Quantum Mechanics II

Exercise 5: Systems of identical particles – Solutions

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1. Permutation operator $\hat{P}_{21}$ for a system of two particles.

(a) Show that this operator possesses two eigenvalues ± 1. What are the properties of corresponding eigenvectors?

Solution: From the the definition of $\hat{P}_{21}$ we have

$$\hat{P}_{21}\hat{P}_{21}|\psi_{12}\rangle = \hat{P}_{21}|\psi_{21}\rangle = |\psi_{12}\rangle, \quad \Rightarrow \hat{P}_{21}\hat{P}_{21} = \mathbb{I}.$$

Let $|\psi_{12}\rangle$ be an eigenstate of $\hat{P}_{21}$ with eigenvalue λ

$$\hat{P}_{21}\hat{P}_{21}|\psi_{12}\rangle = \hat{P}_{21}\lambda|\psi_{12}\rangle = \lambda^2|\psi_{12}\rangle, \quad \Rightarrow \lambda^2 = 1, \quad \Rightarrow \lambda = \pm 1.$$

The eigenvectors corresponding to $\lambda = 1$ are *symmetric*.
The eigenvectors corresponding to $\lambda = -1$ are *antisymmetric*.

(b) Consider two operators $\hat{S}_{\pm} = (1 \pm \hat{P}_{21})/2$ called respectively symmetrizer / antisymmetrizer. Show that they are :

- hermitian,
  Solution: From 1 a) it follows that the eigenvalues of $\hat{S}_{\pm}$ are real, hence $\hat{S}_{\pm}$ are hermitian ■
- projectors,
  Solution: From the definition of $\hat{S}_{\pm}$ we have

$$\begin{aligned}\hat{S}_{\pm}\hat{S}_{\pm} &= \frac{1}{4}(\mathbb{I} \pm \hat{P}_{21})(\mathbb{I} \pm \hat{P}_{21}) = \frac{1}{4}(\mathbb{I} \pm \hat{P}_{21} \pm \hat{P}_{21} + \underbrace{\hat{P}^2_{21})}_{=\mathbb{I}} \\ &= \frac{1}{4}(2\mathbb{I} \pm 2\hat{P}_{21}) = \frac{1}{2}(\mathbb{I} \pm \hat{P}_{21}) = \hat{S}_{\pm} \blacksquare\end{aligned}$$

- they project on the orthogonal subspaces.
  Solution:

$$\begin{aligned}\langle\psi_{12}|\hat{S}^{\dagger}_{-}\hat{S}_{+}|\psi_{12}\rangle &= \frac{1}{4}\langle\psi_{12}|(\mathbb{I} - \hat{P}_{21})(\mathbb{I} + \hat{P}_{21})|\psi_{12}\rangle \\ &= \frac{1}{4}\langle\psi_{12}|(\mathbb{I} + \hat{P}_{21} - \hat{P}_{21} - \underbrace{\hat{P}^2_{21})}_{=\mathbb{I}}|\psi_{12}\rangle = \frac{1}{4}\langle\psi_{12}|0|\psi_{12}\rangle = 0 \blacksquare\end{aligned}$$

(c) Show that $\hat{P}_{21}\hat{S}_\pm = \hat{S}_\pm\hat{P}_{21} = \pm\hat{S}_\pm$ and $\hat{S}_+ + \hat{S}_- = \mathbb{I}$. How can we interpret these results?

<u>Solution:</u> From the definition of $\hat{S}_\pm$ we have immediately

$$\hat{P}_{21}\hat{S}_\pm = \frac{1}{2}\hat{P}_{21}(\mathbb{I} \pm \hat{P}_{21}) = \frac{1}{2}(\hat{P}_{21} \pm \hat{P}_{21}\hat{P}_{21}) = \frac{1}{2}(\mathbb{I} \pm \hat{P}_{21})\hat{P}_{21} = \hat{S}_\pm\hat{P}_{21}\,,$$

then

$$\hat{S}_\pm\hat{P}_{21} = \frac{1}{2}(\mathbb{I} \pm \hat{P}_{21})\hat{P}_{21} = \frac{1}{2}(\hat{P}_{21} \pm \hat{P}_{21}\hat{P}_{21}) = \frac{1}{2}(\hat{P}_{21} \pm \mathbb{I}) = \pm\frac{1}{2}(\mathbb{I} \pm \hat{P}_{21}) = \pm\hat{S}_\pm\,,$$

and finally

$$\hat{S}_+ + \hat{S}_- = \frac{1}{2}(\mathbb{I} + \hat{P}_{21}) + \frac{1}{2}(\mathbb{I} - \hat{P}_{21}) = \mathbb{I}$$

(d) Using the property proven in 1 c) show that $\hat{S}_\pm|\psi\rangle$ is an eigenstate of $\hat{P}_{21}$ with eigenvalue ±1.

