'. In: *Quantum* 5 (Jan. 2021), p. 384. DOI: 10.22331/q-2021-01-20-384. arXiv: 2005.04204 [quant-ph] (cited on pages ix, 2, 14, 34, 42, 56, 64, 107, 148, 158, 159).

[2] Timothée Hoffreumon and Ognyan Oreshkov. *Projective characterization of higher-order quantum transformations*. 2022. arXiv: 2206.06206 [quant-ph] (cited on pages ix, 2, 75, 161).

[3] Titouan Carette, Timothée Hoffreumon, Émile Larroque, and Renaud Vilmart. 'Complete Graphical Language for Hermiticity-Preserving Superoperators'. In: *2023 38th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)* (June 2023). DOI: 10.1109/lics56636.2023.10175712. arXiv: 2302.04212 [quant-ph] (cited on pages ix, 38, 193).

[4] Giulio Chiribella, Giacomo Mauro D'Ariano, Paolo Perinotti, and Benoit Valiron. 'Quantum computations without definite causal structure'. In: *Physical Review A* 88.2 (Aug. 2013). DOI: 10.1103/physreva.88.022318. arXiv: 0912.0195 [quant-ph] (cited on pages 1, 14, 25, 30, 43, 147, 195, 196).

[5] Ognyan Oreshkov, Fabio Costa, and Časlav Brukner. 'Quantum correlations with no causal order'. In: *Nature Communications* 3 (2012), pp. 1–13. DOI: 10.1038/ncomms2076. arXiv: 1105.4464 (cited on pages 1, 3, 7, 12–14, 16, 18, 25–27, 33, 37, 43–47, 53, 54, 71, 88, 89, 92, 101, 103, 107–109, 113, 143, 147, 169, 189, 196).

[6] E. C. George Sudarshan, Piravonu M. Mathews, and Jayaseetha Rau. 'Stochastic Dynamics of Quantum-Mechanical Systems'. In: *Physical Review* 121 (3 Feb. 1961), pp. 920–924. DOI: 10.1103/PhysRev.121.920 (cited on page 1).

[7] Karl Kraus. 'General state changes in quantum theory'. In: *Annals of Physics* 64.2 (1971), pp. 311–335. DOI: https://doi.org/10.1016/0003-4916(71)90108-4 (cited on page 1).

[8] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. 'Transforming quantum operations: Quantum supermaps'. In: *Europhysics Letters* 83.3 (July 2008), p. 30004. DOI: 10.1209/0295-5075/83/30004. arXiv: 0804.0180 [quant-ph] (cited on pages 1, 25, 26, 33–35, 71, 112, 113, 152, 180, 189, 194).

[9] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. 'Theoretical framework for quantum networks'. In: *Physical Review A* 80.2 (Aug. 2009). DOI: 10.1103/physreva.80.022339. arXiv: 0904.4483 [quant-ph] (cited on pages 1, 26, 27, 30, 33, 39, 40, 42, 73, 106, 112, 114, 116, 138–140, 147, 171, 194).

[10] Paolo Perinotti. 'Causal Structures and the Classification of Higher Order Quantum Computations'. In: *Tutorials, Schools, and Workshops in the Mathematical Sciences* (2017), pp. 103–127. DOI: 10.1007/978-3-319-68655-4_7. arXiv: 1612.05099 [quant-ph] (cited on pages 1–3, 23, 25–27, 33, 36, 48, 63, 64, 71, 75, 83, 110, 112–114, 117, 119, 121, 124, 133, 140, 167, 225, 230).

[11] Alessandro Bisio and Paolo Perinotti. 'Theoretical framework for higher-order quantum theory'. In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 475.2225 (2019), p. 20180706. DOI: 10.1098/rspa.2018.0706. arXiv: 1806.09554 [quant-ph] (cited on pages 1–3, 23, 25, 26, 36, 48, 63, 64, 71, 75, 83, 110, 112–114, 116, 117, 119, 121, 124, 133, 139, 140, 167, 170, 171, 230).

*"All these feathers squeaking on the parchment, all that has made a book. All that has made a book – and I, I cannot read."

[12] Gus Gutoski and John Watrous. 'Toward a General Theory of Quantum Games'. In: *Proceedings of the Thirty-Ninth Annual ACM Symposium on Theory of Computing*. STOC '07. San Diego, California, USA: Association for Computing Machinery, 2007, pp. 565–574. ISBN: 9781595936318. DOI: 10.1145/1250790.1250873. arXiv: quant-ph/0611234 [quant-ph] (cited on pages 1, 20, 194).

[13] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. 'Quantum Circuit Architecture'. In: *Physical Review Letters* 101.6 (Aug. 2008). DOI: 10.1103/physrevlett.101.060401. arXiv: 0712.1325 [quant-ph] (cited on pages 1, 30, 71, 113, 138, 140, 194).

[14] Edward Brian Davies and John Trevor Lewis. 'An operational approach to quantum probability'. In: *Communications in Mathematical Physics* 17.3 (1970), pp. 239–260. DOI: 10.1007/BF01647093 (cited on pages 1, 22, 59, 71).

[15] Lucien Hardy. *Probability Theories with Dynamic Causal Structure: A New Framework for Quantum Gravity*. 2005. arXiv: gr-qc/0509120 [gr-qc] (cited on pages 1, 8, 195).

[16] Lucien Hardy. 'Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure'. In: *Journal of Physics A: Mathematical and Theoretical* 40.12 (Mar. 2007), pp. 3081–3099. DOI: 10.1088/1751-8113/40/12/s12. arXiv: gr-qc/0608043 [gr-qc] (cited on pages 1, 8, 195, 196).

[17] Lucien Hardy. 'Quantum Gravity Computers: On the Theory of Computation with Indefinite Causal Structure'. In: *Quantum Reality, Relativistic Causality, and Closing the Epistemic Circle: An International Conference in Honour of Abner Shimony*. Ed. by Wayne C. Myrvold and Joy Christian. The Western Ontario Series in Philosophy of Science. Dordrecht: Springer, Feb. 2009, pp. 379–401. ISBN: 978-1-4020-9107-0. DOI: 10.1007/978-1-4020-9107-0_21. arXiv: quant-ph/0701019 [quant-ph] (cited on page 1).

[18] Lorenzo M. Procopio, Amir Moqanaki, Mateus Araújo, Fabio Costa, Irati Alonso Calafell, Emma G. Dowd, Deny R. Hamel, Lee A. Rozema, Časlav Brukner, and Philip Walther. 'Experimental superposition of orders of quantum gates'. In: *Nature Communications* 6.1 (Aug. 2015). DOI: 10.1038/ncomms8913. arXiv: 1412.4006 [quant-ph] (cited on pages 1, 195).

[19] Timoteo Colnaghi, Giacomo Mauro D'Ariano, Stefano Facchini, and Paolo Perinotti. 'Quantum computation with programmable connections between gates'. In: *Physics Letters A* 376.45 (Oct. 2012), pp. 2940–2943. DOI: 10.1016/j.physleta.2012.08.028. arXiv: 1109.5987 [quant-ph] (cited on pages 2, 195).

[20] Mateus Araújo, Fabio Costa, and Časlav Brukner. 'Computational advantage from quantum-controlled ordering of gates'. In: *Physical Review Letters* 113.25 (Jan. 2014). DOI: 10.1103/PhysRevLett.113.250402. arXiv: 1401.8127 (cited on page 2).

