

Quantum Mechanics II

Exercise 3: Systems of identical particles

9 October 2019

1. Permutation operator \hat{P}_{21} for a system of two particles.

- (a) Show that this operator possesses two eigenvalues ± 1 . What are the properties of corresponding eigenvectors?
- (b) Consider two operators $\hat{S}_{\pm} = (1 \pm \hat{P}_{21})/2$ called respectively symmetrizer / anti-symmetrizer. Show that they are :
 - hermitian,
 - projectors,
 - they project on the orthogonal subspaces.
- (c) Show that $\hat{P}_{21}\hat{S}_{\pm} = \hat{S}_{\pm}\hat{P}_{21} = \pm\hat{S}_{\pm}$ and $\hat{S}_{+} + \hat{S}_{-} = \mathbb{I}$.
How can we interpret these equalities?
- (d) Using the property proven in 1 c) show that $\hat{S}_{\pm}|\psi\rangle$ is an eigenstate of \hat{P}_{21} with eigenvalue ± 1 .

2. Generalization to N particles.

Let permutation operator \hat{P} correspond to a particular permutation P of N particles and p be the parity of the permutation P . Consider operators $\hat{S}_{\pm} = \frac{1}{N!} \sum_P (\pm 1)^p \hat{P}$ respectively called symmetrizer / antisymmetrizer (here the summation is taken over all possible permutations P of N particles). Show that $\hat{P}\hat{S}_{\pm} = \hat{S}_{\pm}\hat{P} = (\pm 1)^p \hat{S}_{\pm}$

and deduce the following facts:

- (a) \hat{S}_{\pm} are projectors,
- (b) \hat{S}_{+} et \hat{S}_{-} project on the orthogonal subspaces,
- (c) $\hat{S}_{\pm}|\psi\rangle$ is an eigenstate of \hat{P} with eigenvalue ± 1 confirming that the eigenstates of \hat{P} are completely symmetric or antisymmetric.

3. Identical particles crossing a beamsplitter.

Consider a particle prepared at the initial moment of time t_0 as a wave packet $\psi(\vec{r}, t_0) = \phi_1(\vec{r})$ arriving at a beamsplitter 50%–50% as shown in Figure 1. When the wave packet already crossed the beamsplitter, at time t_1 , the state of the particle can be written as $\psi_1(\vec{r}, t_1) = \frac{1}{\sqrt{2}} (\phi_3(\vec{r}) + \phi_4(\vec{r}))$. Here ϕ_3 and ϕ_4 denote normalized outgoing wave packets propagating in one or another direction so that their overlap can be neglected: $\langle \phi_3 | \phi_4 \rangle \approx 0$.

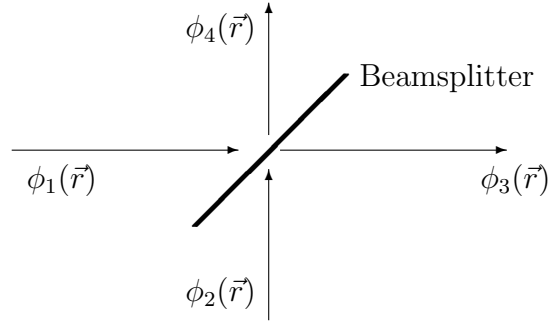


Figure 1: Configuration of incoming and outgoing waves for a beamsplitter.

- (a) If we prepare a particle in the state $\psi(\vec{r}, t_0) = \phi_2(\vec{r})$, coming from the direction which is symmetric to $\phi_1(\vec{r})$ with respect to the beamsplitter, the state of the particle at the moment t_1 can be written as an unknown superposition

$$\psi_2(\vec{r}, t_1) = \alpha\phi_3(\vec{r}) + \beta\phi_4(\vec{r}).$$

Determine (up to a global phase) the coefficients α et β taking into account that crossing the beamsplitter corresponds to a Hamiltonian evolution. Take an example of $\phi_2(x)$ which satisfies $\langle \phi_2 | \phi_1 \rangle = 0$.