<u>Solution:</u> From 1 c) we have immediately

$$\hat{P}_{21}\hat{S}_\pm|\psi_{12}\rangle = \pm\hat{S}_\pm|\psi_{12}\rangle\,\blacksquare$$

2. Generalization to N particles.

Let permutation operator $\hat{P}$ corresponds to a particular permutation P of N particles and p be the parity of permutation P. Consider operators $\hat{S}_\pm = \frac{1}{N!}\sum_P(\pm1)^p\hat{P}$ respectively called symmetrizer / antisymmetrizer (here the summation is taken over all possible permutations P of N particles). Show that $\hat{P}\hat{S}_\pm = \hat{S}_\pm\hat{P} = (\pm1)^p\hat{S}_\pm$

<u>Solution:</u>

$$\hat{P}\hat{S}_\pm = \hat{P}\left(\frac{1}{N!}\sum_{P'}(\pm1)^{p'}\hat{P}'\right) = \frac{1}{N!}\sum_{P'}(\pm1)^{p'}\hat{P}\hat{P}'.$$

Let us note that $\hat{P}\hat{P}' = \hat{P}''$ is a permutation operator which corresponds to new permutation P'' and parity $p'' = p + p'$. Then using new notations we have

$$\hat{P}\hat{S}_\pm = \frac{1}{N!}\sum_{P'}(\pm1)^{p''-p}\hat{P}''.$$

Note that the sum $\sum_{P'}$ is taken over all permutations and therefore is the same as $\sum_{P''}$ which also is taken over all permutations. Then

$$\hat{P}\hat{S}_\pm = (\pm1)^{-p}\frac{1}{N!}\sum_{P''}(\pm1)^{p''}\hat{P}''.$$

Taking into account that p takes only the values $(0,1)$ we have $(\pm1)^{-p} = (\pm1)^p$, then

$$\hat{P}\hat{S}_\pm = (\pm1)^p\frac{1}{N!}\sum_{P''}(\pm1)^{p''}\hat{P}'' = (\pm1)^p\hat{S}_\pm\,\blacksquare$$

and deduce the following facts:

(a) $\hat{S}_{\pm}$ are projectors,

<u>Solution:</u> Let us check the defining property of projector operator.

$$\begin{aligned}
\hat{S}_{\pm}\hat{S}_{\pm} &= \left(\frac{1}{N!}\sum_P (\pm 1)^p \hat{P}\right)\hat{S}_{\pm} = \frac{1}{N!}\sum_P (\pm 1)^p \hat{P}\hat{S}_{\pm} \\
&= \frac{1}{N!}\sum_P (\pm 1)^p (\pm 1)^p \hat{S}_{\pm} = \frac{1}{N!}\sum_P \hat{S}_{\pm} = \hat{S}_{\pm} \blacksquare
\end{aligned}$$

(b) $\hat{S}_+$ et $\hat{S}_-$ project on the orthogonal subspaces,

<u>Solution:</u> The number of all permutations of N elements is $N! = 1 \cdot 2 \cdot (3 \cdot \ldots \cdot N)$ is even number, therefore the numbers of even and odd permutations are equal :

$$\begin{aligned}
\hat{S}_-\hat{S}_+ &= \hat{S}_-\left(\frac{1}{N!}\sum_P (+1)^p \hat{P}\right) = \frac{1}{N!}\sum_P \hat{S}_-\hat{P} \\
&= \frac{1}{N!}\sum_P (-1)^p \hat{S}_- = \frac{1}{N!}\hat{S}_{\pm}\left(\sum_{P_+} - \sum_{P_-}\right) \\
&= \frac{1}{N!}\hat{S}_{\pm}\left(\frac{N!}{2} - \frac{N!}{2}\right) = 0 \blacksquare
\end{aligned}$$

(c) $\hat{S}_{\pm}|\psi\rangle$ is an eigenstate of P with eigenvalue ± 1 confirming that the eigenstates of P are completely symmetric or anitisymmetric.

