Quantum Mechanics II

Exercise 4: Wigner Representation

Wigner representation of quantum states is equivalent to the one by density operators.

1. The Wigner function for a system in the state $\hat{\rho}$ is defined in the *phase space* (x, p) as

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{ipy/\hbar}\langle x - y/2|\hat{\rho}|x + y/2\rangle\, dy.$$

where kets $|x\rangle$ are the eigenstates of the position operator:

$$\hat{x}|x\rangle = x|x\rangle.$$

a) Using the identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-a)}\, dp = \delta(x-a)$$

show that the integral of the Wigner function over p is the probability density for x and *vice versa*.

b) Verify that the expectation value of the kinetic energy operator $\hat{T} = \hat{p}^2/2m$ is given by

$$\langle\hat{T}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T(p)W(x,p)\, dx\, dp,$$

where $T(p)$ is a function of p.

c) Verify that the expectation value of the potential energy operator $\hat{U} = U(\hat{x})$ is given by

$$\langle\hat{U}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x)W(x,p)\, dx\, dp.$$

2. Quantum superposition and mixture of two Gaussian states. Use here $\hbar = 1$.

Let two (non-normalized) Gaussian states be given by the wave functions:

$$\begin{aligned}\psi_1(x) &= \exp\left[-(x-5)^2\right],\\ \psi_2(x) &= \exp\left[-(x+5)^2\right].\end{aligned}$$

a) Find (up to a constant) the Wigner function of equiprobable statistical mixture of the two states.

b) Find (up to a constant) the Wigner function of the superposition of the two states given by equal amplitudes.

c) Compare the two Wigner functions. In which case does the Wigner function show non-classical features of the state?

3. *Coherent state* is an eigenstate of the annihilation operator. Use here $\hbar = 1$.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \text{where} \quad \hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}.$$

The (complex) eigenvalue α is related to the phase space variables x et p as

$$\begin{aligned} x &= \sqrt{2}\,\mathrm{Re}(\alpha), \\ p &= \sqrt{2}\,\mathrm{Im}(\alpha). \end{aligned}$$

a) Find the average number of particles in the coherent state.

b) Find the representation of the coherent state in the eigenbasis of the number operator. What gives this representation for the coherent state with $\alpha = 0$?

c*) Find up to a normalization constant the wave function of the coherent state in x-representation $\varphi_\alpha(x) = \langle x|\alpha\rangle$ using the representation of the annihilation operator in terms of the position and momentum as given above. Remember that in the position representation we have

$$\begin{aligned} \hat{x} &= \int_{-\infty}^{\infty} x|x\rangle\langle x|\,dx \\ \hat{p} &= -i\int_{-\infty}^{\infty} \frac{d}{dx}|x\rangle\langle x|\,dx. \end{aligned}$$

d*) Find the Wigner function of the coherent state using its wave function in the form

$$\varphi_\alpha(x) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{1}{2}(x-x_0)^2 + ip_0 x} = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{1}{2}p_0^2} e^{ip_0 x_0} e^{-\frac{1}{2}(x-x_0-ip_0)^2}.$$

What is the shape of this Wigner function in the phase space?

Reminder

Integrals of Gaussian functions

$$\int_{-\infty}^{+\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}}\,,$$

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}\,.$$

Example

Wigner's quasiprobability distributions for Fock states :

a) $|0\rangle$

b) $|1\rangle$

c) $|5\rangle$.