[21] Philippe Allard Guérin, Adrien Feix, Mateus Araújo, and Časlav Brukner. 'Exponential Communication Complexity Advantage from Quantum Superposition of the Direction of Communication'. In: *Physical Review Letters* 117.10 (May 2016), p. 100502. DOI: 10.1103/PhysRevLett.117.100502. arXiv: 1605.07372 (cited on page 2).

[22] Daniel Ebler, Sina Salek, and Giulio Chiribella. 'Enhanced Communication with the Assistance of Indefinite Causal Order'. In: *Physical Review Letters* 120.12 (Mar. 2018), p. 120502. DOI: 10.1103/PhysRevLett.120.120502. arXiv: 1711.10165 (cited on page 2).

[23] David Felce and Vlatko Vedral. 'Quantum Refrigeration with Indefinite Causal Order'. In: *Physical Review Letters* 125.7 (Aug. 2020). DOI: 10.1103/physrevlett.125.070603. arXiv: 2003.00794 [quant-ph] (cited on page 2).

[24] Tamal Guha, Mir Alimuddin, and Preeti Parashar. 'Thermodynamic advancement in the causally inseparable occurrence of thermal maps'. In: *Physical Review A* 102.3 (Sept. 2020). DOI: 10.1103/physreva.102.032215. arXiv: 2003.01464 [quant-ph] (cited on page 2).

[25] Kyrylo Simonov, Gianluca Francica, Giacomo Guarnieri, and Mauro Paternostro. 'Work extraction from coherently activated maps via quantum switch'. In: *Physical Review A* 105.3 (Mar. 2022). DOI: 10.1103/physreva.105.032217. arXiv: 2009.11265 [quant-ph] (cited on page 2).

[26] Xiaobin Zhao, Yuxiang Yang, and Giulio Chiribella. 'Quantum Metrology with Indefinite Causal Order'. In: *Physical Review Letters* 124.19 (May 2020). DOI: 10.1103/physrevlett.124.190503. arXiv: 1912.02449 [quant-ph] (cited on page 2).

[27] François Chapeau-Blondeau. 'Noisy quantum metrology with the assistance of indefinite causal order'. In: *Physical Review A* 103.3 (Mar. 2021). DOI: 10.1103/physreva.103.032615. arXiv: 2104.06284 [quant-ph] (cited on page 2).

[28] Ognyan Oreshkov. 'Time-delocalized quantum subsystems and operations: on the existence of processes with indefinite causal structure in quantum mechanics'. In: *Quantum* 3 (Dec. 2019), p. 206. DOI: 10.22331/q-2019-12-02-206. arXiv: 1801.07594 [quant-ph] (cited on pages 2, 171, 190, 195).

[29] Nikola Paunković and Marko Vojinović. 'Causal orders, quantum circuits and spacetime: distinguishing between definite and superposed causal orders'. In: *Quantum* 4 (May 2020), p. 275. DOI: 10.22331/q-2020-05-28-275. arXiv: 1905.09682 [quant-ph] (cited on page 2).

[30] Julian Wechs, Cyril Branciard, and Ognyan Oreshkov. 'Existence of processes violating causal inequalities on time-delocalised subsystems'. In: *Nature Communications* 14.1 (Mar. 2023). DOI: 10.1038/s41467-023-36893-3. arXiv: 2201.11832 [quant-ph] (cited on pages 2, 171, 195).

[31] Nick Ormrod, Augustin Vanrietvelde, and Jonathan Barrett. 'Causal structure in the presence of sectorial constraints, with application to the quantum switch'. In: *Quantum* 7 (June 2023), p. 1028. DOI: 10.22331/q-2023-06-01-1028. arXiv: 2204.10273 [quant-ph] (cited on pages 2, 195).

[32] Esteban Castro-Ruiz, Flaminia Giacomini, and Časlav Brukner. 'Dynamics of Quantum Causal Structures'. In: *Physical Review X* 8.1 (2018), pp. 1–18. DOI: 10.1103/PhysRevX.8.011047. arXiv: 1710.03139 [quant-ph] (cited on pages 2, 72, 75, 124, 205).

[33] Aleks Kissinger and Sander Uijlen. 'A categorical semantics for causal structure'. In: *Logical Methods in Computer Science* Volume 15, Issue 3 (Aug. 2019). DOI: 10.23638/LMCS-15(3:15)2019. arXiv: 1701.04732 [quant-ph] (cited on pages 2, 25, 28, 30, 36, 48, 71, 72, 75, 83, 90, 110, 117, 119, 125, 129, 130, 133, 137, 167, 171, 172, 225).

[34] Timothée Hoffreumon. 'Processes with indefinite causal structure in quantum theory: The Multi-Round Process Matrix'. http://quic.ulb.ac.be/_media/publications/2019_mfe_timothee_hoffreumon.pdf. Master Thesis. Université libre de Bruxelles, Aug. 2019 (cited on pages 2, 34, 48, 64, 158).

[35] Mateus Araújo, Cyril Branciard, Fabio Costa, Adrien Feix, Christina Giarmatzi, and časlav Brukner. 'Witnessing causal nonseparability'. In: *New Journal of Physics* 17.10 (Oct. 2015), p. 102001. DOI: 10.1088/1367-2630/17/10/102001. arXiv: 1506.03776 [quant-ph] (cited on pages 2, 3, 43, 47, 56, 60, 64, 109, 148, 160, 161).

[36] Will Simmons and Aleks Kissinger. *Higher-order causal theories are models of BV-logic*. 2022. DOI: 10.48550/ARXIV.2205.11219. arXiv: 2205.11219 [quant-ph]. URL: https://arxiv.org/abs/2205.11219 (cited on pages 2, 83, 90, 117, 119, 130, 147, 167, 170–172, 213, 234).

[37] Simon Milz and Marco Túlio Quintino. *Transformations between arbitrary (quantum) objects and the emergence of indefinite causality*. 2023. arXiv: 2305.01247 [quant-ph] (cited on pages 2, 25, 30, 75, 119, 147, 161, 170).

[38] Ognyan Oreshkov and Christina Giarmatzi. 'Causal and causally separable processes'. In: *New Journal of Physics* 18.9 (Sept. 2016), p. 093020. DOI: 10.1088/1367-2630/18/9/093020. arXiv: 1506.05449 [quant-ph] (cited on pages 3, 7, 8, 11, 14, 17, 47).

[39] Julian Wechs, Alastair A Abbott, and Cyril Branciard. 'On the definition and characterisation of multipartite causal (non)separability'. In: *New Journal of Physics* 21.1 (Jan. 2019), p. 013027. DOI: 10.1088/1367-2630/aaf352. arXiv: 1807.10557 [quant-ph] (cited on pages 3, 14).

[40] Alastair A. Abbott, Christina Giarmatzi, Fabio Costa, and Cyril Branciard. 'Multipartite causal correlations: Polytopes and inequalities'. In: *Physical Review A* 94.3 (Sept. 2016). DOI: 10.1103/physreva.94.032131. arXiv: 1608.01528 [quant-ph] (cited on pages 3, 17, 38, 161).

[41] Lieven Vandenberghe and Stephen Boyd. 'Semidefinite Programming'. In: *SIAM Review* 38.1 (1996), pp. 49–95. DOI: 10.1137/1038003. eprint: https://doi.org/10.1137/1038003 (cited on pages 3, 38, 148, 161).

