

INFORMATION AND CODING THEORY  
Exercise Sheet 2

**Exercise 1.** The random variables  $X$  and  $Y$  have the joint probability distribution given by the table below:

	Y	0	1	2
X				
	0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
	1	$\frac{1}{9}$	0	$\frac{2}{9}$
	2	$\frac{1}{9}$	$\frac{2}{9}$	0

Find:

- (a)  $H(X)$  and  $H(Y)$ .
- (b)  $H(X|Y)$  and  $H(Y|X)$ .
- (c)  $H(X, Y)$ .
- (d)  $I(X; Y)$ .
- (e) Draw a Venn diagram for the quantities obtained above.

**Exercise 2.** An urn contains  $r$  red,  $g$  green and  $b$  black balls. We consider as a random variable  $X$  the color of a ball that is randomly drawn from the urn.

- (a) Show that the entropy of drawing  $k$  balls with replacement is given by  $H(X_1, X_2, \dots, X_k) = k \cdot H(X)$  where  $H(X)$  is the entropy of a single draw.
- (b) If the balls are not replaced, and if we do not look at the color of the balls that were taken out, give an intuitive argument why  $H(X_i) = H(X)$  where  $H(X)$  is the entropy of an arbitrary draw. Without appealing to intuition, it might be useful to start solving (??) to (??).
- (c) The balls are not replaced. Compare the probability that has the colors  $c_1$  at the first draw and  $c_2$  at the second draw to the probability that has the color  $c_2$  at the first draw and  $c_1$  at the second draw.
- (d) The balls are not replaced. What is the probability that a red ball is drawn at the second draw?
- (e) The balls are not replaced. What can we say about the marginal probabilities?
- (f) The balls are not replaced. Compare the entropy of the second draw with that of the first draw.
- (g) Show that the entropy of drawing  $k$  balls without replacements is less than or equal to the entropy of drawing  $k$  balls with replacement.

**Exercise 3.** The random variables  $X$ ,  $Y$  and  $Z$  form a Markov chain  $X \rightarrow Y \rightarrow Z$  if the conditional probability of  $Z$  depends only on  $Y$ , i.e.  $p(X, Y, Z) = p(X)p(Y|X)p(Z|Y)$ .

- (a) Show that  $X$  and  $Z$  are conditionally independent given  $Y$ , i.e.  $p(X, Z|Y) = p(X|Y)p(Z|Y)$ .
- (b) Show that  $I(X; Y) \geq I(X; Z)$  by using the result of (ba) and the chain rule for mutual information  $I(X; Y|Z)$ .
- (c) The random variables  $X$ ,  $Y$  and  $Z$  have probability distributions on sets  $\mathcal{X}$  of cardinality  $n$ ,  $\mathcal{Y}$  of cardinality  $k$  and  $\mathcal{Z}$  of cardinality  $m$ . Show that if  $n > k$  and  $m > k$  then  $I(X; Z) \leq \log_2 k$ .

(d) Explain what happens if  $k = 1$ .

**Exercise 4.** Consider binary sequences of length  $n$  emitted by a source of independent and identically distributed (i.i.d.) random variables. Each bit is emitted independently of the previous one according to the binomial distribution where with probability  $p$  a 1 is emitted. A **typical sequence** contains  $k$  ones and  $(n - k)$  zeros, with  $k \simeq np$  (the mean number of ones is rounded to the closest integer.)

- (a) What is the probability that the source emits a particular typical sequence? Give an approximation to that probability as a function of the entropy of the source.
- (b) What is the number of typical sequences, expressed as a function of the entropy of the source? Compare it with the absolute number of sequences that can be emitted by the source. What is the approximate probability that a typical sequence is emitted? (Use the Stirling approximation  $n! \approx n^n$ )
- (c) What is the most probable sequence that can be emitted by the source? Is it typical?

**Exercise 5.**

**Definition.**  $\rho(x, y)$  is a **metric** if it satisfies the following properties  $\forall x, y, z$ :

- $\rho(x, y) \geq 0$
- $\rho(x, y) = \rho(y, x)$
- $\rho(x, y) = 0$  if and only if  $x = y$
- $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$

- (a) Let  $\rho(X, Y) = H(X|Y) + H(Y|X)$ . Verify that  $\rho(X, Y) = H(X, Y) - I(X; Y) = 2H(X, Y) - H(X) - H(Y)$ , and interpret this relation with the help of a Venn diagram.
- (b) Prove that  $\rho(X, Y)$  satisfies all properties of a distance if the notation  $X = Y$  means that there exists a bijection between  $X$  and  $Y$ .

*Hint:* To show the last property use the strong sub-additivity.

**Exercise 6.** Let  $X, Y$  and  $Z$  be uniformly distributed binary random variables such that  $H(X) = H(Y) = H(Z) = 1$  bit. If  $Z$  is ignored, the mutual information of  $X$  and  $Y$  can be defined as  $I(X; Y) = H(X) - H(X|Y)$ . If  $Z$  is taken into account, the mutual information, **conditioned** on  $Z$ , can be defined as  $I(X; Y|Z) = H(X, Z) - H(X|Y, Z)$ . Finally, the mutual information of all three random variables can be defined as  $I(X; Y; Z) = I(X; Y) - I(X; Y|Z)$ . The last quantity can be positive, negative or zero.

- (a) Give an example for a distribution of  $X, Y$  and  $Z$  such that  $I(X; Y; Z) > 0$ .
- (b) Give an example for a distribution of  $X, Y$  and  $Z$  such that  $I(X; Y; Z) < 0$ .

The exercises and solutions are available at <http://quic.ulb.ac.be/teaching>