# Quantum Mechanics II

## Exercise 4: Second quantization. $N$-body problems.

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1. Express two-particle operator $\hat{F} = \frac{1}{2}\sum_{\alpha\neq\beta} \hat{f}^{(2)}(x_\alpha; x_\beta)$ in terms of creation and annihilation operators for the case of bosons as well as fermions.

2. Show that the number operator $\hat{N} = \sum_i \hat{a}_i^\dagger \hat{a}_i$ (for bosons and fermions) commutes with the Hamiltonian

$$\hat{H} = \sum_{ij} \hat{a}_i^\dagger \langle i|T|j\rangle \hat{a}_j + \frac{1}{2}\sum_{ijkm} \hat{a}_i^\dagger \hat{a}_j^\dagger \langle ij|V|km\rangle \hat{a}_m \hat{a}_k.$$

3. Bose-Einstein Condensate (BEC) in a harmonic trap.

   Consider $N$ bosons of spin zero in an isotropic harmonic trap of angular frequency $\omega$. The interactions between particles are neglected and the mean number of particles at the energy level $E$ is given by the Bose-Einstein law

$$n_E = \frac{1}{e^{(E-\mu)/k_b T} - 1},$$

   where $\mu$ is chemical potential, $T$ is temperature, and $k_b$ is Boltzmann's constant.

   Show that:

   (a) The chemical potential satisfies $\mu < \frac{3}{2}\hbar\omega$.

   (b) The number of particles $N$ outside the fundamental level of the trap satisfies

$$N \leq F(\xi) = \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2(e^{n\xi} - 1)}, \qquad \xi = \frac{\hbar\omega}{k_b T}.$$

   Reminder: The degeneracy of the energy level $E_n = (n + \frac{3}{2})\hbar\omega$ is $g_n = \frac{1}{2}(n+1)(n+2)$.

   (c) In the high temperature limit ($k_\mathrm{B} T \gg \hbar\omega$) the discrete sum in the definition of $F(\xi)$ can be replaced by an integral. Show that the number of particles outside the fundamental level is majorized by

$$N'_{\max} = \zeta(3)\left(\frac{k_b T}{\hbar\omega}\right)^3,$$

   where the Rieman $\zeta(n)$ fonction.

   Reminder:

$$\int_0^\infty \frac{x^{\alpha-1}}{e^x - 1}\mathrm{d}x = \Gamma(\alpha)\zeta(\alpha), \qquad \Gamma(3) = 2!, \qquad \zeta(3) \approx 1.202.$$

(d) What happens if we place in the trap more than $N'_{\rm max}$ particles? At which temperature this phenomenon can be observed in a trap of frequency $\frac{\omega}{2\pi} = 100$ Hz containing $10^6$ atoms?

4. Fermi gas : non-interacting fermions at low (zero) temperatures.

- "non-interacting" particles - the energy of the particles is only kinetic.

- "low temperatures" - the particles occupy the lowest possible energy levels.

Consider $N$ non-interacting fermions of spin $s$ at low temperature confined in a three-dimensional (cubic) box with the edge length $L$:

(a) Find a relation between the density of the Fermi gas $\rho$ and the *Fermi momentum* $p_F$ assuming that the number of fermions $N$ is large. Use the momentum quantization of a free particle in a box with the momentum eigenvalues $\vec{p} = \frac{2\pi\hbar}{L}\vec{n}$, where $\vec{n} = (n_1, n_2, n_3)$ and all $n_i \neq 0$ integer. Take into account the maximal number of fermions of spin $s$ which can occupy the same energy level.

Reminder: Fermi momentum $p_F$ is the maximal absolute value of the momentum of a particle in the Femi gas.

(b) Express the average energy of the fermions in terms of the *Fermi energy* $\varepsilon_F$ corresponding to Fermi momentum $p_F$.

(c) Express the Fermi energy as a function of the density of fermions and deduce an expression of the Fermi energy for electrons.

Reminder: The Fermi energy of electrons can attain large values ($\varepsilon_F = 3eV$ in sodium metal) which is much higher than the kinetic energy of the thermal motion at room temperature ($k_{\rm B}T \approx 0,025$ eV). That is why the "zero temperature" approximation is applicable for the conduction electrons in metals even at room temperature.

Note. Applications of the Fermi gas model:

- conduction electrons in a metal
- semi-conductors
- electronic degenerate gas in white dwarfs
- electronic degenerate gas in neutron stars