[42] Adrien Feix, Mateus Araújo, and Časlav Brukner. 'Quantum superposition of the order of parties as a communication resource'. In: *Physical Review A* 92.5 (Nov. 2015). DOI: 10.1103/physreva.92.052326. arXiv: 1508.07840 [quant-ph] (cited on page 3).

[43] Adrien Feix, Mateus Araújo, and Časlav Brukner. 'Causally nonseparable processes admitting a causal model'. In: *New Journal of Physics* 18.8 (Aug. 2016), p. 083040. DOI: 10.1088/1367-2630/18/8/083040. arXiv: 1604.03391 [quant-ph] (cited on pages 3, 38, 161).

[44] Giulio Chiribella and Daniel Ebler. 'Optimal quantum networks and one-shot entropies'. In: *New Journal of Physics* 18.9 (Sept. 2016), p. 093053. DOI: 10.1088/1367-2630/18/9/093053. arXiv: 1606.02394 [quant-ph] (cited on pages 3, 38, 161).

[45] Jessica Bavaresco, Mateus Araújo, Časlav Brukner, and Marco Túlio Quintino. 'Semi-device-independent certification of indefinite causal order'. In: *Quantum* 3 (Aug. 2019), p. 176. DOI: 10.22331/q-2019-08-19-176. arXiv: 1903.10526 [quant-ph] (cited on pages 3, 14, 38, 161).

[46] Simon Milz, Jessica Bavaresco, and Giulio Chiribella. 'Resource theory of causal connection'. In: *Quantum* 6 (Aug. 2022), p. 788. DOI: 10.22331/q-2022-08-25-788. arXiv: 2110.03233 [quant-ph] (cited on pages 3, 71, 161).

[47] Ognyan Oreshkov and Nicolas J. Cerf. 'Operational quantum theory without predefined time'. In: *New Journal of Physics* 18.7 (July 2016), p. 073037. DOI: 10.1088/1367-2630/18/7/073037. arXiv: 1406.3829 [quant-ph] (cited on pages 7, 12, 88).

[48] Ognyan Oreshkov and Nicolas J. Cerf. 'Operational formulation of time reversal in quantum theory'. In: *Nature Physics* 11.10 (July 2015), pp. 853–858. DOI: 10.1038/nphys3414. arXiv: 1507.07745 [quant-ph] (cited on page 7).

[49] Jonathan Barrett. 'Information processing in generalized probabilistic theories'. In: *Physical Review A* 75.3 (Mar. 2007). DOI: 10.1103/physreva.75.032304. arXiv: quant-ph/0508211 [quant-ph] (cited on page 7).

[50] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. 'Probabilistic theories with purification'. In: *Physical Review A* 81.6 (June 2010). DOI: 10.1103/physreva.81.062348. arXiv: 0908.1583 [quant-ph] (cited on pages 7, 25, 81, 101, 169).

[51] Sally Shrapnel and Fabio Costa. 'Causation does not explain contextuality'. In: *Quantum* 2 (May 2018), p. 63. DOI: 10.22331/q-2018-05-18-63. arXiv: 1708.00137 [quant-ph] (cited on pages 8, 11, 12, 66, 210).

[52] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. 2nd ed. Cambridge, UK: Cambridge University Press, 2009. ISBN: 978-0-511-97666-7 (cited on pages 8, 19, 22, 141, 171, 199, 209, 213).

[53] Lucien Hardy. *Quantum Theory From Five Reasonable Axioms*. 2001. arXiv: quant-ph/0101012 [quant-ph] (cited on pages 8, 71, 195).

[54] Tobias Fritz. 'Beyond Bell's theorem: correlation scenarios'. In: *New Journal of Physics* 14.10 (Oct. 2012), p. 103001. DOI: 10.1088/1367-2630/14/10/103001. arXiv: 1206.5115 [quant-ph] (cited on page 8).

[55] Tobias Fritz. 'Beyond Bell's Theorem II: Scenarios with Arbitrary Causal Structure'. In: *Communications in Mathematical Physics* 341.2 (Nov. 2016), pp. 391–434. DOI: 10.1007/s00220-015-2495-5. arXiv: 1404.4812 [quant-ph] (cited on page 8).

[56] Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner. 'Bell nonlocality'. In: *Reviews of Modern Physics* 86.2 (Apr. 2014), pp. 419–478. DOI: 10.1103/revmodphys.86.419. arXiv: 1303.2849 [quant-ph] (cited on pages 8, 11, 47, 184, 212).

[57] Howard Barnum, Salman Beigi, Sergio Boixo, Matthew B. Elliott, and Stephanie Wehner. 'Local Quantum Measurement and No-Signaling Imply Quantum Correlations'. In: *Physical Review Letters* 104.14 (Apr. 2010). DOI: 10.1103/physrevlett.104.140401. arXiv: 0910.3952 [quant-ph] (cited on pages 9, 196).

[58] Robert W. Spekkens. 'Contextuality for preparations, transformations, and unsharp measurements'. In: *Physical Review A* 71.5 (May 2005). DOI: 10.1103/physreva.71.052108. arXiv: quant-ph/0406166 [quant-ph] (cited on pages 12, 171, 210).

[59] Robert W. Spekkens. 'The Status of Determinism in Proofs of the Impossibility of a Noncontextual Model of Quantum Theory'. In: *Foundations of Physics* 44.11 (Sept. 2014), pp. 1125–1155. DOI: 10.1007/s10701-014-9833-x. arXiv: 1312.3667 [quant-ph] (cited on pages 12, 171, 210).

[60] Sally Shrapnel, Fabio Costa, and Gerard Milburn. 'Updating the Born rule'. In: *New Journal of Physics* 20.5 (May 2018), p. 053010. DOI: 10.1088/1367-2630/aabe12. arXiv: 1702.01845 [quant-ph] (cited on pages 12, 44, 53, 62, 71, 210).

[61] David Beckman, Daniel Gottesman, Michael A. Nielsen, and John Preskill. 'Causal and localizable quantum operations'. In: *Physical Review A* 64.5 (Oct. 2001). DOI: 10.1103/physreva.64.052309. arXiv: quant-ph/0102043 [quant-ph] (cited on pages 13, 95, 104, 105, 108, 112, 154, 172, 212, 213).

[62] Tilo Eggeling, Dirk Schlingemann, and Reinhard F. Werner. 'Semicausal operations are semilocalizable'. In: *Europhysics Letters (EPL)* 57.6 (Mar. 2002), pp. 782–788. DOI: 10.1209/epl/i2002-00579-4. arXiv: quant-ph/0104027 [quant-ph] (cited on pages 13, 105, 154, 212).

[63] Sandu Popescu and Daniel Rohrlich. 'Quantum nonlocality as an axiom'. In: *Foundations of Physics* 24.3 (1994), pp. 379–385. DOI: 10.1007/BF02058098 (cited on pages 13, 108, 184).

[64] Marco Piani, Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. 'Properties of quantum nonsignaling boxes'. In: *Physical Review A* 74.1 (July 2006). DOI: 10.1103/physreva.74.012305. arXiv: quant-ph/0505110 [quant-ph] (cited on pages 13, 105, 106, 112, 154, 212, 213).

[65] Giacomo Mauro D'Ariano, Stefano Facchini, and Paolo Perinotti. 'No Signaling, Entanglement Breaking, and Localizability in Bipartite Channels'. In: *Physical Review Letters* 106.1 (Jan. 2011). DOI: 10.1103/physrevlett.106.010501. arXiv: 1007.1177 [quant-ph] (cited on pages 14, 105, 108, 154, 172, 195, 212).

