Quantum Mechanics II

Exercise 5: Systems of identical particles

3 November 2017

1. Permutation operator $\hat{P}_{21}$ for a system of two particles.

   (a) Show that this operator possesses two eigenvalues ± 1. What are the properties of corresponding eigenvectors?

   (b) Consider two operators $\hat{S}_{\pm} = (1 \pm \hat{P}_{21})/2$ called respectively symmetrizer / antisymmetrizer. Show that they are :

   - hermitian,
   - projectors,
   - they project on the orthogonal subspaces.

   (c) Show that $\hat{P}_{21}\hat{S}_{\pm} = \hat{S}_{\pm}\hat{P}_{21} = \pm S_{\pm}$ and $\hat{S}_{+} + \hat{S}_{-} = \mathbb{I}$.
   How can we interpret these equalities?

   (d) Using the property proven in 1 c) show that $\hat{S}_{\pm}|\psi\rangle$ is an eigenstate of $\hat{P}_{21}$ with eigenvalue ± 1.

2. Generalization to N particles.

   Let permutation operator $\hat{P}$ correspond to a particular permutation P of N particles and p be the parity of the permutation P. Consider operators $\hat{S}_{\pm} = \frac{1}{N!}\sum_{P}(\pm 1)^{p}\hat{P}$ respectively called symmetrizer / antisymmetrizer (here the summation is taken over all possible permutations P of N particles). Show that $\hat{P}\hat{S}_{\pm} = \hat{S}_{\pm}\hat{P} = (\pm 1)^{p}\hat{S}_{\pm}$

   and deduce the following facts:

   (a) $\hat{S}_{\pm}$ are projectors,

   (b) $\hat{S}_{+}$ et $\hat{S}_{-}$ project on the orthogonal subspaces,

   (c) $\hat{S}_{\pm}|\psi\rangle$ is an eigenstate of $\hat{P}$ with eigenvalue ± 1 confirming that the eigenstates of $\hat{P}$ are completely symmetric or antisymmetric.

3. Identical particles crossing a beamsplitter.

   If we consider a particle prepared at the initial moment of time t_0 as a wave packet $\psi(\vec{r}, t_0) = \phi_1(\vec{r})$ arriving at a beamsplitter $50\% - 50\%$ as shown in Figure 1 then, when the wave packet already crossed the beamsplitter, at time t_1, the state of the particle can be written as $\psi_1(\vec{r}, t_1) = \frac{1}{\sqrt{2}}(\phi_3(\vec{r}) + \phi_4(\vec{r}))$. Here ϕ_3 and ϕ_4 denote normalized outgoing wave packets propagating in one or another direction so that their overlap can be neglected: $\langle\phi_3|\phi_4\rangle \approx 0$.

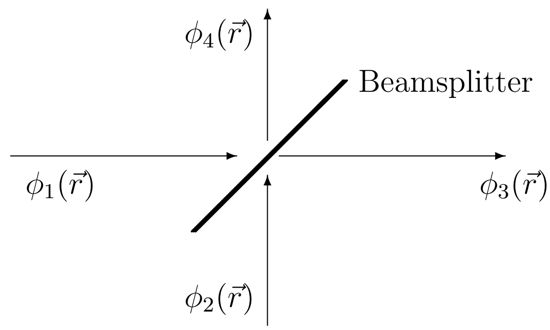

Figure 1: Configuration of incoming and outgoing waves for a beamsplitter.

(a) If we prepare a particle in the state $\psi(\vec{r}, t_0) = \phi_2(\vec{r})$, coming from the direction which is symmetric to $\phi_1(\vec{r})$ with respect to the beamsplitter, the state of the particle at the moment t_1 can be written as an unknown superposition

$$\psi_2(\vec{r}, t_1) = \alpha\phi_3(\vec{r}) + \beta\phi_4(\vec{r}).$$

Determine (up to a global phase) the coefficients α et β taking into account that crossing the beamsplitter corresponds to a Hamiltonian evolution. Take an example of $\phi_2(x)$ which satisfies $\langle\phi_2|\phi_1\rangle = 0$.

