Quantum Mechanics II

Exercise 2: Wigner Representation

Wigner representation of quantum states is equivalent to the one by density operators.

1. The Wigner function for a system in the state $\hat{\rho}$ defined in the *phase space* (x, p) is given by
$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ipy/\hbar}\langle x - y/2|\hat{\rho}|x + y/2\rangle\, dy.$$

   a) Using the identity
   $$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-a)}\, dp = \delta(x-a)$$
   show that the integral of the Wigner function over p is a probability distribution for x and *vice versa*.

   b) Verify that the expectation value of the operator of kinetic energy $\hat{T} = \hat{p}^2/2m$ is given by
   $$\langle\hat{T}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(p)W(x,p)\, dx\, dp,$$

   c) Verify that the expectation value of operator of potential energy $\hat{U} = U(\hat{x})$ is given by
   $$\langle\hat{U}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x)W(x,p)\, dx\, dp.$$

2. Quantum superposition and mixture of two Gaussian states. Use here $\hbar = 1$.

   Let two (non-normalized) Gaussian states be given by the wave functions:
   $$\begin{aligned}\psi_1(x) &= \exp\left[-(x-5)^2\right],\\ \psi_2(x) &= \exp\left[-(x+5)^2\right].\end{aligned}$$

   a) Find (up to a constant) the Wigner function of equiprobable statistical mixture of the two states.

   b) Find (up to a constant) the Wigner function of the superposition of the two states given by equal amplitudes.

   c) Comparer the two Wigner functions. In which case does the Wigner function show some non-classical features of the state? Why is the Wigner function called *quasi probability* distribution?

3. *Coherent state* is an eigenstate of the annihilation operator. Use here $\hbar = 1$.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \text{where} \quad \hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}.$$

The (complex) eigenvalue α is related to the phase space variables x et p as

$$\begin{aligned} x &= \sqrt{2}\,\mathrm{Re}(\alpha), \\ p &= \sqrt{2}\,\mathrm{Im}(\alpha). \end{aligned}$$

a) Find the average number of particles in the coherent state.

b) Find the representation of the coherent state in the eigenbasis of the number operator. What gives this representation for the coherent state with $\alpha = 0$?

c) Find the wave function of the coherent state in x-representation $\varphi_\alpha(x) = \langle x|\alpha\rangle$ using the representation of the annihilation operator in terms of position and momentum as given above. Remember that in the position representation we have

$$\begin{aligned} \hat{x} &= \int_{-\infty}^{\infty} x|x\rangle\langle x|\,dx \\ \hat{p} &= -i\int_{-\infty}^{\infty} \frac{d}{dx}|x\rangle\langle x|\,dx. \end{aligned}$$

d) Find the Wigner function of the coherent state. What is the shape of this function in the phase space?

Reminder

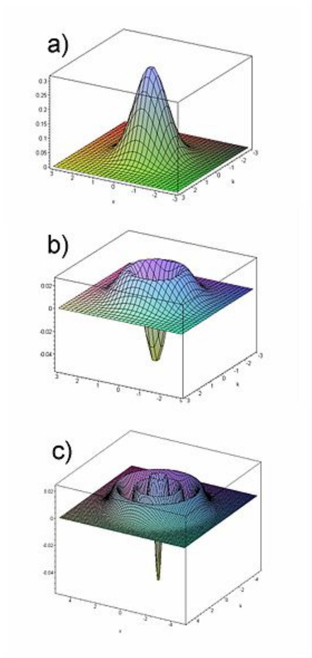
Integral of a Gaussian function

$$\int_{-\infty}^{+\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}.$$

Example

The Wigner quasi probability distribution of the Fock states :



a) $|0\rangle$

b) $|1\rangle$

c) $|5\rangle$.