[66] Paolo Perinotti. 'Causal influence in operational probabilistic theories'. In: *Quantum* 5 (Aug. 2021), p. 515. DOI: 10.22331/q-2021-08-03-515. arXiv: 2012.15213 [quant-ph] (cited on pages 14, 108).

[67] Giacomo M. D'Ariano, Franco Manessi, and Paolo Perinotti. 'Determinism without causality'. In: *Physica Scripta* T163 (Dec. 2014), p. 014013. DOI: 10.1088/0031-8949/2014/t163/014013. arXiv: 1301.7578 [quant-ph] (cited on pages 14, 108, 184).

[68] Flaminia Giacomini, Esteban Castro-Ruiz, and Časlav Brukner. 'Indefinite causal structures for continuous-variable systems'. In: *New Journal of Physics* 18.11 (Nov. 2016), p. 113026. DOI: 10.1088/1367-2630/18/11/113026. arXiv: 1510.06345 [quant-ph] (cited on page 18).

[69] Asher Peres. *Quantum Theory: Concepts and Methods*. Vol. 57. Fundamental Theories of Physics. Dordrecht: Springer, 1993. ISBN: 978-0-7923-2549-9 (cited on pages 18, 19).

[70] Mário Ziman. 'Process positive-operator-valued measure: A mathematical framework for the description of process tomography experiments'. In: *Physical Review A* 77.6 (June 2008). DOI: 10.1103/physreva.77.062112. arXiv: 0802.3862 [quant-ph] (cited on pages 18, 34, 44, 54, 194).

[71] John von Neumann. *Mathematical Foundations of Quantum Mechanics*. Ed. by Nicholas A. Wheeler. New Edition (2018) of Robert T. Beyer's authorized English translation (1955) of "Mathematische Grunglagen der Quantenmechanik" by John von Neumann, Springer 1932. Princeton: Princeton University Press, 1932. ISBN: 978-06-9117-856-1 (cited on pages 18, 201).

[72] Karl Kraus. *States, Effects, and Operations: Fundamental Notions of Quantum Theory*. Ed. by A. Böhm, J.D. Dollard, and W.H. Wootters. 1st ed. Vol. 190. Lecture notes in physics. Berlin Heidelberg: Springer-Verlag, 1983. ISBN: 978-3-540-38725-1 (cited on pages 19, 20, 59, 71, 141).

[73] Alexander Holevo. *Probabilistic and Statistical Aspect of Quantum Theory*. 2nd ed. Quaderni Monographs. First published in Russian in 1980. Scuola Normale Superiore Pisa: Edizioni Della Normale, 2011. ISBN: 978-88-7642-375-8 (cited on pages 20, 184).

[74] Ingemar Bengtsson and Karol Życzkowski. *Geometry of Quantum States: An Introduction to Quantum Entanglement*. 2nd ed. Cambridge, UK: Cambridge University Press, 2017. ISBN: 978-1-107-02625-4 (cited on pages 20, 37, 38, 47, 78, 85, 179, 184, 192, 199, 200).

[75] Gus Gutoski. 'Quantum Strategies and Local Operations'. PhD thesis. University of Waterloo, 2010. arXiv: 1003.0038 [quant-ph] (cited on page 20).

[76] Masanao Ozawa. 'Quantum measuring processes of continuous observables'. In: *Journal of Mathematical Physics* 25.1 (Jan. 1984), pp. 79–87. DOI: 10.1063/1.526000. eprint: https://pubs.aip.org/aip/jmp/article-pdf/25/1/79/8153372/79_1_online.pdf (cited on pages 22, 42, 71, 171, 209).

[77] Ämin Baumeler and Stefan Wolf. 'The space of logically consistent classical processes without causal order'. In: *New Journal of Physics* 18.1 (Jan. 2016), p. 013036. DOI: 10.1088/1367-2630/18/1/013036. arXiv: 1507.01714 [quant-ph] (cited on pages 25, 165).

[78] Giacomo Mauro D'Ariano, Giulio Chiribella, and Paolo Perinotti. *Quantum Theory from First Principles: An Informational Approach*. Cambridge, UK: Cambridge University Press, 2017. ISBN: 978-1-107-33834-0 (cited on pages 25, 81, 85, 141, 172).

[79] Martin Plávala. *General probabilistic theories: An introduction*. 2021. arXiv: 2103.07469 [quant-ph] (cited on pages 25, 63, 79, 85, 169, 172).

[80] Giulio Chiribella, Giacomo Mauro D'Ariano, and Paolo Perinotti. 'Informational derivation of quantum theory'. In: *Physical Review A* 84.1 (July 2011). DOI: 10.1103/physreva.84.012311. arXiv: 1011.6451 [quant-ph] (cited on page 25).

[81] Bob Coecke and Aleks Kissinger. *Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning*. Cambridge, UK: Cambridge University Press, 2017. ISBN: 978-1-107-10422-8 (cited on pages 25, 63, 81, 101, 141, 175, 180, 184, 186, 187, 193, 226).

[82] Matt Wilson, Giulio Chiribella, and Aleks Kissinger. *Quantum Supermaps are Characterized by Locality*. 2023. arXiv: 2205.09844 [quant-ph] (cited on pages 26, 71).

[83] Anna Jenčová. 'Generalized channels: Channels for convex subsets of the state space'. In: *Journal of Mathematical Physics* 53.1 (Jan. 2012). DOI: 10.1063/1.3676294. arXiv: 1105.1899 [quant-ph] (cited on pages 30, 71, 75).

[84] Matt Wilson and Giulio Chiribella. *A Mathematical Framework for Transformations of Physical Processes*. 2022. arXiv: 2204.04319 [quant-ph] (cited on pages 30, 36, 71).

[85] Andrej Jamiołkowski. 'Linear transformations which preserve trace and positive semidefiniteness of operators'. In: *Reports on Mathematical Physics* 3.4 (Dec. 1972), pp. 275–278. DOI: 10.1016/0034-4877(72)90011-0 (cited on pages 30, 37, 169).

[86] Man-Duen Choi. 'Positive Linear Maps on Complex Matrices'. In: *Linear Algebra and its Applications* 10.3 (1975), pp. 285–290. DOI: 10.1016/0024-3795(75)90075-0 (cited on pages 30, 37, 169).

[87] Bertrand Russell. 'On the Notion of Cause'. In: *Proceedings of the Aristotelian Society* 13 (1912), pp. 1–26. eprint: http://www.jstor.org/stable/4543833 (cited on page 33).

[88] Luca Apadula, Alessandro Bisio, and Paolo Perinotti. *No-signalling constrains quantum computation with indefinite causal structure*. 2022. arXiv: 2202.10214 [quant-ph] (cited on pages 35, 71, 119, 124, 171).

[89] John de Pillis. 'Linear transformations which preserve hermitian and positive semidefinite operators.' In: *Pacific Journal of Mathematics* 23.1 (1967), pp. 129–137 (cited on pages 37, 169).

[90] John Watrous. *The Theory of Quantum Information*. Cambridge, UK: Cambridge University Press, 2018. ISBN: 978-1-107-18056-7 (cited on pages 37, 141, 171, 200, 209).