<u>Solution:</u> We have:

$$PS_{\pm}|\psi\rangle = (\pm 1)^p S_{\pm}|\psi\rangle \blacksquare$$

3. Identical particles crossing a beamsplitter.

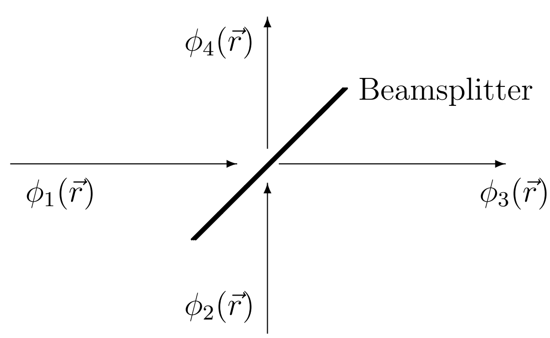

Figure 1: Configuration of incoming and outgoing waves for a beamsplitter.

If we consider a particle prepared at the initial moment of time t_0 as a wave packet $\psi(\vec{r}, t_0) = \phi_1(\vec{r})$ arriving at a balanced beamsplitter as shown in Figure 1 then, when the wave packet already crossed the beamsplitter, at time t_1, the state of the particle can be written as $\psi_1(\vec{r}, t_1) = \frac{1}{\sqrt{2}}(\phi_3(\vec{r}) + \phi_4(\vec{r}))$. Here ϕ_3 and ϕ_4 denote normalized outgoing wave packets propagating in one or another direction so that their overlap can be neglected: $\langle\phi_3|\phi_4\rangle \approx 0$.

(a) If we prepare a particle in the state $\psi(\vec{r}, t_0) = \phi_2(\vec{r})$, coming from the direction which is symmetric to $\phi_1(\vec{r})$ with respect to the beamsplitter, the state of the particle at the moment t_1 can be written as an unknown superposition

$$\psi_2(\vec{r}, t_1) = \alpha\phi_3(\vec{r}) + \beta\phi_4(\vec{r}).$$

Determine (up to a global phase) the coefficients α et β taking into account that crossing the beamsplitter corresponds to a Hamiltonian evolution. Take an example of $\phi_2(x)$ which satisfies $\langle\phi_2|\phi_1\rangle = 0$.

Solution: For the balanced beam splitter the transmission and reflection coefficients are equal and the inputs 1 and 2 are symmetric therefore the absolute values of the amplitudes at the output should be the same $|\alpha|^2 = |\beta|^2 = 1/\sqrt{2}$. Then denoting $\beta - \alpha = \varphi$ we have up to a global phase

$$\psi_2(\vec{r}, t_1) = \frac{1}{\sqrt{2}}\left(\phi_3(\vec{r}) + e^{i\varphi}\phi_4(\vec{r})\right).$$

The hamiltonian evolution is unitary, therefore it preserves orthogonality of initial states:

$$|\psi_{1,2}(\vec{r}, t_1)\rangle = U(t_1)|\phi_{1,2}\rangle \Rightarrow \langle\psi_1(\vec{r}, t_1)|\psi_2(\vec{r}, t_1)\rangle = \langle\phi_1|U^\dagger(t_1)U(t_1)|\phi_2\rangle = \langle\phi_1|\phi_2\rangle \approx 0.$$

This implies

$$\begin{aligned} \langle\phi_1|U^\dagger(t_1)U(t_1)|\phi_2\rangle = 0 \quad &\Leftrightarrow \quad \frac{1}{2}\left(\langle\phi_3| + \langle\phi_4|\right)\left(|\phi_3\rangle + e^{i\varphi}|\phi_4\rangle\right) = 0 \\ &\Leftrightarrow \quad 1 + e^{i\varphi} = 0 \Leftrightarrow e^{i\varphi} = -1 \\ &\Leftrightarrow \quad \varphi = \pi\,. \end{aligned}$$

The we have finally

$$\psi_2(\vec{r}, t_1) = \frac{1}{\sqrt{2}}\left(\phi_3(\vec{r}) - \phi_4(\vec{r})\right).$$

(b) Prepare at the initial moment t_0 two fermions with the same state of spin, one in the state $\phi_1(\vec{r})$, and another in the state $\phi_2(\vec{r})$. What is the final state of the system? Is it possible to detect both fermions in the same output direction?

