

INFORMATION AND CODING THEORY  
Exercise Sheet 5

**Exercise 1. Channel with additive noise.** We consider a memoryless channel with additive noise, with input  $X$  and output  $Y = X + Z$ , where  $Z$  is a random variable with  $P(Z = 0) = P(Z = a) = \frac{1}{2}$ ,  $a \in \mathbb{N}$ . We assume that the alphabet of the input is  $\mathcal{X} = \{0, 1\}$  and that  $Z$  is independent of  $X$ . Calculate the capacity of this channel as a function of  $a$ .

**Exercise 2.** We consider a memoryless channel  $S$  with input  $X = \{0, 1, 2, 3\}$  and output  $Y = X + Z \pmod{4}$ , where  $Z = \{-1, 0, 1\}$  with probabilities  $\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$ , respectively. In addition,  $X$  and  $Z$  are independent.

- (a) What is the probability distribution  $p^*(x)$  that maximizes the mutual information?
- (b) Calculate the capacity of this channel.
- (c) Calculate the capacity of a channel system that consists of two concatenations of the channel.
- (d) Calculate the capacity of a channel system that consists of two times the channel in parallel. Compare the result with the results of (b) and (c).

**Exercise 3.** We are given a noisy channel with a binary alphabet for the input and output, with transition matrix

$$p(y|x) = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix} \quad x, y \in \{0, 1\}.$$

- (a) Calculate the capacity and the probability distribution  $p^*(x)$  that attains this maximum.
- (b) Find the symmetric binary channel that gives the same capacity as the previous channel.
- (c) Calculate the capacity of a channel with transition matrix

$$p(y|x) = \begin{pmatrix} 1 & 0 \\ 1-q & q \end{pmatrix}, \quad x, y \in \{0, 1\}, q \in [0, 1]$$

**Exercise 4. Suboptimal codes.** We reuse the channel of exercises 5-3 and consider a sequence of random error correction codes  $(2^{nR}, n)$ , where each codeword is a sequence of  $n$  equiprobable random bits.

- (a) This sequence does not attain the capacity that we have calculated above. Why?
- (b) Determine the maximal transmission rate  $R$  for which the average error probability  $P_e^{(n)}$  of the random codes tends to zero for a block length  $n$  tending to infinity.

**Exercise 5. Preprocessing of the output.** We consider a channel characterized by the transition matrix  $p(y|x)$  and a given capacity  $C$ . We want to increase  $C$  by introducing a “preprocessing” of the output,  $\tilde{Y} = f(Y)$ . The resulting channel is thus  $X \rightarrow Y \rightarrow \tilde{Y}$ , with capacity  $\tilde{C}$ .

- (a) Show that  $\tilde{C} \leq C$ . What does this imply?
- (b) When does this preprocessing not decrease the capacity of the channel?