[91] Marco Túlio Quintino, Qingxiuxiong Dong, Atsushi Shimbo, Akihito Soeda, and Mio Murao. 'Probabilistic exact universal quantum circuits for transforming unitary operations'. In: *Physical Review A* 100.6 (Dec. 2019). DOI: 10.1103/physreva.100.062339. arXiv: 1909.01366 [quant-ph] (cited on pages 38, 161).

[92] Jessica Bavaresco, Mio Murao, and Marco Túlio Quintino. 'Strict Hierarchy between Parallel, Sequential, and Indefinite-Causal-Order Strategies for Channel Discrimination'. In: *Physical Review Letters* 127.20 (Nov. 2021). DOI: 10.1103/physrevlett.127.200504. arXiv: 2011.08300 [quant-ph] (cited on pages 38, 64, 161).

[93] Jessica Bavaresco, Mio Murao, and Marco Túlio Quintino. 'Unitary channel discrimination beyond group structures: Advantages of sequential and indefinite-causal-order strategies'. In: *Journal of Mathematical Physics* 63.4 (Apr. 2022). DOI: 10.1063/5.0075919. arXiv: 2105.13369 [quant-ph] (cited on pages 38, 161).

[94] Giulio Chiribella, Giacomo M. D'Ariano, and Paolo Perinotti. 'Memory Effects in Quantum Channel Discrimination'. In: *Physical Review Letters* 101.18 (Oct. 2008). DOI: 10.1103/physrevlett.101.180501. arXiv: 0803.3237 [quant-ph] (cited on pages 41, 44, 53, 54, 194).

[95] Cyril Branciard. 'Witnesses of causal nonseparability: an introduction and a few case studies'. In: *Scientific Reports* 6.1 (May 2016). DOI: 10.1038/srep26018. arXiv: 1603.00043 [quant-ph] (cited on page 43).

[96] Steven Roman. *Advanced Linear Algebra*. 3rd ed. New York: Springer-Verlag, 2008. ISBN: 978-0-387-72828-5 (cited on pages 44, 175, 197, 201, 217, 218).

[97] Mafalda L. Almeida, Jean-Daniel Bancal, Nicolas Brunner, Antonio Acín, Nicolas Gisin, and Stefano Pironio. 'Guess Your Neighbor's Input: A Multipartite Nonlocal Game with No Quantum Advantage'. In: *Physical Review Letters* 104.23 (June 2010). DOI: 10.1103/physrevlett.104.230404. arXiv: 1003.3844 [quant-ph] (cited on pages 45, 46, 89).

[98] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. 'Quantum entanglement'. In: *Reviews of Modern Physics* 81.2 (June 2009), pp. 865–942. DOI: 10.1103/revmodphys.81.865. arXiv: quant-ph/0702225 [quant-ph] (cited on pages 47, 161).

[99] Ämin Baumeler, Adrien Feix, and Stefan Wolf. 'Maximal incompatibility of locally classical behavior and global causal order in multiparty scenarios'. In: *Physical Review A* 90.4 (Oct. 2014). DOI: 10.1103/physreva.90.042106. arXiv: 1403.7333 [quant-ph] (cited on page 47).

[100] Ravi Kunjwal and Ognyan Oreshkov. *Nonclassicality in correlations without causal order*. 2023. arXiv: 2307.02565 [quant-ph] (cited on page 47).

[101] Tomoyuki Morimae. *The process matrix framework for a single-party system*. 2014. arXiv: 1408.1464 [quant-ph] (cited on page 54).

[102] Richard V. Kadison. 'Unitary Invariants for Representations of Operator Algebras'. In: *Annals of Mathematics* 66.2 (1957), pp. 304–379. DOI: 10.2307/1970002. (Visited on 11/28/2023) (cited on page 57).

[103] Man-Duen Choi and Edward G. Effros. 'Injectivity and operator spaces'. In: *Journal of Functional Analysis* 24.2 (1977), pp. 156–209. DOI: 10.1016/0022-1236(77)90052-0 (cited on pages 57, 71).

[104] David M. Topping. 'Jordan Algebras of Self-Adjoint Operators'. In: *Memoirs of the American Mathematical Society* 1.53 (1957), pp. 1–48 (cited on page 57).

[105] Edward G. Effros and Erling Stormer. 'Jordan Algebras of Self-Adjoint Operators'. In: *Transactions of the American Mathematical Society* 127.2 (1967), pp. 313–316. DOI: 10.2307/1994650. (Visited on 11/28/2023) (cited on page 57).

[106] Allan Sinclair and Roger Smith. *Finite von Neumann Algebras and Masas*. Vol. 351. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 2008. ISBN: 978-0-521-71919-3 (cited on pages 57, 201).

[107] Fumio Hiai and Dénes Petz. *Introduction to Matrix Analysis and Applications*. Universitext. Springer, Cham, 2014. ISBN: 978-3-319-04149-0 (cited on pages 57, 68, 175).

[108] Günther Ludwig. *Foundations of Quantum Mechanics I*. Berlin Heidelberg: Springer-Verlag, 1983. ISBN: 978-3-642-86751-4 (cited on page 59).

[109] Matt S. Leifer and Robert W. Spekkens. 'Towards a formulation of quantum theory as a causally neutral theory of Bayesian inference'. In: *Physical Review A* 88.5 (Nov. 2013). DOI: 10.1103/physreva.88.052130. arXiv: 1107.5849 [quant-ph] (cited on page 60).

[110] Andrew J. Gleason. 'Measures on the Closed Subspaces of a Hilbert Space'. In: *Indiana University Mathematics Journal* 6.4 (1957), pp. 885–893. DOI: 10.1512/iumj.1957.6.56050 (cited on pages 62, 63).

[111] Paul Busch. 'Quantum States and Generalized Observables: A Simple Proof of Gleason's Theorem'. In: *Physical Review Letters* 91.12 (Sept. 2003). DOI: 10.1103/physrevlett.91.120403. arXiv: quant-ph/9909073 [quant-ph] (cited on pages 62, 63, 196).

[112] Carlton M. Caves, Christopher A. Fuchs, Kiran K. Manne, and Joseph M. Renes. 'Gleason-Type Derivations of the Quantum Probability Rule for Generalized Measurements'. In: *Foundations of Physics* 34.2 (Feb. 2004), pp. 193–209. DOI: 10.1023/b:foop.0000019581.00318.a5. arXiv: quant-ph/0306179 [quant-ph] (cited on pages 62, 196, 211).

[113] Stephen M. Barnett, James D. Cresser, John Jeffers, and David T. Pegg. 'Quantum probability rule: a generalization of the theorems of Gleason and Busch'. In: *New Journal of Physics* 16.4 (Apr. 2014), p. 043025. DOI: 10.1088/1367-2630/16/4/043025. arXiv: 1308.0946 [quant-ph] (cited on page 62).

[114] Kieran Flatt, Stephen M. Barnett, and Sarah Croke. *Sequential measurements: Busch-Gleason theorem and its applications*. 2017. arXiv: 1704.07137 [quant-ph] (cited on page 62).

[115] Victoria J. Wright and Stefan Weigert. 'A Gleason-type theorem for qubits based on mixtures of projective measurements'. In: *Journal of Physics A: Mathematical and Theoretical* 52.5 (Jan. 2019), p. 055301. DOI: 10.1088/1751-8121/aaf93d. arXiv: 1808.08091 [quant-ph] (cited on pages 62, 171, 211).

