

Quantum Mechanics II

Exercise 3: Density matrix

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1. Let $\hat{\rho}$ be a mixed state $\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k|$, where $\forall k : p_k \geq 0$ and $\sum_k p_k = 1$.

Prove that

- a) $\hat{\rho}$ is hermitian,
- b) $\text{Tr } \hat{\rho} = 1$,
- c) $\hat{\rho} \geq 0$,
- d) $1 - \hat{\rho} \geq 0$.

How the eigenvalues of $\hat{\rho}$ can be interpreted ?

Reminder: Operator $\hat{\rho}$ is positive if and only if $\forall |\psi\rangle : \langle\psi|\hat{\rho}|\psi\rangle \geq 0$.

2. Demonstrate (prove) that $\text{Tr } \hat{\rho}^2 \leq 1$. When does the equality $\text{Tr } \hat{\rho}^2 = 1$ hold ?
3. Knowing that the evolution of $\hat{\rho}(t)$ obeys Liouville's equation

$$i\hbar \frac{d\hat{\rho}}{dt} = [H, \hat{\rho}],$$

show that if the initial state $\hat{\rho}(0)$ is pure, it stays pure for all t .

4. Derive Ehrenfest's theorem using Liouville's evolution.

Reminder: Ehrenfest's theorem gives the evolution of the mean value of observable.

5. In two-dimensional Hilbert space with orthonormal basis $\{|a\rangle, |b\rangle\}$, is it possible to distinguish by measurements the states defined below?
- a) Superposition of two basis states $|a\rangle$ and $|b\rangle$ given by corresponding amplitudes α and β .
 - b) Statistical mixture of basis states $|a\rangle$ and $|b\rangle$ taken with weights $|\alpha|^2$ and $|\beta|^2$ correspondingly.
 - c) Equally weighted mixture of pure states $|\psi\rangle$ and $|\phi\rangle$ where state $|\psi\rangle$ is the same as in item a) and state $|\phi\rangle$ is given by the amplitudes α and $-\beta$.

Hint: Analyze the density matrices of these states.