Solution: The wave function of fermions is antisymmetric

$$|\psi(\vec{r}, t_0)\rangle_{\text{fermion}} = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle|\phi_2\rangle - |\phi_2\rangle|\phi_1\rangle\right)$$

Knowing the evolution of input state going through the beam splitter we have at the output

$$\begin{aligned} |\psi(\vec{r}, t_1)\rangle_{\text{fermion}} &= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|\phi_3\rangle + |\phi_4\rangle\right)\frac{1}{\sqrt{2}}\left(|\phi_3\rangle - |\phi_4\rangle\right)\right. \\ &- \left.\frac{1}{\sqrt{2}}\left(|\phi_3\rangle - |\phi_4\rangle\right)\frac{1}{\sqrt{2}}\left(|\phi_3\rangle + |\phi_4\rangle\right)\right] \\ &= \frac{1}{\sqrt{2}}\left[|\phi_4\rangle|\phi_3\rangle - |\phi_3\rangle|\phi_4\rangle\right] \end{aligned}$$

The output state is a superposition of states corresponding to the fermions exiting from the beamsplitter in different output directions.

(c) Take the conditions of the previous question and apply them to two bosons, also initially prepared in the same state of spin, one boson being initially in the state $\phi_1(\vec{r})$, and another in the state $\phi_2(\vec{r})$. Show that the two bosons always exit at the same output [1].

<u>Solution:</u> The wave function of bosons is symmetric

$$|\psi(\vec{r},t_0)\rangle_{\text{boson}} = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle|\phi_2\rangle + |\phi_2\rangle|\phi_1\rangle\right)$$

Knowing the evolution of input state going through the beam splitter we have at the output

$$\begin{aligned}
|\psi(\vec{r},t_1)\rangle_{\text{boson}} &= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(|\phi_3\rangle + |\phi_4\rangle\right)\frac{1}{\sqrt{2}}\left(|\phi_3\rangle - |\phi_4\rangle\right)\right.\\
&+ \left.\frac{1}{\sqrt{2}}\left(|\phi_3\rangle - |\phi_4\rangle\right)\frac{1}{\sqrt{2}}\left(|\phi_3\rangle + |\phi_4\rangle\right)\right]\\
&= \frac{1}{\sqrt{2}}\left[|\phi_3\rangle|\phi_3\rangle + |\phi_4\rangle|\phi_4\rangle\right]
\end{aligned}$$

The output state is a superposition of states corresponding to both bosons exiting together from the beamsplitter in one or another output directions.

4. <u>Bose-Einstein Condensate (BEC)</u> in a harmonic trap.

Consider N bosons of spin zero in an isotropic harmonic trap of angular frequency ω. The interactions between particles are neglected and the mean number of particles at the energy level E is given by the Bose-Einstein law

$$n_E = \frac{1}{e^{(E-\mu)/k_bT} - 1},$$

where μ is chemical potential and T is temperature.

Show that:

(a) The chemical potential satisfies $\mu < \frac{3}{2}\hbar\omega$.

<u>Solution:</u> The number of partticles is a non negative number, therefore, from the definition of n_E we have

$$n_E \geq 0 \Leftrightarrow e^{(E-\mu)/k_bT} - 1 \geq 0 \Leftrightarrow (E-\mu)/k_bT \geq 0 \Leftrightarrow E \geq \mu.$$

The energy levels, which for bosons in the three dimensional trap are

$$E_n = \left(n + \frac{3}{2}\right)\hbar\omega.$$

The last inequality should hold for all levels therefore,

$$E_n \geq E_0 = \frac{3}{2}\hbar \geq \mu \blacksquare$$

[1] This experiment has been realized with photons by C.K. Hong *et al*, Phys. Rev. Lett. 59 (1987) 2044.

(b) The number of particles N outside the fundamental level of the trap satisfies

$$N \leq F(\xi) = \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2(e^{n\xi}-1)}, \qquad \xi = \frac{\hbar\omega}{k_b T}.$$

Reminder: The degeneracy of the energy level $E_n = (n+\frac{3}{2})\hbar\omega$ is
$g_n = \frac{1}{2}(n+1)(n+2)$.