(b) Prepare at the initial moment t_0 two fermions with the same state of spin, one in the state $\phi_1(\vec{r})$, and another in the state $\phi_2(\vec{r})$. What is the final state of the system? Is it possible to detect both fermions in the same output direction?

(c) Take the conditions of the previous question and apply them to two bosons, also initially prepared in the same state of spin, one boson being initially in the state $\phi_1(\vec{r})$, and another in the state $\phi_2(\vec{r})$. Show that the two bosons always exit at the same output [1].

4. Bose-Einstein Condensate (BEC) in a harmonic trap.

Consider N bosons of spin zero in an isotropic harmonic trap of angular frequency ω. The interactions between particles are neglected and the mean number of particles at the energy level E is given by the Bose-Einstein law

$$n_E = \frac{1}{e^{(E-\mu)/k_b T} - 1},$$

where μ is chemical potential, T is temperature, and k_b is Boltzmann's constant.

Show that:

(a) The chemical potential satisfies $\mu < \frac{3}{2}\hbar\omega$.

[1] This experiment has been realized with photons by C.K. Hong *et al*, Phys. Rev. Lett. 59 (1987) 2044.

(b) The number of particles N outside the fundamental level of the trap satisfies

$$N \leq F(\xi) = \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2(e^{n\xi}-1)}, \qquad \xi = \frac{\hbar\omega}{k_b T}.$$

Reminder: The degeneracy of the energy level $E_n = (n + \frac{3}{2})\hbar\omega$ is $g_n = \frac{1}{2}(n+1)(n+2)$.

(c) In the high temperature limit ($k_{\mathrm{B}}T \gg \hbar\omega$) the discrete sum in the definition of $F(\xi)$ can be replaced by an integral. Show that the number of particles outside the fundamental level is majorized by

$$N'_{\max} = \zeta(3)\left(\frac{k_b T}{\hbar\omega}\right)^3,$$

where the Rieman $\zeta(n)$ fonction.

Reminder:

$$\int_0^{\infty} \frac{x^{\alpha-1}}{e^x - 1}\mathrm{d}x = \Gamma(\alpha)\zeta(\alpha), \qquad \Gamma(3) = 2!, \qquad \zeta(3) \approx 1.202.$$

(d) What happens if we place in the trap more than $N'_{\max}$ particles? At which temperature this phenomenon can be observed in a trap of frequency $\frac{\omega}{2\pi} = 100$ Hz containing 10^6 atoms?

5. Fermi gas : non-interacting fermions at low (zero) temperatures.

- "non-interacting" particles - the energy of the particles is only kinetic.
- "low temperatures" - the particles occupy the lowest possible energy levels.

Consider N non-interacting fermions of spin s at low temperature confined in a three-dimensional (cubic) box with the edge length L:

(a) Find a relation between the density of the Fermi gas ρ and the *Fermi momentum* p_F assuming that the number of fermions N is large. Use the momentum quantization of a free particle in a box with the momentum eigenvalues $\vec{p} = \frac{2\pi\hbar}{L}\vec{n}$, where $\vec{n} = (n_1, n_2, n_3)$ and all $n_i \neq 0$ integer. Take into account the maximal number of fermions of spin s which can occupy the same energy level.

Reminder: Fermi momentum p_F is the maximal absolute value of the momentum of a particle in the Femi gas.

(b) Express the average energy of the fermions in terms of the *Fermi energy* ε_F corresponding to Fermi momentum p_F.

(c) Express the Fermi energy as a function of the density of fermions and deduce an expression of the Fermi energy for electrons.

Reminder: The Fermi energy of electrons can attain large values ($\varepsilon_F = 3eV$ in sodium metal) which is much higher than the kinetic energy of the thermal motion at room temperature ($k_{\mathrm{B}}T \approx 0,025$ eV). That is why the "zero temperature" approximation is applicable for the conduction electrons in metals even at room temperature.

Note. Applications of the Fermi gas model:

- conduction electrons in a metal
- semi-conductors
- electronic degenerate gas in white dwarfs
- electronic degenerate gas in neutron stars