MQ I Particule chargée dans un champ magnétique uniforme

$$\vec{B} = \mathrm{rot}\,\vec{A} = \vec{\nabla} \times \vec{A}$$

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m}$$

1) $[\vec{p}, \vec{A}] = \vec{p}\vec{A} - \vec{A}\vec{p}$, $\vec{p} = -i\hbar\vec{\nabla}$

$$\Leftrightarrow [\vec{p}, \vec{A}] f = \left(-i\hbar\vec{\nabla}(\vec{A} f)\right) - \left(\vec{A}((-i\hbar\vec{\nabla}) f)\right)$$

$$= -i\hbar(\vec{\nabla}\vec{A})f \cancel{- i\hbar\vec{A}(\vec{\nabla} f)} \cancel{+ i\hbar\vec{A}(\vec{\nabla} f)} = -i\hbar\,\mathrm{div}(\vec{A})\, f$$

$$\Rightarrow [\vec{p}, \vec{A}] = -i\hbar\,\mathrm{div}\,\vec{A}$$

$\Rightarrow$ jauge de Coulomb : $\mathrm{div}\,\vec{A} = 0 \Rightarrow [\vec{p}, \vec{A}] = 0$

$$\Rightarrow \vec{p}\vec{A} = \vec{A}\vec{p}$$

$$\Rightarrow H = \frac{1}{2m}(\vec{p} - q\vec{A})(\vec{p} - q\vec{A})$$

$$= \frac{1}{2m}\left(\vec{p}^2 - q\vec{p}\vec{A} - q\vec{A}\vec{p} + q^2\vec{A}^2\right)$$

$$= \frac{1}{2m}\left(\vec{p}^2 - 2q\vec{A}\vec{p} + q^2\vec{A}^2\right)$$

2) $\vec{A} = Bx\,\vec{1\!\!1}_y$, $\vec{B} = \mathrm{rot}\,\vec{A}$

$$\vec{\nabla} \times \vec{A} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} 0 \\ Bx \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_y \cdot 0 - \partial_z \cdot Bx \\ \partial_z \cdot 0 - \partial_x \cdot 0 \\ \partial_x Bx - \partial_y 0 \end{pmatrix} = B\,\vec{1\!\!1}_z$$

$\rightarrow$ uniforme ✓

Séance 6 - ②

$\text{div}\,\vec{A} = 0$?

$$\vec{\nabla}\cdot\vec{A} = \partial_x A_x + \partial_y A_y + \partial_z A_z$$
$$= \partial_x \cdot 0 + \partial_y (Bx) + \partial_z \cdot 0 = 0 \checkmark$$

$$H = \frac{1}{2m}\left(\vec{p}^{\,2} - 2q\vec{A}\cdot\vec{p} + q^2\vec{A}^2\right)$$

$$\vec{A}\cdot\vec{p} = Bx\,p_y \ , \quad \vec{A}^2 = B^2x^2$$

$$\Leftrightarrow \quad H = \frac{1}{2m}\left(-\hbar^2\Delta + i\hbar 2qBx\,\partial_y + q^2B^2x^2\right)$$

3)

$$[H, p_x] = \frac{1}{2m}\Big(-\hbar^2\cancel{[\vec{p}^{\,2}, p_x]} - 2qB[xp_y, p_x] + q^2B^2[x^2, p_x]\Big)$$

$$= \frac{1}{2m}\Big(-2qB\big(x\cancel{[p_y, p_x]} + \underbrace{[x, p_x]}_{= i\hbar}\,p_y\big) + q^2B^2\,2i\hbar x\Big)$$

$$= i\hbar\frac{qB}{m}\left(-p_y + qBx\right) \neq 0$$

$$[H, p_y] = \frac{1}{2m}\left(-\hbar\cancel{[\vec{p}^{\,2}, p_y]} - 2qB\cancel{[xp_y, p_y]} + q^2B^2\cancel{[x^2, p_y]}\right)$$
$$= 0$$

$[H, p_z] = \dots = 0 \Rightarrow$ ECOC : H, p_y, p_z , car $[p_y, p_z] = 0$

(ensemble complet d'observables qui commutent)

MQ I

4) $\Psi(x,y,z) = \Phi(x)\, e^{ik_y y}\, e^{ik_z z}$

$H\Psi = E\Psi$

$$\frac{1}{2m}\left(-\hbar^2\Delta + i\hbar 2qBx\,\partial_y + q^2B^2x^2\right)\Phi(x)e^{ik_y y}e^{ik_z z} = E\Psi(x,y,z)$$

$$= \frac{1}{2m}\Big(-\hbar^2\Phi''(x)e^{ik_y y}e^{ik_z z} - \hbar^2\Phi(x)(-k_y^2e^{ik_y y})e^{ik_z z}$$

$$-\hbar^2\Phi(x)e^{ik_y y}(-k_z^2)e^{ik_z z} + i\hbar 2iqk_yBx\,\Phi(x)e^{ik_y y}e^{ik_z z}$$
$$+ q^2B^2x^2\Phi(x)e^{ik_y y}e^{ik_z z}\Big)$$

$$= \Big(-\frac{\hbar^2}{2m}\Phi''(x) + \frac{\hbar^2}{2m}(k_y^2+k_z^2)\Phi(x) - \hbar\frac{qk_y}{m}Bx\,\Phi(x)$$

$$+\frac{q^2B^2}{2m}x^2\Phi(x)\Big)e^{ik_y y}e^{ik_z z} = E\Psi(x,y,z)$$

$$-\frac{\hbar^2}{2m}\Phi''(x) + \frac{q^2B^2}{2m}(x-x_0)^2\Phi(x) = E_{n_x}\Phi(x) \Rightarrow$$ Oscillateur harmonique en x "shifté" par x_0