Solution: The number of particles N outside the fundamental level in the trap is given by the sum over all other energy levels

$$N' = \sum_{n=1}^{\infty} g_n n_{E_n} = \frac{1}{2}\sum_{n=1}^{\infty}(n+1)(n+2)(e^{(E_n-\mu)}-1)^{-1}$$

where g_n and n_{E_n} are degeneracy and the mean number of particles at level n. Starting from the inequality on μ proven in 4 a) we obtain

$$\begin{aligned}
-\mu \geq -\frac{3}{2}\hbar \quad &\Leftrightarrow \quad E_n - \mu \geq E_n - \frac{3}{2}\hbar\omega = n\hbar\omega \\
&\Leftrightarrow \quad e^{(E_n-\mu)/k_\mathrm{B}T} \geq e^{n\hbar\omega/k_\mathrm{B}T} \\
&\Leftrightarrow \quad \left(e^{(E_n-\mu)/k_\mathrm{B}T}-1\right)^{-1} \leq \left(e^{n\xi}-1\right)^{-1}, \quad \xi = \frac{\hbar\omega}{k_\mathrm{B}T}.
\end{aligned}$$

Using the last inequality in the expression for N' we have:

$$N' = \frac{1}{2}\sum_{n=1}^{\infty}(n+1)(n+2)(e^{(E_n-\mu)}-1)^{-1} \leq \sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{2(e^{n\xi}-1)} = F(\xi) \blacksquare$$

(c) In the limit $k_\mathrm{B}T \gg \hbar\omega$ the discrete sum in the definition of $F(\xi)$ can be replaced by an integral. Show that the number of particles outside the fundamental level is majorized by

$$N'_{\mathrm{max}} = \zeta(3)\left(\frac{k_b T}{\hbar\omega}\right)^3,$$

where the Rieman $\zeta(n)$ fonction.

Reminder:

$$\int_0^\infty \frac{x^{\alpha-1}}{e^x-1}\mathrm{d}x = \Gamma(\alpha)\zeta(\alpha), \qquad \Gamma(3) = 2!, \qquad \zeta(3) \approx 1.202.$$

Solution: In the limit $k_\mathrm{B}T \gg \hbar\omega$, variable $x = \frac{\hbar\omega}{k_\mathrm{B}T}n = \xi n$ changes almost continuously with n and therefore the sum in the definition of $F(\xi)$ can be replaced by the integral

$$F(\xi) = \int_0^\infty \frac{\left(\frac{x}{\xi}+1\right)\left(\frac{x}{\xi}+2\right)}{2(e^x-1)}\frac{dx}{\xi} = \frac{1}{2\xi}\int_0^\infty \frac{\left(\frac{x^2}{\xi^2}+3\frac{x}{\xi}+2\right)}{e^x-1}dx.$$

Note that in the limit $k_B T \gg \hbar\omega$ we have $\xi \ll 1$ and therefore

$$\frac{1}{\xi^2} \gg \frac{1}{\xi} \gg 1.$$

Keeping only the highest order term we obtain

$$F(\xi) \approx \frac{1}{2\xi^3} \int_0^\infty \frac{x^2}{e^x - 1} dx = \frac{1}{2\xi^3} \Gamma(3)\zeta(3)$$

Then using $\Gamma(3) = 2! = 2$ we come to

$$F(\xi) \approx \zeta(3) \left(\frac{k_B T}{\hbar\omega} \right)^3,$$

which gives us

$$N'_{\max} = \zeta(3) \left(\frac{k_B T}{\hbar\omega} \right)^3$$

(d) What happens if we place in the trap more than $N'_{\max}$ particles? At which temperature this phenomenon can be observed in a trap of frequency $\frac{\omega}{2\pi} = 100$ Hz containing 10^6 atoms?

Solution: If more than N' atoms are in the trap the excess number of bosons will be at the fundamental level. This phenomenon is called Bose-Einstein condensation. By inverting the result of 4 c) we obtain the temperature of such transition:

$$T = \left(\frac{N'}{\zeta(3)} \right)^{1/3} \frac{\hbar\omega}{k_B} \simeq 4,52 \cdot 10^{-7} K,$$

taking into account the following numbers

$$\begin{cases} N'_{\max} = 10^6 \\ \zeta(3) \approx 1.202 \\ \hbar\omega = h\nu = 6,63 \cdot 10^{-34} \cdot 100 [\mathrm{J \cdot s \cdot Hz}] \\ k_B = 1.38 \cdot 10^{-23} [\mathrm{J \cdot K^{-1}}]. \end{cases}$$

The relation shows us that by decreasing the temperature we decrease the number of bosons which can be outside the fundamental level. Even if initially we had $N < N'$ by decreasing temperature we can decrease N' such that $N > N'$, and therefore, $N - N'$ bosons will necessarily occupy the fundamental level (go to the BE condensate).

5. Fermi gas : non-interacting fermions at low (zero) temperature.

- "non-interacting" particles - the energy of the particles is only kinetic.
- "low temperature" - the particles occupy the lowest possible energy levels.

