

Quantum Mechanics II

Exercise 4: Wigner Representation

Wigner representation of quantum states is equivalent to the one by density operators.

1. The Wigner function for a system in the state $\hat{\rho}$ is defined in the *phase space* (x, p) as

$$W(x, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ipy/\hbar} \langle x - y/2 | \hat{\rho} | x + y/2 \rangle dy.$$

where kets $|x\rangle$ are the eigenstates of the position operator:

$$\hat{x}|x\rangle = x|x\rangle.$$

- a) Using the identity

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip(x-a)} dp = \delta(x-a)$$

show that the integral of the Wigner function over p is a probability distribution for x and *vice versa*.

- b) Verify that the expectation value of the kinetic energy $\hat{T} = \hat{p}^2/2m$ is given by

$$\langle \hat{T} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T(p)W(x, p) dx dp,$$

- c) Verify that the expectation value of the potential energy $\hat{U} = U(\hat{x})$ is given by

$$\langle \hat{U} \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x)W(x, p) dx dp.$$

2. Quantum superposition and mixture of two Gaussian states. Use here $\hbar = 1$.

Let two (non-normalized) Gaussian states be given by the wave functions:

$$\begin{aligned} \psi_1(x) &= \exp[-(x-5)^2], \\ \psi_2(x) &= \exp[-(x+5)^2]. \end{aligned}$$

- a) Find (up to a constant) the Wigner function of equiprobable statistical mixture of the two states.
- b) Find (up to a constant) the Wigner function of the superposition of the two states given by equal amplitudes.
- c) Compare the two Wigner functions. In which case does the Wigner function show non-classical features of the state?

3. *Coherent state* is an eigenstate of the annihilation operator. Use here $\hbar = 1$.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \text{where} \quad \hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}.$$

The (complex) eigenvalue α is related to the phase space variables x et p as

$$\begin{aligned} x &= \sqrt{2} \operatorname{Re}(\alpha), \\ p &= \sqrt{2} \operatorname{Im}(\alpha). \end{aligned}$$

- a) Find the average number of particles in the coherent state.
- b) Find the representation of the coherent state in the eigenbasis of the number operator. What gives this representation for the coherent state with $\alpha = 0$?
- c*) Find the wave function of the coherent state in x -representation $\varphi_\alpha(x) = \langle x|\alpha\rangle$ using the representation of the annihilation operator in terms of the position and momentum as given above. Remember that in the position representation we have

$$\begin{aligned} \hat{x} &= \int_{-\infty}^{\infty} x|x\rangle\langle x| dx \\ \hat{p} &= -i \int_{-\infty}^{\infty} \frac{d}{dx}|x\rangle\langle x| dx. \end{aligned}$$

- d) Find the Wigner function of the coherent state using its wave function in the form

$$\varphi_\alpha(x) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{1}{2}(x-x_0)^2 + ip_0x} = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{1}{2}p_0^2} e^{ip_0x_0} e^{-\frac{1}{2}(x-x_0-ip_0)^2}.$$

What is the shape of this Wigner function in the phase space?

Reminder

Integrals of Gaussian functions

$$\int_{-\infty}^{+\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}.$$

Example

Wigner's quasiprobability distributions for Fock states :

a) $|0\rangle$

b) $|1\rangle$

c) $|5\rangle$.