[116] Costantino Budroni, Adán Cabello, Otfried Gühne, Matthias Kleinmann, and Jan-Åke Larsson. 'Kochen-Specker contextuality'. In: *Reviews of Modern Physics* 94.4 (Dec. 2022). DOI: 10.1103/revmodphys.94.045007. arXiv: 2102.13036 [quant-ph] (cited on pages 63, 171).

[117] Dénes Petz. 'Complementarity in quantum systems'. In: *Reports on Mathematical Physics* 59.2 (Apr. 2007), pp. 209–224. DOI: 10.1016/S0034-4877(07)00010-9. arXiv: quant-ph/0610189 [quant-ph] (cited on page 68).

[118] Rudolf Haag and Daniel Kastler. 'An Algebraic Approach to Quantum Field Theory'. In: *Journal of Mathematical Physics* 5.7 (July 1964), pp. 848–861. DOI: 10.1063/1.1704187. eprint: https://pubs.aip.org/aip/jmp/article-pdf/5/7/848/8174205/848_1_online.pdf (cited on page 68).

[119] Paolo Zanardi. 'Virtual Quantum Subsystems'. In: *Physical Review Letters* 87.7 (July 2001). DOI: 10.1103/physrevlett.87.077901. arXiv: quant-ph/0103030 [quant-ph] (cited on page 70).

[120] Paolo Zanardi, Daniel A. Lidar, and Seth Lloyd. 'Quantum Tensor Product Structures are Observable Induced'. In: *Physical Review Letters* 92.6 (Feb. 2004). DOI: 10.1103/physrevlett.92.060402. arXiv: quant-ph/0308043 [quant-ph] (cited on pages 70, 98).

[121] Paul Busch, Marian Grabowski, and Pekka J. Lahti. *Operational Quantum Physics*. Berlin Heidelberg: Springer-Verlag, 1995. ISBN: 978-3-540-49239-9 (cited on page 71).

[122] Christopher A. Fuchs. *Quantum Mechanics as Quantum Information (and only a little more)*. 2002. arXiv: quant-ph/0205039 [quant-ph] (cited on page 71).

[123] Howard Barnum, Christopher A. Fuchs, Joseph M. Renes, and Alexander Wilce. *Influence-free states on compound quantum systems*. 2005. arXiv: quant-ph/0507108 [quant-ph] (cited on pages 71, 108, 176, 196, 213).

[124] Steven Roman. *An Introduction to the Language of Category Theory*. Compact Textbooks in Mathematics. Birkhäuser, Cham, 2017. ISBN: 978-3-319-41916-9 (cited on pages 72, 125, 175).

[125] Chris Heunen and Jamie Vicary. *Categories for Quantum Theory: An Introduction*. Oxford: Oxford University Press, Nov. 2019. ISBN: 978-0-19-873961-6. eprint: https://academic.oup.com/book/43710/book-pdf/50991591/9780191060069_web.pdf (cited on pages 72, 125, 180).

[126] Peter Selinger. 'Dagger Compact Closed Categories and Completely Positive Maps: (Extended Abstract)'. In: *Electronic Notes in Theoretical Computer Science* 170 (2007). Proceedings of the 3rd International Workshop on Quantum Programming Languages (QPL 2005), pp. 139–163. DOI: https://doi.org/10.1016/j.entcs.2006.12.018 (cited on pages 72, 187).

[127] Bob Coecke. 'Axiomatic Description of Mixed States From Selinger's CPM-construction'. In: *Electronic Notes in Theoretical Computer Science* 210 (2008). Proceedings of the 4th International Workshop on Quantum Programming Languages (QPL 2006), pp. 3–13. DOI: https://doi.org/10.1016/j.entcs.2008.04.014 (cited on page 72).

[128] Matthias P. Kläy, Charles H. Randall, and David J. Foulis. 'Tensor products and probability weights'. In: *International Journal of Theoretical Physics* 26.3 (Mar. 1987), pp. 199–219. DOI: 10.1007/BF00668911 (cited on pages 78, 79).

[129] Guillaume Aubrun, Ludovico Lami, Carlos Palazuelos, and Martin Plávala. 'Entangleability of cones'. In: *Geometric and Functional Analysis* 31.2 (2021), pp. 181–205. DOI: 10.1007/s00039-021-00565-5. arXiv: 1911.09663 [math.FA] (cited on page 78).

[130] Aleks Kissinger, Matty Hoban, and Bob Coecke. *Equivalence of relativistic causal structure and process terminality*. 2017. arXiv: 1708.04118 [quant-ph] (cited on pages 81, 141).

[131] Scott Aaronson. 'Quantum computing, postselection, and probabilistic polynomial-time'. In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 461.2063 (Sept. 2005), pp. 3473–3482. DOI: 10.1098/rspa.2005.1546. arXiv: quant-ph/0412187 [quant-ph] (cited on page 85).

[132] Seth Lloyd, Lorenzo Maccone, Raul Garcia-Patron, Vittorio Giovannetti, Yutaka Shikano, Stefano Pirandola, Lee A. Rozema, Ardavan Darabi, Yasaman Soudagar, Lynden K. Shalm, and Aephraim M. Steinberg. 'Closed Timelike Curves via Postselection: Theory and Experimental Test of Consistency'. In: *Physical Review Letters* 106.4 (Jan. 2011). DOI: 10.1103/physrevlett.106.040403. arXiv: 1005.2219 [quant-ph] (cited on page 85).

[133] Ralph Silva, Yelena Guryanova, Nicolas Brunner, Noah Linden, Anthony J. Short, and Sandu Popescu. 'Pre- and postselected quantum states: Density matrices, tomography, and Kraus operators'. In: *Physical Review A* 89.1 (Jan. 2014). DOI: 10.1103/physreva.89.012121. arXiv: 1308.2089 [quant-ph] (cited on page 85).

[134] Ralph Silva, Yelena Guryanova, Anthony J. Short, Paul Skrzypczyk, Nicolas Brunner, and Sandu Popescu. 'Connecting processes with indefinite causal order and multi-time quantum states'. In: *New Journal of Physics* 19.10 (Oct. 2017), p. 103022. DOI: 10.1088/1367-2630/aa84fe. arXiv: 1701.08638 [quant-ph] (cited on pages 85, 88).

[135] Mateus Araújo, Philippe Allard Guérin, and Ämin Baumeler. 'Quantum computation with indefinite causal structures'. In: *Physical Review A* 96.5 (Nov. 2017). DOI: 10.1103/physreva.96.052315. arXiv: 1706.09854 [quant-ph] (cited on page 88).

[136] Yelena Guryanova, Ralph Silva, Anthony J. Short, Paul Skrzypczyk, Nicolas Brunner, and Sandu Popescu. 'Exploring the limits of no backwards in time signalling'. In: *Quantum* 3 (Dec. 2019), p. 211. DOI: 10.22331/q-2019-12-09-211. arXiv: 1708.00669 [quant-ph] (cited on page 88).

[137] Samson Abramsky and Bob Coecke. 'A Categorical Semantics of Quantum Protocols'. In: *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science*. Los Alamitos, CA, USA: IEEE Computer Society, July 2004, pp. 415–425. DOI: 10.1109/LICS.2004.1319636. arXiv: quant-ph/0402130 [quant-ph] (cited on pages 90, 137).

[138] Tom Purves and Anthony J. Short. 'Quantum Theory Cannot Violate a Causal Inequality'. In: *Physical Review Letters* 127.11 (Sept. 2021). DOI: 10.1103/physrevlett.127.110402. arXiv: 2101.09107 [quant-ph] (cited on page 92).