- (b) Prepare at the initial moment t_0 two fermions with the same state of spin, one in the state $\phi_1(\vec{r})$, and another in the state $\phi_2(\vec{r})$. What is the final state of the system? Is it possible to detect both fermions in the same output direction?
- (c) Take the conditions of the previous question and apply them to two bosons, also initially prepared in the same state of spin, one boson being initially in the state $\phi_1(\vec{r})$, and another in the state $\phi_2(\vec{r})$. Show that the two bosons always exit at the same output ¹.

4. * Bose-Einstein Condensate (BEC) in a harmonic trap.

Consider N bosons of spin zero in an isotropic harmonic trap of angular frequency ω . The interactions between particles are neglected and the mean number of particles at the energy level E is given by the Bose-Einstein law

$$n_E = \frac{1}{e^{(E-\mu)/k_b T} - 1},$$

where μ is chemical potential, T is temperature, and k_b is Boltzmann's constant.

Show that:

- (a) The chemical potential satisfies $\mu < \frac{3}{2}\hbar\omega$.

¹This experiment has been realized with photons by C.K. Hong *et al*, Phys. Rev. Lett. 59 (1987) 2044.

(b) The number of particles N outside the fundamental level of the trap satisfies

$$N \leq F(\xi) = \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2(e^{n\xi} - 1)}, \quad \xi = \frac{\hbar\omega}{k_B T}.$$

Reminder: The degeneracy of the energy level $E_n = (n + \frac{3}{2})\hbar\omega$ is

$$g_n = \frac{1}{2}(n+1)(n+2).$$

(c) In the high temperature limit ($k_B T \gg \hbar\omega$) the discrete sum in the definition of $F(\xi)$ can be replaced by an integral. Show that the number of particles outside the fundamental level is majorized by

$$N'_{\max} = \zeta(3) \left(\frac{k_B T}{\hbar\omega} \right)^3,$$

where the Riemann $\zeta(n)$ function.

Reminder:

$$\int_0^{\infty} \frac{x^{\alpha-1}}{e^x - 1} dx = \Gamma(\alpha)\zeta(\alpha), \quad \Gamma(3) = 2!, \quad \zeta(3) \approx 1.202.$$

(d) What happens if we place in the trap more than N'_{\max} particles? At which temperature this phenomenon can be observed in a trap of frequency $\frac{\omega}{2\pi} = 100$ Hz containing 10^6 atoms?

5. * Fermi gas : non-interacting fermions at low (zero) temperatures.

- “non-interacting” particles - the energy of the particles is only kinetic.
- “low temperatures” - the particles occupy the lowest possible energy levels.

Consider N non-interacting fermions of spin s at low temperature confined in a three-dimensional (cubic) box with the edge length L :

(a) Find a relation between the density of the Fermi gas ρ and the *Fermi momentum* p_F assuming that the number of fermions N is large. Use the momentum quantization of a free particle in a box with the momentum eigenvalues $\vec{p} = \frac{2\pi\hbar}{L}\vec{n}$, where $\vec{n} = (n_1, n_2, n_3)$ and all $n_i \neq 0$ integer. Take into account the maximal number of fermions of spin s which can occupy the same energy level.

Reminder: Fermi momentum p_F is the maximal absolute value of the momentum of a particle in the Fermi gas.

- (b) Express the average energy of the fermions in terms of the *Fermi energy* ε_F corresponding to Fermi momentum p_F .
- (c) Express the Fermi energy as a function of the density of fermions and deduce an expression of the Fermi energy for electrons.

Reminder: The Fermi energy of electrons can attain large values ($\varepsilon_F = 3eV$ in sodium metal) which is much higher than the kinetic energy of the thermal motion at room temperature ($k_B T \approx 0,025$ eV). That is why the “zero temperature” approximation is applicable for the conduction electrons in metals even at room temperature.

Note. Applications of the Fermi gas model:

- conduction electrons in a metal
- semi-conductors
- electronic degenerate gas in white dwarfs
- electronic degenerate gas in neutron stars