Consider N non-interacting fermions of spin s at low temperature confined in a three-dimensional (cubic) box with the edge length L:

(a) Find the relation between the density of the Fermi gas ρ and the *Fermi momentum* p_F asuming that the number of fermions N is large. Use the momentum quantization of a free particle in a box with the momentum eigenvalues $\vec{p} = \frac{2\pi\hbar}{L}\vec{n}$, where $\vec{n} = (n_1, n_2, n_3)$ and all $n_i \neq 0$ integer. Take into account the maximal number of fermions of spin s which can occupy the same energy level.

Reminder: Fermi momentum is the maximal absolute value of the momentum of a fermionic particle in the Femi gas.

Solution: The total hamiltonian of N independent fermions is a sum of N individual hamiltonians of single fermion in a box (in our case), because there is no interaction. The eigenstates of each individual hamiltonian are plane waves $\phi_{\mathbf{p}}(\mathbf{r}) = e^{i\mathbf{pr}}/L^3$, to which is associated one of $2s+1$ spin states corresponding to a particular projection $m_s\hbar$ ($m_s = -s, -s+1, \ldots, s$) of spin on a chosen axis. The three-dimensional momentum is given by $\mathbf{p} = (2\pi\hbar/L)\mathbf{n}$, where $\mathbf{n} = (n_1, n_2, n_3)$ is a three-dimensional vector of integers (positive or negative). The fundamental level of the total system is achieved when the particles fill in all possible states with lowest energies and therefore momenta. Then the absolute values of all momenta are bounded by *Fermi momentum* $|\mathbf{p}| < p_F$. The sum of the filled states is equal to N.

$$N = \sum_{\mathbf{p}(|\mathbf{p}|<p_F)} (2s+1).$$

For large N the sum may be replaced by integral over the sphere of radius p_F:

$$N \simeq (2s+1)\frac{L^3}{(2\pi\hbar)^3}\int_{|\mathbf{p}|<p_F} \mathrm{d}^3\mathbf{p} = (2s+1)\frac{L^3}{(2\pi\hbar)^3}\frac{4}{3}\pi p_F^3 = \frac{2s+1}{6\pi^2}\left(\frac{Lp_F}{\hbar}\right)^3.$$

The the gas density $\rho = N/L^3$ is related to P_F as

$$\rho = \frac{2s+1}{6\pi^2}\left(\frac{p_F}{\hbar}\right)^3.$$

(b) Express the average energy of the fermions in terms of the *Fermi energy* ε_F corresponding to Fermi momentum p_F.

Solution: The energy of non-interacting particles is only kinetic. For particles with mass m we have:

$$\langle E \rangle = \frac{\langle|\mathbf{p}|^2\rangle}{2m} = \frac{1}{N}\sum_{\mathbf{p}(|\mathbf{p}|<p_F)} (2s+1)\frac{|\mathbf{p}|^2}{2m} \simeq \frac{2s+1}{N}\left(\frac{L}{2\pi\hbar}\right)^3 \int_{|\mathbf{p}|<p_F} \frac{|\mathbf{p}|^2}{2m}\mathrm{d}^3\mathbf{p}.$$

Hence

$$\langle E \rangle = \frac{\langle|\mathbf{p}|^2\rangle}{2m} = \frac{3}{5}\frac{p_F^2}{2m} = \frac{3}{5}\epsilon_F$$

where $\epsilon_F = p_F^2/2m_e$

(c) Express the Fermi energy as a function of the density of fermions and deduce an expression of the Fermi energy for electrons.

NB: The Fermi energy of electrons can attain large values ($\varepsilon_F = 3eV$ in sodium metal) which is much higher than the kinetic energy of the thermal motion at room temperature ($k_{\mathrm{B}}T \approx 0,025$ eV). That is why the "zero temperature" approximation is applicable for the conduction electrons in metals even at room temperature.

Solution: Using the relation between the density of particles and the Fermi momentum we have for electrons ($s = 1/2$):

$$\rho_e = \frac{1}{3\pi^2}\left(\frac{p_F}{\hbar}\right)^3 = \frac{1}{3\pi^2\hbar^3}(2m\epsilon_F)^{3/2},$$

and finally

$$\epsilon_F = \frac{\hbar^2\left(3\rho_e\pi^2\right)^{3/2}}{2m_e}.$$

Note. Applications of the Fermi gas model:

- conduction electrons in a metal
- semi-conductors
- electronic degenerate gas in white dwarfs
- electronic degenerate gas in neutron stars