

INFORMATION AND CODING THEORY
Exercise Sheet 2

Exercise 1. The random variables X and Y have the joint probability distribution given by the table below:

X \ Y	0	1	2
0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
1	$\frac{1}{9}$	0	$\frac{2}{9}$
2	$\frac{1}{9}$	$\frac{2}{9}$	0

Find:

- (a) $H(X)$ and $H(Y)$.
- (b) $H(X, Y)$.
- (c) $H(X|Y)$ and $H(Y|X)$.
- (d) $I(X; Y)$.
- (e) Draw a Venn diagram for the quantities obtained above.

Exercise 2. An urn contains r red, g green, and b black balls. We consider as a random variable X the color of a ball that is randomly drawn from the urn.

- (a) Show that the entropy of drawing k balls with replacement (i.e., putting the ball back to the urn after retrieving its color) is given by $H(X_1, X_2, \dots, X_k) = k \cdot H(X)$ where $H(X)$ is the entropy of a single draw.
- (b) The balls are not replaced, and we do not look at the color of the balls that were taken out during the first $i - 1$ draws (or, in other words, the colors of the preceding $i - 1$ balls are forgotten). Give an intuitive argument why $H(X_i) = H(X)$ where $H(X)$ is the entropy of an arbitrary draw and X_i is the color of the ball at the i th draw. Without appealing to intuition, the rigorous proof of this claim follows after solving (c) to (f).
- (c) The balls are not replaced. Compare the probability of having the colors c_1 at the first draw and c_2 at the second draw to the probability of having the color c_2 at the first draw and c_1 at the second draw.
- (d) The balls are not replaced. What is the probability that a red ball is drawn at the second draw?
- (e) The balls are not replaced. What can we say about the marginal probabilities of drawing a specific color at the second draw?
- (f) The balls are not replaced. Compare the entropy of the second draw with that of the first draw.
- (g) Show that the entropy of drawing k balls without replacements is less than or equal to the entropy of drawing k balls with replacement.

Exercise 3. The random variables X , Y and Z form a Markov chain $X \rightarrow Y \rightarrow Z$ if the conditional probability of Z depends only on Y , i.e. $p(X, Y, Z) = p(X)p(Y|X)p(Z|Y)$.

- (a) Show that X and Z are conditionally independent given Y , i.e. $p(X, Z|Y) = p(X|Y)p(Z|Y)$.

- (b) Show that $I(X; Y) \geq I(X; Z)$ (data processing inequality) by using the result of (a) and the chain rule for mutual information $I(X; Y|Z)$.
- (c) The random variables X , Y and Z have probability distributions on sets \mathcal{X} of cardinality n , \mathcal{Y} of cardinality k and \mathcal{Z} of cardinality m . Show that if $n > k$ and $m > k$ then $I(X; Z) \leq \log_2 k$.
- (d) Explain what happens if $k = 1$.

Exercise 4. Consider binary sequences of length n emitted by a source of independent and identically distributed (i.i.d.) random variables. Each bit is emitted independently of the previous one according to the binomial distribution where with probability p a 1 is emitted. A **typical sequence** contains k ones and $(n - k)$ zeros, with $k \simeq np$ (the mean number of ones is rounded to the closest integer.)

- (a) What is the probability that the source emits a particular typical sequence? Give an approximation to that probability as a function of the entropy of the source.
- (b) What is the number of typical sequences, expressed as a function of the entropy of the source? Compare it with the absolute number of sequences that can be emitted by the source. What is the approximate probability that a typical sequence is emitted? (Use the Stirling approximation $n! \approx n^n$)
- (c) What is the most probable sequence that can be emitted by the source? Is it typical?

Exercise 5.

Definition. A real-valued map $\rho(x, y)$ is a **metric** if it satisfies the following properties $\forall x, y, z$:

- $\rho(x, y) \geq 0$
- $\rho(x, y) = \rho(y, x)$
- $\rho(x, y) = 0$ if and only if $x = y$
- $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$

- (a) Let $\rho(X, Y) = H(X|Y) + H(Y|X)$. Verify that $\rho(X, Y) = H(X, Y) - I(X; Y) = 2H(X, Y) - H(X) - H(Y)$, and interpret this relation with the help of a Venn diagram.
- (b) Prove that $\rho(X, Y)$ satisfies all properties of a distance if the notation $X = Y$ means that there exists a bijection between X and Y .

Hint: To show the last property, use the strong subadditivity.

Exercise 6. Let X, Y and Z be uniformly distributed binary random variables such that $H(X) = H(Y) = H(Z) = 1$ bit. If Z is ignored, the mutual information of X and Y can be defined as $I(X; Y) = H(X) - H(X|Y)$. If Z is taken into account, the mutual information, **conditioned** on Z , can be defined as $I(X; Y|Z) = H(X, Z) - H(X|Y, Z)$. Finally, the mutual information of all three random variables can be defined as $I(X; Y; Z) = I(X; Y) - I(X; Y|Z)$. The last quantity can be positive, negative, or zero.

- (a) Give an example of a distribution of X , Y , and Z , such that $I(X; Y; Z) > 0$.
- (b) Give an example of a distribution of X , Y , and Z , such that $I(X; Y; Z) < 0$.

The exercises and solutions are available at <http://quic.ulb.ac.be/teaching>