Quantum Mechanics II

Exercise 1: Density matrix

30 September 2019

1. Let state $\hat{\rho}$ be given by $\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k|$, where $\forall k \in \mathbf{N}_0 : p_k \geq 0$ and $\sum_k p_k = 1$. Prove that

   a) $\hat{\rho}$ is hermitian,

   b) $\mathrm{Tr}\, \hat{\rho} = 1$,

   c) $\hat{\rho} \geq 0$,

   d) $1 - \hat{\rho} \geq 0$.

   How the eigenvalues of $\hat{\rho}$ can be interpreted ?
   Reminder: Operator $\hat{\rho}$ is positive if and only if $\forall|\psi\rangle : \langle\psi|\hat{\rho}|\psi\rangle \geq 0$.

2. Demonstrate (prove) that $\mathrm{Tr}\, \hat{\rho}^2 \leq 1$. When does the equality $\mathrm{Tr}\, \hat{\rho}^2 = 1$ hold ?

3. Knowing that the evolution of $\hat{\rho}(t)$ obeys Liouville's equation

   $$i\hbar \frac{d\hat{\rho}}{dt} = [H, \hat{\rho}],$$

   show that if the initial state $\hat{\rho}(0)$ is pure, it stays pure for all t.

4. Derive Ehrenfest's theorem using Liouville's evolution.

   Reminder: Ehrenfest's theorem gives the evolution of the mean value of observable.

5. In two-dimensional Hilbert space with orthonormal basis $\{|a\rangle, |b\rangle\}$, is it possible to discriminate by measurements the states (a), (b) and (c) defined below?

   a) Superposition of two basis states $|a\rangle$ and $|b\rangle$ given by corresponding amplitudes α and β.

   b) Statistical mixture of basis states $|a\rangle$ and $|b\rangle$ taken with weights $|\alpha|^2$ and $|\beta|^2$ correspondingly.

   c) Equally weighted statistical mixture of pure states $|\psi\rangle$ and $|\phi\rangle$ where state $|\psi\rangle$ is the same as in item a) and state $|\phi\rangle$ is given by the amplitudes α and $-\beta$.

   Hint: Analyze the density matrices of these states.