[139] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. 'Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels'. In: *Physical Review Letters* 70 (13 Mar. 1993), pp. 1895–1899. DOI: 10.1103/PhysRevLett.70.1895 (cited on page 93).

[140] Andris Ambainis, Ashwin Nayak, Ammon Ta-Shma, and Umesh Vazirani. 'Dense quantum coding and a lower bound for 1-way quantum automata'. In: *Proceedings of the thirty-first annual ACM symposium on Theory of Computing* (May 1999). DOI: 10.1145/301250.301347. arXiv: quant-ph/9804043 [quant-ph] (cited on page 93).

[141] Andris Ambainis, Ashwin Nayak, Amnon Ta-Shma, and Umesh Vazirani. 'Dense Quantum Coding and Quantum Finite Automata'. In: *Journal of the ACM* 49.4 (July 2002), pp. 496–511. DOI: 10.1145/581771.581773. eprint: https://people.eecs.berkeley.edu/~vazirani/pubs/qc.pdf (cited on page 93).

[142] Nicolas Brunner, Miguel Navascués, and Tamás Vértesi. 'Dimension Witnesses and Quantum State Discrimination'. In: *Physical Review Letters* 110.15 (Apr. 2013). DOI: 10.1103/physrevlett.110.150501. arXiv: 1209.5643 [quant-ph] (cited on page 94).

[143] Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and Types*. Translated in English and with appendices by Paul Taylor and Yves Lafont. Cambridge UK: Cambridge University Press, 1989. ISBN: 978-0-521-37181-3. eprint: https://www.paultaylor.eu/stable/prot.pdf (cited on pages 97, 110, 129, 233).

[144] Peter Selinger and Benoît Valiron. 'A lambda calculus for quantum computation with classical control'. In: *Mathematical Structures in Computer Science* 16.3 (June 2004), pp. 527–552. DOI: 10.1017/s0960129506005238. arXiv: cs/0404056 [cs.LO] (cited on page 110).

[145] John C. Baez and Mike Stay. 'Physics, Topology, Logic and Computation: A Rosetta Stone'. In: *Lecture Notes in Physics* (2010), pp. 95–172. DOI: 10.1007/978-3-642-12821-9_2. arXiv: 0903.0340 [quant-ph] (cited on page 110).

[146] Jean-Yves Girard. 'Linear logic'. In: *Theoretical Computer Science* 50.1 (1987), pp. 1–101. DOI: 10.1016/0304-3975(87)90045-4 (cited on pages 119, 129, 167, 233, 234).

[147] Christian Rétoré. 'Réseaux et Séquents Ordonnés'. PhD thesis. Université Paris 7, 1993 (cited on pages 119, 130, 167).

[148] Christian Retoré. 'Pomset Logic'. In: *Joachim Lambek: The Interplay of Mathematics, Logic, and Linguistics*. Ed. by Claudia Casadio and Philip J. Scott. Cham: Springer International Publishing, 2021, pp. 299–345. ISBN: 978-3-030-66545-6. DOI: 10.1007/978-3-030-66545-6{_}9. arXiv: 2001.02155 [cs.LO] (cited on pages 119, 130).

[149] Alessio Guglielmi. 'A System of Interaction and Structure'. In: *ACM Trans. Comput. Logic* 8.1 (Jan. 2007), 1–es. DOI: 10.1145/1182613.1182614 (cited on pages 119, 130, 167, 170).

[150] Nathan Jacobson. *Basic Algebra*. 2nd ed. Vol. I. Reprint of the second edition published by W. H. Freeman in 1985. Garden City, New York: Dover, 2009. ISBN: 978-0-486-47189-1 (cited on pages 119, 215).

[151] Garrett Birkhoff and John Von Neumann. 'The Logic of Quantum Mechanics'. In: *Annals of Mathematics* 37.4 (1936), pp. 823–843. (Visited on 08/08/2023) (cited on page 129).

[152] Richard Blute, Prakash Panangaden, and Sergey Slavnov. 'Deep Inference and Probabilistic Coherence Spaces'. In: *Applied Categorical Structures* 20.3 (2012), pp. 209–228. DOI: 10.1007/s10485-010-9241-0 (cited on pages 130, 167).

[153] Alastair A. Abbott, Julian Wechs, Fabio Costa, and Cyril Branciard. 'Genuinely multipartite non-causality'. In: *Quantum* 1 (Dec. 2017), p. 39. DOI: 10.22331/q-2017-12-14-39. arXiv: 1708.07663 [quant-ph] (cited on pages 157, 159).

[154] Ämin Baumeler and Stefan Wolf. 'Perfect signaling among three parties violating predefined causal order'. In: *2014 IEEE International Symposium on Information Theory* (June 2014). DOI: 10.1109/isit.2014.6874888. arXiv: 1312.5916 [quant-ph] (cited on page 165).

[155] Andrzej Grudka and Paweł Kurzyński. 'Is There Contextuality for a Single Qubit?' In: *Physical Review Letters* 100.16 (Apr. 2008). DOI: 10.1103/physrevlett.100.160401. arXiv: 0705.0181 [quant-ph] (cited on pages 171, 210).

[156] Teiko Heinosaari and Mário Ziman. *The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement*. Cambridge, UK: Cambridge University Press, 2011. ISBN: 978-0-521-19583-6 (cited on pages 171, 179).

[157] Julian Wechs, Hippolyte Dourdent, Alastair A. Abbott, and Cyril Branciard. 'Quantum Circuits with Classical Versus Quantum Control of Causal Order'. In: *PRX Quantum* 2.3 (Aug. 2021). DOI: 10.1103/prxquantum.2.030335. arXiv: 2101.08796 [quant-ph] (cited on page 172).

[158] Tom Lancaster and Stephen J. Blundell. *Quantum Field Theory for the Gifted Amateur*. Oxford: Oxford University Press, Apr. 2014. ISBN: 978-0-19-969932-2 (cited on page 175).

[159] Brian C. Hall. *Quantum Theory for Mathematicians*. Vol. 267. Graduate Texts in Mathematics. New York: Springer, 2013. ISBN: 978-1-461-47116-5 (cited on pages 175, 198, 199).

[160] Judea Pearl. *Causality. Models, Reasoning, and Inference*. 2nd ed. Cambridge, UK: Cambridge University Press, 2009. ISBN: 978-0-521-89560-6 (cited on page 175).

[161] Dennis D. Wackerly, William Mendenhall III, and Richard L. Scheaffer. *Mathematical Statistics with Applications*. 7th ed. Duxbury, Thomson Brooks/Cole, 2008. ISBN: 978-0-495-38508-0 (cited on page 175).

[162] Douglas R. Farenick. *Algebras of Linear Transformations*. New York: Springer, 2000. ISBN: 978-1-4613-0097-7 (cited on pages 179, 189, 197, 201).

[163] Jürgen Audretsch. *Entangled Systems: New Directions in Quantum Physics*. Weinheim: John Wiley and Sons, Ltd, 2007. ISBN: 978-3-527-40684-5 (cited on page 179).

[164] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. 2nd ed. Cambridge UK: Cambridge University Press, 2013. ISBN: 978-0-521-83940-2 (cited on pages 180, 198).

[165] Alexander Philip Dawid. 'Conditional Independence in Statistical Theory'. In: *Journal of the Royal Statistical Society. Series B (Methodological)* 41.1 (1979), pp. 1–31 (cited on page 182).

