Information and Coding Theory

Exercise Sheet 5

Exercise 1. Channel with additive noise. We consider a memoryless channel with additive noise, with input X and output $Y = X + Z$, where Z is a random variable with $P(Z = 0) = P(Z = a) = \frac{1}{2}$, $a \in \mathbb{N}$. We assume that the alphabet of the input is $\mathcal{X} = \{0, 1\}$ and that Z is independent of X. Calculate the capacity of this channel as a function of a.

Exercise 2. We consider a memoryless channel S with input $X = \{0, 1, 2, 3\}$ and output $Y = X + Z \pmod 4$, where $Z = \{-1, 0, 1\}$ with probabilities $\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$, respectively. In addition, X and Z are independent.

(a) What is the probability distribution $p^*(x)$ that maximizes the mutual information?

(b) Calculate the capacity of this channel.

(c) Calculate the capacity of a channel system that consists of two concatenations of the channel.

(d) Calculate the capacity of a channel system that consists of two times the channel in parallel. Compare the result with the results of (b) and (c).

Exercise 3. We are given a noisy channel with a binary alphabet for the input and output, with transition matrix

$$p(y|x) = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}, \qquad x, y \in \{0, 1\}.$$

(a) Calculate the capacity and the probability distribution $p^*(x)$ that attains this maximum.

(b) Find the symmetric binary channel that gives the same capacity as the previous channel.

(c) Calculate the capacity of a channel with transition matrix

$$p(y|x) = \begin{pmatrix} 1 & 0 \\ 1-q & q \end{pmatrix}, \qquad x, y \in \{0, 1\}, q \in [0, 1].$$

Exercise 4. Suboptimal codes. We reuse the channel of exercises 5-3 and consider a sequence of random error correction codes $(2^{nR}, n)$, where each codeword is a sequence of n equiprobable random bits.

(a) This sequence does not attain the capacity that we have calculated above. Why?

(b) Determine the maximal transmission rate R for which the average error probability $P_e^{(n)}$ of the random codes tends to zero for a block length n tending to infinity.

Exercise 5. Preprocessing of the output. We consider a channel characterized by the transition matrix $p(y|x)$ and a given capacity C. We want to increase C by introducing a “preprocessing” of the output, $\tilde{Y} = f(Y)$. The resulting channel is thus $X \to Y \to \tilde{Y}$, with capacity $\tilde{C}$.

(a) Show that $\tilde{C} \leq C$. What does this imply?

(b) When does this preprocessing not decrease the capacity of the channel?

The exercises and solutions are available at `http://quic.ulb.ac.be/teaching`