5) $x_0 = \frac{\hbar k_y}{Bq} = \frac{p_y}{qB}$

$E_{n_x} = E - \frac{\hbar^2k_z^2}{2m} \Rightarrow E_{n_x} = (n+\frac{1}{2})\hbar\omega$ $(n=n_x)$ $\Rightarrow$

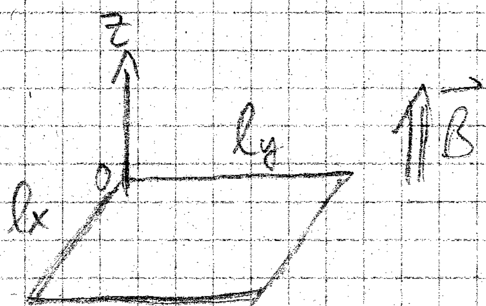
$\omega = \frac{qB}{m}$ = fréquence cyclotron

$$E = E_{n k_z k_y} = \left(n+\frac{1}{2}\right)\underbrace{\frac{\hbar q B}{m}}_{\omega_c} + \frac{\hbar^2 k_z^2}{2m}$$

Fonction d'ondes pour l'oscillateur harmonique donner par les polynômes d'Hermite

$$\Rightarrow \Psi_{n k_y k_z}(x,y,z) \propto e^{i(k_z z + k_y y)}\, e^{-\left(x-\frac{\hbar k_y}{qB}\right)^2 \frac{qB}{2\hbar}}\, H_n\left(\sqrt{\frac{qB}{\hbar}}\left(x-\frac{\hbar k_y}{qB}\right)\right)$$

6)



Pas de movement en $z \Rightarrow k_z = 0$

Conditions aux bords : $\Psi(y=0) = \Psi(y=l_y) = 0$

$\Rightarrow \Psi \propto \sin(k_y y)$ avec $k_y = \frac{n_y \pi}{l_y}$, $n_y \in \mathbb{Z}$

Dégénérescence : $0 < x_0 \leq l_x$

$$\Leftrightarrow \quad 0 < \frac{\hbar k_y}{Bq} \leq l_x$$

Flux magnétique

$$\Leftrightarrow \quad 0 < n_y \leq \frac{B l_x l_y}{\frac{\pi\hbar}{q}} = \frac{\Phi_B}{\Phi_0}$$

quantum de flux magnétique

pour $B = 2{,}605\,T$, $l_x = l_y = 10^{-2}\,m$

(longueur caractéristique puits : $\sqrt{\frac{\hbar}{qB}} \simeq 2{,}57 \cdot 10^{-8}\,m \ll l_x, l_y$)

$\Rightarrow n_y \lesssim 2{,}0048 \cdot 10^{10}$

$\Rightarrow$ Mesurer ω (par absorption de micro-ondes) permet de mesurer la masse effective des électrons dans un matériau

Séance 6-(5)

laser $\lambda = 337\,\mu m$, raie abs. pour $B = 2{,}605\,T$

$$\frac{qB}{m_f} = \omega = \frac{c2\pi}{\lambda} \Rightarrow \text{masse effective } m_f \approx 0{,}080\,m_e$$

7)

Changement de jauge:

$$\vec{A} \to \tilde{\vec{A}} = \vec{A} + \vec{\nabla} f$$

$$\Rightarrow \psi \to \tilde{\psi} = e^{i\varphi(\vec{x})}\psi$$

phase

Montrer que $\langle \psi | (\hat{p} - q\vec{A}) | \psi \rangle = \langle \tilde{\psi} | (\hat{p} - q\tilde{\vec{A}}) | \tilde{\psi} \rangle$

(ça implique $\langle \psi | \frac{(\hat{p} - q\vec{A})^2}{2m} | \psi \rangle = \langle \tilde{\psi} | \frac{(\hat{p} - q\tilde{\vec{A}})^2}{2m} | \tilde{\psi} \rangle$)

$$\langle \tilde{\psi} | (\hat{p} - q\tilde{\vec{A}}) | \tilde{\psi} \rangle$$

$$= \langle \psi | e^{-i\varphi}\Big[\underbrace{-i\hbar\vec{\nabla}}_{=p}(e^{i\varphi}|\psi\rangle) + e^{i\varphi}(-i\hbar\vec{\nabla})|\psi\rangle - q\vec{A}e^{i\varphi}|\psi\rangle - q(\vec{\nabla} f)e^{i\varphi}|\psi\rangle\Big]$$

$$= \underbrace{\langle \psi | e^{-i\varphi}\left(-i\hbar(\nabla e^{i\varphi}) - q(\vec{\nabla} f)e^{i\varphi}\right)|\psi\rangle}_{} + \langle \psi | (\hat{p} - q\vec{A}) | \psi \rangle$$

$$\perp \langle \psi | e^{-i\varphi}\left(-i\hbar i\vec{\nabla}\varphi e^{i\varphi} - q\nabla f e^{i\varphi}\right)|\psi\rangle \overset{!}{=} 0$$

$$\Rightarrow \hbar\nabla\varphi = q\nabla f \Rightarrow \boxed{\varphi(\vec{x}) = \frac{qf(\vec{x})}{\hbar}}$$

$$\Rightarrow \boxed{\tilde{\psi} = e^{iqf(\vec{x})/\hbar}\psi}$$

Ex: $\tilde{\vec{A}} = -By\mathbb{1}_x = \vec{A} + \vec{\nabla} f \Rightarrow f(\vec{x}) = Bxy$

$$\Rightarrow \tilde{\psi} = e^{iqBxy/\hbar}\psi$$