[166] John S. Bell. 'On the Einstein Podolsky Rosen paradox'. In: *Physics Physique Fizika* 1.3 (Nov. 1964), pp. 195–200. DOI: 10.1103/PhysicsPhysiqueFizika.1.195 (cited on page 183).

[167] Boris S. Tsirelson. 'Quantum generalizations of Bell's inequality'. In: *Letters in Mathematical Physics* 4.2 (1980), pp. 93–100. DOI: 10.1007/BF00417500 (cited on page 183).

[168] Sarah Croke and Stephen M. Barnett. 'Difficulty of distinguishing product states locally'. In: *Physical Review A* 95.1 (Jan. 2017). DOI: 10.1103/physreva.95.012337. arXiv: 1701.02579 [quant-ph] (cited on page 184).

[169] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. 'Mixed-State Entanglement and Distillation: Is there a "Bound" Entanglement in Nature?' In: *Physical Review Letters* 80.24 (June 1998), pp. 5239–5242. DOI: 10.1103/physrevlett.80.5239. arXiv: quant-ph/9801069 [quant-ph] (cited on pages 184, 212).

[170] Albert Einstein, Boris Podolsky, and Nathan Rosen. 'Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?' In: *Physical Review* 47.10 (May 1935), pp. 777–780. DOI: 10.1103/PhysRev.47.777 (cited on page 184).

[171] Philippe H. Eberhard. 'Bell's theorem and the different concepts of locality'. In: *Il Nuovo Cimento B (1971-1996)* 46.2 (Aug. 1978), pp. 392–419. DOI: 10.1007/BF02728628 (cited on page 184).

[172] Augustin Vanrietvelde, Hlér Kristjánsson, and Jonathan Barrett. 'Routed quantum circuits'. In: *Quantum* 5 (July 2021), p. 503. DOI: 10.22331/q-2021-07-13-503. arXiv: 2011.08120 [quant-ph] (cited on pages 190, 195).

[173] Armin Uhlmann. 'Anti- (conjugate) linearity'. In: *Science China Physics, Mechanics & Astronomy* 59.3 (Jan. 2016). DOI: 10.1007/s11433-015-5777-1 (cited on page 192).

[174] Chiara Macchiavello and G. Massimo Palma. 'Entanglement-enhanced information transmission over a quantum channel with correlated noise'. In: *Physical Review A* 65.5 (Apr. 2002). DOI: 10.1103/physreva.65.050301. arXiv: quant-ph/0107052 [quant-ph] (cited on page 194).

[175] Dennis Kretschmann and Reinhard F. Werner. 'Quantum channels with memory'. In: *Physical Review A* 72.6 (Dec. 2005). DOI: 10.1103/physreva.72.062323. arXiv: quant-ph/0502106 [quant-ph] (cited on page 194).

[176] Nilanjana Datta and Tony C. Dorlas. 'The coding theorem for a class of quantum channels with long-term memory'. In: *Journal of Physics A: Mathematical and Theoretical* 40.28 (June 2007), p. 8147. DOI: 10.1088/1751-8113/40/28/S20. arXiv: quant-ph/0610049 [quant-ph] (cited on page 194).

[177] Marco B. Plenio and Shashank Virmani. 'Spin Chains and Channels with Memory'. In: *Physical Review Letters* 99.12 (Sept. 2007). DOI: 10.1103/physrevlett.99.120504. arXiv: quant-ph/0702059 [quant-ph] (cited on page 194).

[178] Daniel K. L. Oi. 'Interference of Quantum Channels'. In: *Physical Review Letters* 91.6 (Aug. 2003). DOI: 10.1103/physrevlett.91.067902. arXiv: quant-ph/0303178 [quant-ph] (cited on page 195).

[179] Giulia Rubino, Lee A. Rozema, Adrien Feix, Mateus Araújo, Jonas M. Zeuner, Lorenzo M. Procopio, Časlav Brukner, and Philip Walther. 'Experimental verification of an indefinite causal order'. In: *Science Advances* 3.3 (Mar. 2017). DOI: 10.1126/sciadv.1602589. arXiv: 1608.01683 [quant-ph] (cited on page 195).

[180] Jonathan Barrett, Robin Lorenz, and Ognyan Oreshkov. *Quantum Causal Models*. 2019. arXiv: 1906.10726 [quant-ph] (cited on page 195).

[181] Jonathan Barrett, Robin Lorenz, and Ognyan Oreshkov. 'Cyclic quantum causal models'. In: *Nature Communications* 12.1 (Feb. 2021). DOI: 10.1038/s41467-020-20456-x. arXiv: 2002.12157 [quant-ph] (cited on page 195).

[182] Mateus Araújo, Adrien Feix, Miguel Navascués, and Časlav Brukner. 'A purification postulate for quantum mechanics with indefinite causal order'. In: *Quantum* 1 (Apr. 2017), p. 10. DOI: 10.22331/q-2017-04-26-10. arXiv: 1611.08535 [quant-ph] (cited on page 195).

[183] Reinhold A. Bertlmann and Philipp Krammer. 'Bloch vectors for qudits'. In: *Journal of Physics A: Mathematical and Theoretical* 41.23 (May 2008), p. 235303. DOI: 10.1088/1751-8113/41/23/235303. arXiv: 0806.1174 [quant-ph] (cited on page 199).

[184] Pablo Arrighi and Christophe Patricot. 'Conal representation of quantum states and non-trace-preserving quantum operations'. In: *Physical Review A* 68.4 (Oct. 2003). DOI: 10.1103/physreva.68.042310. arXiv: quant-ph/0212062 [quant-ph] (cited on page 199).

[185] Paweł Horodecki and Ravishankar Ramanathan. *Relativistic Causality vs. No-Signaling as the limiting paradigm for correlations in physical theories*. 2016. arXiv: 1611.06781 [quant-ph] (cited on page 212).

[186] Seiseki Akibue, Masaki Owari, Go Kato, and Mio Murao. 'Entanglement-assisted classical communication can simulate classical communication without causal order'. In: *Physical Review A* 96.6 (Dec. 2017). DOI: 10.1103/physreva.96.062331. arXiv: 1602.08835 [quant-ph] (cited on page 213).

[187] Ravi Kunjwal and Ämin Baumeler. 'Trading Causal Order for Locality'. In: *Physical Review Letters* 131.12 (Sept. 2023). DOI: 10.1103/physrevlett.131.120201. arXiv: 2202.00440 [quant-ph] (cited on page 213).

[188] Asher Peres. 'Separability Criterion for Density Matrices'. In: *Physical Review Letters* 77.8 (Aug. 1996), pp. 1413–1415. DOI: 10.1103/physrevlett.77.1413. arXiv: quant-ph/9604005 [quant-ph] (cited on page 214).

[189] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. 'Separability of mixed states: necessary and sufficient conditions'. In: *Physics Letters A* 223.1–2 (Nov. 1996), pp. 1–8. DOI: 10.1016/s0375-9601(96)00706-2. arXiv: quant-ph/9605038 [quant-ph] (cited on page 214).

[190] Robert Piziak, Patrick L. Odell, and R. Hahn. 'Constructing projections on sums and intersections'. In: *Computers & Mathematics with Applications* 37.1 (Jan. 1999), pp. 67–74. DOI: 10.1016/S0898-1221(98)00242-9 (cited on page 218).